Minimal varieties of graded PI algebras

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Polynomial identities

- $X := \{x_1, x_2, \ldots\}$ is a countable set and F is a field
- $F\langle X\rangle$ is the free associative algebra over F generated by X

Definition

An element $f(x_1, \ldots, x_s) \in F\langle X\rangle$ is a *polynomial identity* for an F-algebra A if $f(a_1, \ldots, a_s) = 0_A$ for all $a_i \in A$.
If an F-algebra A satisfies a non-trivial polynomial identity, then we say that A is a *PI algebra*.

Examples

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- A *finite dimensional* algebra of dimension n is PI since it satisfies

$$\mathrm{St}_{n+1}(x_1, x_2, \ldots, x_{n+1}) := \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n+1)}$$

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- The *Grassmann algebra* on a countable dimension F-vector space (char $F \neq 2$) is PI

A first line of research

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Theorem [Isaacs-Passmann, 1973]

Let FG be the group algebra of a group G over a field F of characteristic p. Then FG is PI if, and only if, G has a p-abelian subgroup of finite index.

Focus on Id(A)

Describe

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The algebraic structure of $\mathrm{Id}(A)$

$\mathrm{Id}(A)$ is a *T-ideal* of $F\langle X\rangle$, namely a two-sided ideal closed under endomorphisms of $F\langle X\rangle$.

The characteristic zero case and Specht's Problem

Theorem [Kemer, 1991]

Let F be a field of characteristic zero and A a PI algebra. Then

$$\mathrm{Id}(A) = \langle f_i \mid f_i \in F\langle X\rangle \quad f_i \text{ multilinear}\rangle_T.$$

Furthermore $\mathrm{Id}(A)$ is finitely generated (*Specht's Problem*).

- So it is enough to consider

$$\oplus_{n\in\mathbb{N}}(P_n \cap \mathrm{Id}(A)),$$

where $P_n := \mathrm{span}_F\{x_{\tau(1)} \cdots x_{\tau(n)} \mid \tau \in \mathcal{S}_n\}$.

Polynomial identities for matrices

$\mathrm{Id}(M_n(F))$

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- $\mathrm{Id}(M_n(F)) = \langle\, ?\, \rangle_T$ if $n \geq 3$

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Definition

Given a T-ideal (set) S of $F\langle X\rangle$, the class of all algebras A such that $S \subseteq \mathrm{Id}(A)$ for all $f \in S$ is called *the variety $\mathcal{V} = \mathcal{V}(S)$ determined by S*. Let us write $\mathrm{Id}(\mathcal{V}) = S$.
An algebra A generates $\mathcal{V}$ if $\mathrm{Id}(A) = \mathrm{Id}(\mathcal{V})$ (write $\mathcal{V} = \mathrm{var}(A)$).

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Main Goal

To classify varieties

Codimension Sequence

Definition [Regev, 1972]

Let A be an F-algebra. The non-negative integer

$$c_n(A) := \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}$$

is said to be the *n-th codimension* of the algebra A.

- The sequence $(c_n(A))_{n\in\mathbb{N}}$ depends on $\mathrm{Id}(A)$ rather than A , thus it is constant on PI-equivalence classes and can therefore be used as an invariant.

The exponent of a PI algebra

Theorem [Regev, 1972]

If the algebra A satisfies an identity of degree $d \geq 1$, then

$$c_n(A) \leq (d-1)^{2n} \qquad n \geq 1.$$

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$$\exp(A) := \lim_{m \to +\infty} \sqrt[m]{c_m(A)}.$$

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Conjecture [Amitsur, '80]

For any PI algebra A, $\exp(A)$ exists and is an integer.

Existence of the exponent

Theorem [Giambruno-Zaicev, 1998-1999]

Let A be a PI algebra. Then the exponent of A exists and is an integer.

- They provide a method to compute it.

