RATIONALITY OF GROUPS AND INTEGRAL GROUP RINGS

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Notation.

G	*finite* group
$\mathbb{Z}G$	integral group ring of G
$\mathrm{U}(\mathbb{Z}G)$	group of units of $\mathbb{Z}G$

CONTENTS

DEFINITIONS. $x \in G$.

x rational in G $:\Leftrightarrow \quad \forall_{(j,o(x))=1} j \in \mathbb{Z}: \; x^j \sim x$

x semi-rational in G $:\Leftrightarrow \exists\, m \in \mathbb{Z} \; \forall_{(j,o(x))=1} j \in \mathbb{Z}: \; x^j \sim x \;\text{ or }\; x^j \sim x^m$

x inverse semi-rational in G $:\Leftrightarrow \quad \forall_{(j,o(x))=1} j \in \mathbb{Z}: \; x^j \sim x \;\text{ or }\; x^j \sim x^{-1}$

G is called *rational* $:\Leftrightarrow \; \forall\, x \in G$: x is rational in G
etc.

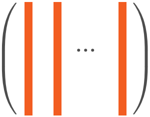
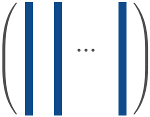
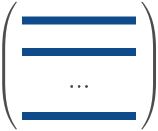
For $\chi \in \mathrm{Irr}(G)$, $x \in G$ set

$$\mathbb{Q}(\chi) := \mathbb{Q}(\{\chi(y) : y \in G\})$$
$$\mathbb{Q}(x) := \mathbb{Q}(\{\psi(x) : \psi \in \mathrm{Irr}(G)\}).$$

G rational $\Leftrightarrow$ $\mathrm{CT}(G) \in \mathbb{Q}^{h \times h}$

G semi-rational $\Leftrightarrow$ $\forall x \in G$: $[\mathbb{Q}(x) : \mathbb{Q}] \leq 2$

G inverse semi-rational $\Leftrightarrow$ $\forall x \in G$: $\mathbb{Q}(x) \subseteq \mathbb{Q}(\sqrt{-d_x})$, $d_x \in \mathbb{Z}_{\geq 0}$
$\Leftrightarrow$ $\forall \chi \in \mathrm{Irr}(G)$: $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\sqrt{-d_\chi})$, $d_\chi \in \mathbb{Z}_{\geq 0}$

$\mathrm{CT}(G) =$  $\mathrm{CT}(G) =$  $\mathrm{CT}(G) =$ 

Examples.

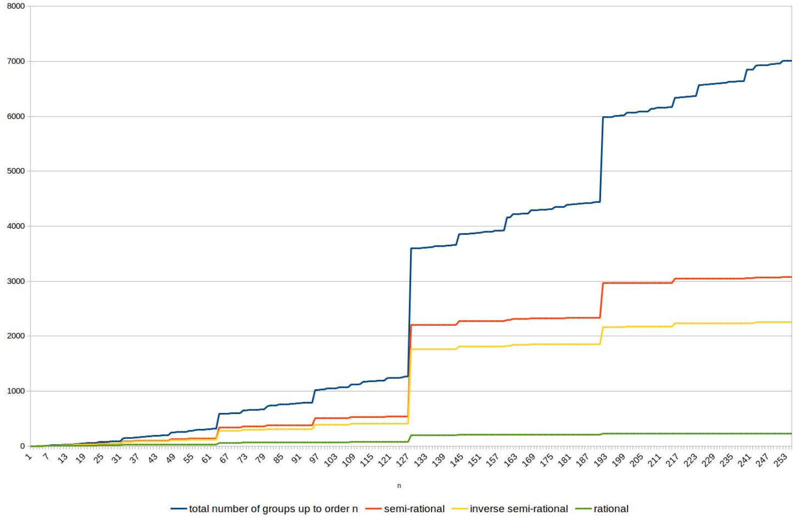
- S_n is rational.
- $P \in \mathrm{Syl}_p(S_n)$.
  P rational $\Leftrightarrow$ $p = 2$.
  P inverse semi-rational $\Leftrightarrow$ $p \in \{2, 3\}$.
- $P \in \mathrm{Syl}_p(\mathrm{GL}(n, p^f))$.
  P rational $\Leftrightarrow$ $p = 2$ and $n \leq 12$.
  P inverse semi-rational
  $\Rightarrow$ $p = 2$ and $n \leq 24$ or $p = 3$ and $n \leq 18$.

Definition.

$\pi(G) = \{p \text{ prime} : p \mid |G|\}$, the *prime spectrum of G*.

Then $|\pi(S_n)| \longrightarrow \infty$ for $n \to \infty$.

