Unipotent automorphisms of solvable groups

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1. Introduction.
2. Groups acting n-unipotently on solvable groups.
3. Examples and proofs.

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(2) G is an Engel (n-Engel) group if and only if all the elements in $\mathrm{Inn}\,(G)$ are unipotent (n-unipotent).

Definition. Let G be group with a finite series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = \{1\}.$$

The stability group of the series is the subgroup S of Aut (G) consisting of the automorphisms a where for each $i = 1, \ldots, m$ and $g \in G_{i-1}$ we have $[g, a] \in G_i$.

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Theorem (W. Burnside) Let V be a a finite dimensional vector space and $H \leq \mathrm{GL}(V)$ where H consists of unipotent automorphisms. Then H stabilises a finite series of subspaces
$V = V_0 \geq V_1 \geq \cdots \geq V_m = \{0\}$.

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Theorem(Frati 2014). Let G be a solvable group and H a finitely generated nilpotent subgroup of Aut (G) consisting of n-unipotent automorphisms. Then H stabilizes a finite series in G. Moreover, the nilpotency class of H is (n, r)-bounded if H is generated by r elements.

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Remark. In fact one can deduce the following stronger variant of the one given in last remark. If $\mathcal{P}_n$ is the set of prime divisors of $e(n)$ in the following result then it suffices that G is $\mathcal{P}_n$-torsion free.

Theorem 2(Puglisi,& T). Let $\mathcal{C}$ be a solvable variety of groups in which every finitely group is residually finite. Let G be any finitely generated group in $\mathcal{C}$ and H be a finitely generated subgroup of $\mathrm{Aut}(G)$ that acts n-unipotently on G. Then H stabilizes a finite series for G. In particular H is nilpotent.

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Proposition(Puglisi,& T). Let G be a group with a characteristic nilpotent supgroup N and suppose that $H \leq$ Aut(G) acts unipotently on G. If H stabilizes a finite series of $G/[N,N]$, then H stabilizes a finite series for G. In particular H is nilpotent.

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Theorem 3(Puglisi,& T). Let G be a finitely generated metanilpotent group and let $H \leq$ Aut(G) be finitely generated acting n-unipotently on G. Then H stabilizes a finite series for G. In particular H is nilpotent.

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Theorem. (Crosby & T 2010). Let H be a normal right n-Engel subgroup of a group G that belongs to some term of the upper central series. Then there exist positive integers $c(n), e(n)$ such that

$$[H^{e(n)},_{c(n)} G] = [H,_{c(n)} G]^{e(n)}.$$

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Sketchproof of Theorem 1(1). Let G be any solvable group and H be a finitely generated subgroup of $\mathrm{Aut}(G)$ acting n-unipotently on G. We want to show that H stabilizes a finite series in G.

Step 1. Using induction on the derived length of G, we can assume that $G = A$ is abelian.

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$$A^{e(n)} \geq [A^{e(n)}, H'] \geq \cdots \geq [A^{e(n)},_{c(n)} H'] = 1.$$

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Step 4. Now H/H' acts naturally on each factor in the series above and by Frati H/H' (and thus H) stabilizes a finite series in each factor. Adding up we see that H stabilizes a finite series in A

Example 1. Let $G = A \operatorname{wr} H$ where A is a cyclic group of order 2 and H is a countable elementary abelian 2-group. The group $\operatorname{Inn}(G)$ then acts on G as a group of 3-unipotent automorphisms but it still doesn't stabilize any finite series.

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Example 2. Let m be the smallest positive integer such that the Burnside variety $\mathcal{B}(2^m)$ is not locally finite. Choose any finitely generated infinite group G in $\mathcal{B}(2^m)$. It is well known that every involution $a \in G$ induces an $(m+1)$-unipotent automorphism in $\mathrm{Inn}(G)$.

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For any involution a that stabilizes a finite series in G we have that $\langle a \rangle$ is in the locally finite radical of G. Thus if all involutions stabilize a finite series it would follow that $G^{2^{m-1}}$ is locally finite. But as $G/G^{2^{m-1}}$ is finite, it then follows that G is locally finite (and thus finite). A contradiction.