Importance of the exponent in classifying varieties

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- Now we start by distributing all varieties into layers according to their exponent.
- Let $S \subseteq F\langle X \rangle$. It could be that if we consider a subset $S \subset T \subseteq F\langle X \rangle$ we get a strictly smaller variety with a strictly smaller exponent.
- If this always happens, $\mathcal{V}(S)$ is called a *minimal variety*.

Minimal varieties

Definition

A variety $\mathcal{V}$ is *minimal* of exponent $d \geq 2$ if $\exp(\mathcal{V}) = d$ and $\exp(\mathcal{U}) < d$ for any proper subvariety $\mathcal{U} \subset \mathcal{V}$.

Theorem [Giambruno-Zaicev, 2003]

Let $\mathcal{V}$ be a variety of algebras of exponent $d \geq 2$. The following statements are equivalent:

(i) $\mathcal{V}$ is minimal;

(ii) $\mathrm{Id}(\mathcal{V})$ is the product of verbally prime T-ideals;

(iii) $\mathcal{V} = \mathrm{var}(Gr(A))$, where A is a suitable *minimal superalgebra* and $Gr(A)$ is its Grassmann envelope.

Graded algebras

Definition

Let G be a group and F be a field. An (associative) F-algebra A is called *G-graded* if

$$A = \oplus_{g \in G} A^{(g)},$$

where $A^{(g)}$ is an F-supaspace of A and $A^{(g)}A^{(h)} \subseteq A^{(gh)}$ for every $g, h \in G$.
When $G = \mathbb{Z}_2$, A is said to be a *superalgebra*.

Examples

- The Grassmann algebra

$$Gr = \langle 1, e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \text{ for all } i, j \geq 1 \rangle$$

so a basis is given by

$$B := \{1, e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < i_2 < \ldots < i_k\}.$$

Set

$$Gr^{(0)} := \mathrm{span}\{e_{i_1} \cdots e_{i_{2k}} \mid 1 \leq i_1 < \ldots < i_{2k},\ k \geq 0\},$$

$$Gr^{(1)} := \mathrm{span}\{e_{i_1} \cdots e_{i_{2k+1}} \mid 1 \leq i_1 < \ldots < i_{2k+1},\ k \geq 0\}.$$

Then $Gr = Gr^{(0)} \oplus Gr^{(1)}$. In particular, $Gr^{(0)} = Z(Gr)$.

Elementary grading on $M_n(F)$

- Let G be a group and $(g_1, \ldots, g_n) \in G^n$. Consider the algebra $M_n(F)$ graded by

$$\begin{pmatrix} g_1^{-1}g_1 = 1_G & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 = 1_G & g_2^{-1}g_3 & \cdots & g_2^{-1}g_n \\ \vdots & \vdots & & & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & \cdots & \cdots & g_n^{-1}g_n = 1_G \end{pmatrix}.$$

This grading on $M_n(F)$ is called *elementary*.

Graded polynomial identities

- G is a group
- $F\langle X\rangle^G$ is the free associative G-graded algebra of countable rank over F (here, if $G = \{g_1, g_2, \ldots\}$, the set X decomposes as $X = \cup_{i=1}^{s} X^{(g_i)}$, where $X^{(g_i)} = \{x_1^{(g_i)}, x_2^{(g_i)}, \ldots\}$)

Definition

An element $f(x_1^{(g_1)}, \ldots, x_{t_1}^{(g_1)}, \ldots, x_1^{(g_s)}, \ldots, x_{t_s}^{(g_s)}) \in F\langle X\rangle^{gr}$ is a *graded polynomial identity* for the G-graded algebra A if $f(a_1^{(g_1)}, \ldots, a_{t_1}^{(g_1)}, \ldots, a_1^{(g_s)}, \ldots, a_{t_s}^{(g_s)}) = 0_A$ for all $a_i \in A$.

