Bounded Engel elements in groups satisfying an identity

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From the positive answer of the RBP

A residually finite group of finite exponent is locally finite.

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Golod, Gupta-Sidki, Grigorchuck

A residually finite *periodic* group is NOT locally finite.

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A group G is said to satisfy the identity $w \equiv 1$ if $w(g_1, \ldots, g_m) = 1$ for all $g_1, \ldots, g_m \in G$.

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If $w = x^n$ then G satisfies the identity $w \equiv 1$ iff the exponent of G divides n.

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Let G be a residually finite periodic group satisfying an identity.
Is G locally finite?

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Zelmanov, 2016

A residually finite p-group, for p a prime, which satisfies an identity is locally finite.

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Engel version of the question

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Engel version of the question

Let G be a residually finite Engel group satisfying an identity.
Is G locally nilpotent?

A group G is an Engel group (resp. an n-Engel group) if all its elements are Engel (resp. n-Engel).

An element $g \in G$ is called a (left) Engel element if for any $x \in G$ there exists a positive integer $n = n(x, g)$ such that $[x,_n g] = 1$.

If n can be chosen independently of x, then g is called a (left) n-Engel element, or a bounded (left) Engel element.

Engel version of the question

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Periodic case

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Positive answer in another special case (Wilson)

A residually finite n-Engel group is locally nilpotent.

Positive answer in a more general case (Bastos, Mansuroglu, Tortora, T.)

Let G be a residually finite group satisfying an identity.
Suppose that G is generated by a commutator closed set X of bounded Engel elements.
Then G is locally nilpotent.

A subset X of a group is commutator closed if $[x, y] \in X$ for any $x, y \in X$.

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Theorem (Bastos, Mansuroglu, Tortora, T.)

Let G be a locally graded group satisfying an identity.
Suppose that G is generated by a *normal* commutator closed set X of bounded Engel elements.
Then G is locally nilpotent.

A group is locally graded if every nontrivial finitely generated subgroup has a proper subgroup of finite index.

A group is a *nil group* if all its elements are bounded Engel.

Corollary

Let G be a locally graded nil group satisfying an identity.
Then G is locally nilpotent.

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Let G be a locally graded *periodic* Engel group satisfying an identity.
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A nil group satisfying an identity might not be locally nilpotent.

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Open Question

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Let $E(G)$ be the set of all bounded Engel elements of G.

Is $E(G)$ a subgroup?

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Let G be a locally graded group satisfying an identity.

Let $HP(G)$ the Hirsch-Plotkin radical.

$$E(G)\ subgroup \ \Rightarrow \ E(G) \subseteq HP(G)$$

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Problem

Find a locally graded group G satisfying an identity such that $E(G) \neq \{1\}$ and $HP(G) = \{1\}$.

THANK YOU!

R. Bastos, N. Mansuroglu, A. Tortora, and M. Tota,
Bounded Engel elements in groups satisfying an identity, Arch. Math. (2017), accepted.