	G rational	G inverse semi-rational	G semi-rational
G solvable	$\Longrightarrow$ $\pi(G) \subseteq \{2,3,5\}$ Gow, 1976	$\Longrightarrow$? $\pi(G) \subseteq \{2,3,5,7,13\}$ Chillag-Dolfi, 2010	$\Longrightarrow$? $\pi(G) \subseteq \{2,3,5,7,13,17\}$
$\lvert G\rvert$ odd	$G = 1$	G semi-rational $\Longleftrightarrow$ G inverse semi-rational Classified by Chillag-Dolfi, 2010	
G simple	3 groups Feit-Seitz, 1989	25 groups	all A_n + 41 groups Alavi-Daneshkhah, 2016
$\lvert G\rvert \leq 511$	$\approx 1\%$	$\approx 46\%$	$\approx 61\%$



8000
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6000
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145
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169
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181
187
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205
211
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223
229
235
241
247
253
n
total number of groups up to order n
semi-rational
inverse semi-rational
rational

$\pm G \subseteq \mathrm{U}(\mathbb{Z}G)$ – "trivial units"

$\pm G = \mathrm{U}(\mathbb{Z}G)$ $\Leftrightarrow$ G abelian with $\exp G \mid 4$ or $\exp G \mid 6$ or
G Hamiltonian 2-group
(Higman, 1940)

$\pm \mathrm{Z}(G) \subseteq \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ – "trivial central units"

$\pm \mathrm{Z}(G) = \mathrm{Z}(\mathrm{U}(\mathbb{Z}G))$ $\Leftrightarrow$: G *cut group*
(all central units trivial)

$$\left[\mathrm{U}(\mathbb{Z}G) : \left\langle (\mathbb{Z}G)^1, \mathrm{Z}(\mathrm{U}(\mathbb{Z}G)) \right\rangle\right] < \infty$$

↗ often up to f.i. by "bicyclic units"

↖ covered by "bicyclic units" & "Bass units"

Theorem (Ritter-Sehgal, et.al.) For a finite group G TFAE

(1) G is cut.

(2) $\forall \chi \in \mathrm{Irr}(G)$: $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\sqrt{-d_\chi})$, $d_\chi \in \mathbb{Z}_{\geq 0}$.

(3) G is inverse semi-rational.

(4) $K_1(\mathbb{Z}G)$ is finite.

In particular: G cut $\Rightarrow$ G/N cut for all $N \trianglelefteq G$.

Theorem (Bakshi-Maheshwary-Passi, 2016) $G \neq 1$ cut-group

(1) $2 \in \pi(G)$ or $3 \in \pi(G)$.

(2) If G is nilpotent, then G is a $\{2,3\}$-group.

(3) If G is metacyclic, then G is in a list of 52 groups.

Theorem (Maheshwary, 2016) Let G be a solvable cut group.

(1) If $|G|$ is odd $\Longrightarrow \pi(G) \subseteq \{3,7\}$ and all elements of G are of prime power order.

(2) If $|G|$ is even and all elements of G are of prime power order $\Longrightarrow \pi(G) \subseteq \{2,3,5,7\}$.

Theorem (B., 2017) Let G be a solvable cut group.
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Strategy of proof.

- $\pi(G) \subseteq \{2, 3, 5, 7, 13\}$ (Chillag-Dolfi).
- Let G be a minimal counterexample, $V \trianglelefteq G$ minimal.
- Then $G \simeq V \rtimes G/V$, G/V is again cut.
- The $\mathbb{F}_{13}[G/V]$-module V has the "12-eigenvalue property".
- Derive restrictions on field of character values of V.
- By a result of Farias e Soares such a module cannot exist for a solvable group G/V. □

Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even ...

(2) If $|K|$ is odd ...

Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even and the compelement of a cut Frobenius group G, then G is isomorphic to a group in the series on the left ($b, c, d \in \mathbb{Z}_{\geq 1}$) or one of the groups on the right.

(a) $C_3^b \rtimes C_2$
(b) $C_3^{2b} \rtimes C_4$
(c) $C_3^{2b} \rtimes Q_8$
(d) $C_5^c \rtimes C_4$
(e) $C_7^d \rtimes C_6$
(f) $C_7^{2d} \rtimes (Q_8 \times C_3)$

(α) $C_5^2 \rtimes Q_8$
(β) $C_5^2 \rtimes (C_3 \rtimes C_4)$
(γ) $C_5^2 \rtimes \mathrm{SL}(2,3)$
(δ) $C_7^2 \rtimes \mathrm{SL}(2,3)$

Conversely, for each of the above structure descriptions, there is a unique cut Frobenius group.

(2) If $|K|$ is odd ...

Theorem (B., 2017). Let K be a Frobenius complement.

(1) If $|K|$ is even ...

(2) If $|K|$ is odd, then there is a cut Frobenius group G if and only if $K \simeq C_3$ and the kernel F is a group admitting a fixed-point free automorphism σ of order 3 such that

  (a) F is a cut 2-group.
  In particular, $|F| = 2^{2a}$, $a \in \mathbb{Z}_{\geq 1}$ and F is an extension of an abelian group of exponent a divisor of 4 by an an abelian group of exponent a divisor of 4.

  (b) F is an extension of an elementary abelian 7-group by an elementary abelian 7-group, $\exp F = 7$ and σ fixes each cyclic subgroup of F.

Strategy of proof. G cut Frobenius group with complement K.

- K is also cut.
- Show that K is solvable, so $\pi(G) \subseteq \{2, 3, 5, 7\}$.
- Determine possible structures of $P \in \mathrm{Syl}_p(K)$.
- Determine possible structures of K.
- Use irreducible representations of these complements to describe structure of some G.
- Decide which subdirect products of the groups above are cut Frobenius groups. □

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