

On endotrivial modules for
finite reductive groups

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k field of char $l > 0$

G fin gr
 $\text{mod}(kG) \xrightarrow{f, g} kG\text{-mod}$

$\text{mod}(kG)$ = stable cat.

$$\frac{\text{Hom}_{kG}(\Pi, N)}{\text{PHom}_{kG}(\Pi, N)} = \frac{\text{Hom}_{kG}(\Pi, N)}{\text{PHom}_{kG}(\Pi, N)}$$

Def: A f.g. kG -mod Π
is endo-trivial if

$$\text{End}_k \Pi \cong k \text{ in } \underline{\text{mod}}(kG)$$

$$\Leftrightarrow \Pi^* \otimes_k \Pi \cong k \oplus \text{proj in } \text{mod}(kG)$$

$$\Pi^* = \text{Hom}_k(\Pi, k)$$

$T(G)$ = set of stable isoclasses
of e-t kG -mod

$$T(G) = \text{all } g \uparrow$$

$$[M] + [N] = [M \oplus N]$$

$$0 = [R]$$

$$-[M] = [M^*]$$

Thm $T(G)$ is fin. gen

$$T(G) \cong TT(G) \oplus TF(G)$$

finite

$$\mathbb{Z}^{\oplus m}$$

$$\geq X(G) = G / G'S$$

$$S \in \text{Syl}_e(G)$$

$$m = \# \mathbb{Z}_{\geq 2}^e(G)$$

Most often

$$T(G) = X(G) \oplus \mathbb{Z}$$

$$\langle [\Omega(k)] \rangle$$

$$\Omega(k) = \ker(P_k \rightarrow k)$$

"THE" Question:

When is $TT(G) \neq X(G)$

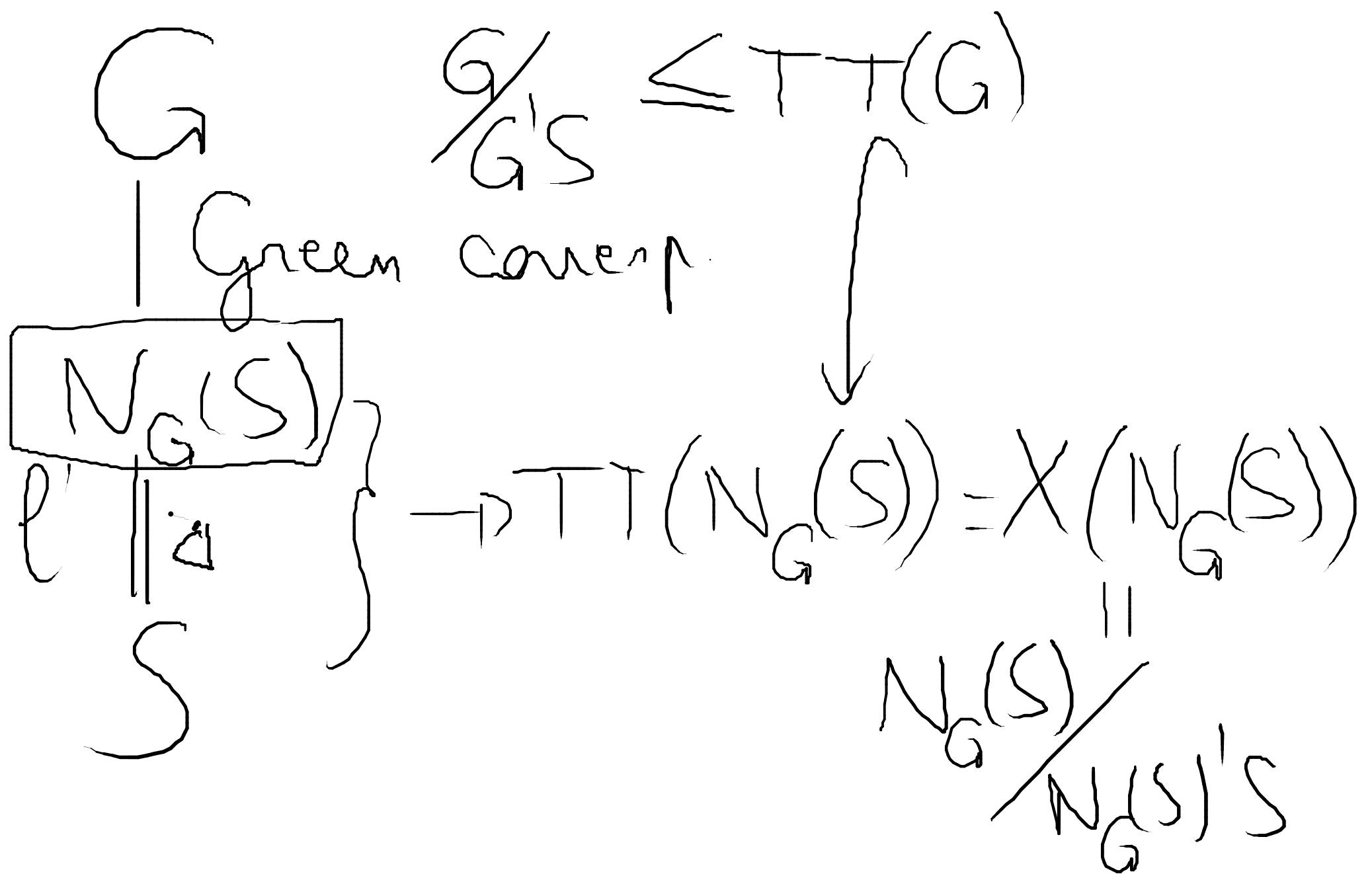
Today:

