On endotrivial modules for finite reductive groups

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k field of char $\ell > 0$

G fin gp

$\operatorname{mod}(kG) =$ f.g. kG-mod

$\underline{\operatorname{mod}}(kG) =$ stable cat.

$$\underline{\operatorname{Hom}}_{kG}(M,N) = \frac{\operatorname{Hom}_{kG}(M,N)}{\operatorname{PHom}_{kG}(M,N)}$$

Def. A f.g. kG-mod M is endotrivial if

$$\operatorname{End}_k M \cong k \text{ in } \underline{\operatorname{mod}}(kG)$$

$$\Longleftrightarrow \underbrace{M^*}_{= \operatorname{Hom}_k(M,k)} \otimes_k M \cong k \oplus \text{proj in } \operatorname{mod}(kG)$$

$T(G)$ = set of stable iso classes of e-t kG-mod

$T(G) = $ ab gp

$$[M] + [N] = [M \oplus N]$$

$$0 = [k]$$

$$-[M] = [M^*]$$

<u>Thm</u>. $T(G)$ is fin. gen

$$T(G) \cong TT(G) \oplus TF(G)$$

finite $\qquad\qquad \mathbb{Z}^{n}$

$\geq X(G) = G/G$'s $\qquad\qquad n = \# \mathcal{C}_{\geq 2}(G)$

$S \in Syl_\ell(G)$

Most often $T(G) = X(G) \oplus \mathbb{Z}$

$\langle [\Omega(k)] \rangle$ $\quad \Omega(k) = \ker(P_k \twoheadrightarrow k)$

"THE" Question: When is $T(G) \neq X(G)$

Today: G is fin. reductive$/\mathbb{F}_q$

$\ell \nmid q$

$SL_n(q)$

Strategy (general)

$$\begin{tikzcd}
G \arrow[d, dash, "\text{Green corresp}"] & G/G'S \le \pi(G) \arrow[d] \\
\boxed{N_G(S)} \arrow[d, dash, "p'"', "\trianglelefteq"] & \pi(N_G(S)) = X(N_G(S)) \\
S &
\end{tikzcd}$$

$$X(N_G(S)) = N_G(S) / N_G(S)'S$$

The main problem FIND $K(G)$

$$K(G) = \ker\left(\mathrm{Res}^G_S : T(G) \to T(S)\right)$$

$\uparrow$ $SESyl_\ell(G)$

$$= \{[M] \in T(G) \mid M\downarrow_S = k \oplus proj\}$$

Conj [CGMN] = Thm 80%

If $\ell \geq 5$, S noncyclic and $(G = Sp_8(2), \ell = 5) \neq (G, \ell)$ then

$$K(G) = X(G)$$

Further, we've list of all cases

$$K(G) \gneq X(G)$$

<u>Thm:</u> Suppose

(1) $C_G(x) = C_G(x)' Q_x$ $x \in S$, $|x| = \ell$

$Q_x \in Syl_\ell(C_G(x))$

(2) $N_G(S) = S\langle y_1, \dots, y_m \rangle$ s.t.

$y_i \in C_G(x_i)$ for some $1 \neq x_i \in S$

Then $K(G) = 0$

$\Rightarrow$ $G = G'S$

$\underline{G}$ connected reductive algebraic gp $/ \overline{\mathbb{F}_q}$

F Steinberg (Frobenius)

$\underline{T}$ max F-stable torus in $\underline{G}$

$W = N_{\underline{G}}(\underline{T}) / \underline{T}$

$G = \underline{G}^F \qquad |\underline{G}| = x^{|\Phi^+|} \prod_{d \geq 1} \Phi_d(x)^{a_d}$

$a_d \geq 0$

Poly order of $\underline{G}$

<u>Def</u>: d-torus = F-stable torus of $\underline{G}$ of poly order $\Phi_d(x)^a$

$a \leq a_d$

max d-torus if $a = a_d$

Thm: (1) $\exists$ max d-tori in $\underline{G}$ and they are all conjugate

(2) Every d-torus is contained in a max d-torus

Eg. $\underline{G} = SL_6$ $\qquad G = SL_6(q)$

$$|G| = q^{15}\,\Phi_1(q)^5\,\Phi_2(q)^3\,\Phi_3(q)^2\,\Phi_4(q)\,\Phi_5(q)\,\Phi_6(q)$$

$d = 2$ $\qquad \underline{T}_2 = GL_1 \times GL_1 \times GL_1$

$$T_2 = \underline{T}_2^F \cong GL_1(q^2)^3$$

$$\begin{pmatrix} \boxed{GL_1} & & \\ & \boxed{\cdot} & \\ & & \boxed{\cdot} \end{pmatrix}$$

$$W_2 = N_{\underline{G}}(\underline{T}_2)/\underline{T}_2 \qquad \text{CRG}$$

$$C_2 \wr \Sigma_3$$

$\underline{G}$ simply conn. connected reductive alg. gp

wma $\underline{G} = \underline{G}'$ semisimple

wma $\underline{G}$ simple

So $G = G'$ except for ...

$\underline{T}$ max torus W as before

d = mult. order of q mod ℓ.

$q^d - 1 = \ell^e e$ $\ell \nmid e$. $S = C_{\ell^a} \wr Q_a$

G has rank $Q_a \in \mathrm{Syl}_\ell(\Sigma_a)$

$n = ad + b$ $0 \le b < d$

Thm: If T_d = max d-torus of G, then $\overline{T_d}$ contains a char. sgp of S el. ab of max rk and each el. ab sgp of S is subconj to it

$N_G(T_d) \geq N_G(S)$ except

$\Rightarrow$ look at the elts of $N_G(S)$

WORK IN PROGRESS

$\rightarrow$ if S has ℓ-rk ≥ 2 then every elt of $N_G(S)$ centralises a non triv. elt of S

and $C_G(x) = C_G(x)' Q_x$
$x \in S$, $|x| = \ell$

$Sp_8(2) = G$ $\quad \ell = 5$ $\quad K(G) = \mathbb{Z}/2$