The graded version of Specht's Problem

Let us consider the T_G-ideal of graded polynomial identities of a G-graded algebra A

$$\mathrm{Id}_G(A) := \{f |\ f \in F\langle X\rangle^G \quad f \equiv 0 \text{ on } A\}.$$

Theorem [Aljadeff-Kanel Belov, 2010]

Let F be a field of characteristic zero and A be a PI algebra graded by a finite group G. Then

$$\mathrm{Id}_G(A) = \langle f_i \mid f_i \in F\langle X\rangle^{gr} \ \ f_i \text{ multilinear}\rangle_{T_G}.$$

Furthermore $\mathrm{Id}_G(A)$ is finitely generated (*Graded Specht's Problem*).

Graded codimensions

Assume that char $F = 0$. As in the ungraded case, it is enough to consider the spaces of multilinear G-graded polynomials in the variables $x_1^{(g_{i_1})}, \dots, x_n^{(g_{i_n})}$

$$P_n^G := \mathrm{Span}\{x_{\sigma(1)}^{(g_{i_1})} \cdots x_{\sigma(n)}^{(g_{i_n})} \mid \sigma \in S_n \quad g_{i_1}, \dots, g_{i_n} \in G\}.$$

Definition

The non-negative integer

$$c_n^G(A) := \dim_F \frac{P_n^G}{P_n^G \cap \mathrm{Id}_G(A)}$$

is said to be the *n-th G-graded codimension* of the algebra A.

Polynomial identities versus graded polynomial identities

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Theorem [Giambruno-Regev, 1985]

If A is a PI-algebra graded by a finite group G, then

$$c_n(A) \leq c_n^G(A) \leq |G|^n c_n(A) \qquad n \geq 1.$$

The graded exponent

Definition

Let A be a PI algebra graded by a finite group G. Then *the graded exponent* of A is (if there exists)

$$\exp^G(A) := \lim_{m \to +\infty} \sqrt[m]{c_m^G(A)}.$$

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- Benanti-Giambruno-Pipitone [J. Algebra **269** (2003), 422–438]: $G = \mathbb{Z}_2$ and A is finitely generated
- Aljadeff-Giambruno-La Mattina [J. Reine Angew. Math. **650** (2011), 83–100]: G is finite abelian and A is finite-dimensional
- Giambruno-La Mattina [Adv. Math. **225** (2010), 859–881]: G is finite abelian and A is PI

Theorem [Aljadeff-Giambruno, 2013]

Let A be a PI algebra graded by a finite group G. Then the graded exponent of A exists and is an integer.

The graded problem

Classify the minimal varietes of PI algebras graded by a finite group G of fixed graded exponent

Definition

A variety $\mathcal{V}^G$ of G-graded PI algebras is said to *minimal* of graded exponent d if $\exp^G(\mathcal{V}^G) = d$ and $\exp^G(\mathcal{U}^G) < d$ for every proper subvariety $\mathcal{U}^G \subset \mathcal{V}^G$.

Motivations

- To construct a theory which generalizes that of ordinary polynomial identities.

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- To construct a theory which generalizes that of ordinary polynomial identities.
- The additional graded structure and realted objects may provide significant information on quite general objects. For instance, Kemer in his fundamental work [Transl. Math. Monograph, vol. 87, Amer. Math. Soc. , Providence, RI, 1991] proved that for any non-trivial variety there exists a finite-dimensional superalgebras A (on an extension of F) such that the T-ideal of the variety coincides with the T-ideal of polynomial identities of $Gr(A)$.

Generators of minimal supervarieties

Theorem [Di Vincenzo-S., 2012]

Let $\mathcal{V}^{\mathbb{Z}_2}$ be a variety PI superalgebras of finite basic rank. If $\mathcal{V}^{\mathbb{Z}_2}$ is minimal of $\mathbb{Z}_2$-graded exponent $d \geq 2$, then $\mathcal{V}^{\mathbb{Z}_2} = \mathrm{var}^{\mathbb{Z}_2}(B)$, where B is a suitable minimal superalgebra.