G is fin. reductive ~~if~~

$\ell + q$

$SL_n(q)$

Strategy (general)



The main problem FIND $K(G)$

$$K(G) = \ker(\text{Res}_S^G: T(G) \rightarrow T(S))$$

$$= \{[M] \in T(G) \mid M|_S = k \oplus mg\}^{\text{SESyl}_\ell(G)}$$

Conj[CGMN] = Thm 80%

If $\ell \geq 5$, S noncyclic and

$(G = \text{Sp}_8(2), \ell=5) \neq (G, \ell)$ then

$$K(G) = X(G)$$

Further, we've list of all cases

$$K(G) \neq X(G)$$

Thm: Suppose

$$(1) C_G(x) = C_G(x)' Q_x \quad x \in S, |x| = l$$

$$Q_x \in \text{Syl}_l(C_G(x))$$

$$(2) N_G(S) = S \langle y_1, \dots, y_m \rangle \text{ s.t.}$$

$$y_i \in C_G(x_i) \text{ for some } 1 \neq x_i \in S$$

$$\text{Then } K(G) = 0$$

$$\Rightarrow G = G'S$$

G connected reductive gp / $\overline{\mathbb{F}}_q$
 $\uparrow$
 algebraic

F Steinberg (Frobenius)

T max F-stable torus in G

$$W = N_G(\underline{T}) / \underline{T}$$

$$G = \underline{G}^F \quad |\underline{G}| = \alpha \prod_{d \geq 1} \phi(d)^{a_d}$$

$a_d \geq 0$

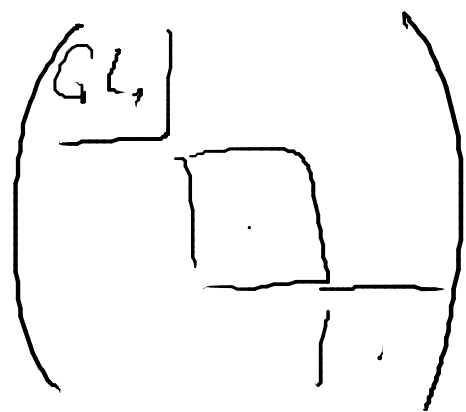
Poly order of G

Def: d-torus = F-stable torus of G
 of poly order $\prod_{d \leq a} \phi(d)^{a_d}$
 max d-torus if $a = a_d$

Thm: (1) $\exists$ max d-tori in G and
 they are all conjugate
 (2) Every d-torus is contained
 in a max d-torus

E.g. $\underline{G} = SL_6$ $G = SL_6(q)$
 $|G| = q^{15} \prod_{i=1}^5 \phi(q^i)$

$d=2$ $I_2 = GL_1 \times GL_1 \times GL_1$
 $T_2 = I_2^F \cong GL_1(q^2)^3$



$W_2 = N_G(I_2) / I_2$
 $C_2 \wr \Sigma_3$

CRC

G simply conn. connected reductive alg. gp

$\swarrow$
wma G = G' semisimple

$\searrow$
wma G simple

So $G = G'$ except for ...

I max torus W as before

$d = \text{mult order of } q \bmod \ell$

$q^d - 1 = \ell^c \ell^e$ lte. $S = C_{\ell^c} \wr Q_a$

G has rank $Q_a \in \text{Syl}_{\ell}(\Sigma_a)$

$n = Qd + b$

$0 \leq b < d$

Thm: If $T_d = \max$ d-torus of G , then
 T_d contains a char. sgp of S
 of max rk and
 each el. ab sgp of S is subconj
 to it

$$N_G(T_d) \geq N_G(S) \quad \text{except}$$

look at the elts of $N_G(S)$

Work in Progress $\Rightarrow$ if S has l-rk ≥ 2 , then
 every elt of $N_G(S)$ centralises
 a non triv. elt of S .

and
$$C_G(x) = C_G(x)' Q_x$$

 $x \in S, |x| = l$

$$Sp_8(2) = G \quad l = 5 \quad K(G) = \mathbb{Z}/2$$