Minimal superalgebras

Definition [Giambruno-Zaicev, 2003]

Let F be an algebraically closed field. An F-superalgebra A is called *minimal* if it is finite dimensional and $A = A_{ss} + J$ where

(1) $A_{ss} = A_1 \oplus \cdots \oplus A_n$ with $A_1, \ldots, A_n$ simple superalgebras;

(2) there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J^{(0)} \cup J^{(1)}$ and minimal graded idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1}, \qquad i = 1, \ldots, n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0;$$

(3) $w_{12}, \ldots, w_{n-1,n}$ generate J as two-sided ideal of A.

Simple Superalgebras

- $M_{k,l} := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $k \geq l \geq 0$, $k \neq 0$, $A \in M_k$, $D \in M_l$, $B \in M_{k\times l}$ and $C \in M_{l\times k}$, endowed with the grading $M_{k,l}^{(0)} := \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $M_{k,l}^{(1)} := \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$;
- $M_m(F \oplus tF)$, where $t^2 = 1$ with grading (M_m, tM_m),

where, for any pair of positive integers m and s, the symbol $M_{m\times s}$ means the F-vector space of all rectangular matrices with m rows and s columns, and $M_m := M_{m\times m}$.

Simple graded simple components

Proposition

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$, where $A_j = M_{k_j,l_j}$ for all j, then A is isomorphic (as superalgebra) to $UT(k_1 + l_1, \ldots, k_n + l_n)$ equipped with a suitable elementary grading.

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Open question

Is it true that the supervariety generated by a minimal superalgebra is minimal with respect to its graded exponent?

Strategy

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- Let $\mathcal{U}^{\mathbb{Z}_2} \subseteq \mathcal{V}^{\mathbb{Z}_2}$ such that $\exp^{\mathbb{Z}_2}(\mathcal{V}^{\mathbb{Z}_2}) = \exp^{\mathbb{Z}_2}(\mathcal{U}^{\mathbb{Z}_2})$.

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- We aim to show that $\mathcal{U}^{\mathbb{Z}_2} = \mathcal{V}^{\mathbb{Z}_2}$.
- Now $\mathcal{U}^{\mathbb{Z}_2}$ has finite basic rank. Hence, by Kemer's result, $\mathcal{U}^{\mathbb{Z}_2}$ is generated by a finite dimensional superalgebra B'.

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- Consequently, $\mathrm{Id}_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}_{\mathbb{Z}_2}(B)$ and $\exp^{\mathbb{Z}_2}(A) = \exp^{\mathbb{Z}_2}(B)$.

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- This implies that $A_{ss} = B_{ss}$.
- We have to prove that $\mathrm{Id}_{\mathbb{Z}_2}(A) = \mathrm{Id}_{\mathbb{Z}_2}(B)$.

Proposition [Di Vincenzo-S., 2012]

Let $A := (UT(\alpha_1, \dots, \alpha_n), |\ |_A)$ and $B := (UT(\alpha_1, \dots, \alpha_n), |\ |_B)$. If $\mathrm{Id}_{\mathbb{Z}_2}(A) \subseteq \mathrm{Id}_{\mathbb{Z}_2}(B)$, then $A \cong B$.

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- This result has been generalized in Di Vincenzo-Spinelli [J. Algebra **415** (2014), 50–64] for gradings on upper block triangular matrix algebras induced by a finite abelian group under suitable restrictions
- Aljadeff-Haile [Trans. Amer. Math. Soc. **366** (2014), 1749–1771]: simple G-graded algebras
- David [J. Algebra **367** (2012), 120–141]: semisimple G-graded algebras

A first positive result

Theorem [Di Vincenzo-S., 2012]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$, where $A_j = M_{k_j,l_j}$ for all j, then $\mathrm{var}^{\mathbb{Z}_2}(A)$ is minimal of graded exponent $\dim_F A_{ss}$.

Minimal varieties vs minimal supervarieties

- Giambruno-Zaicev [Adv. Math. **174** (2003), 310–323] proved that a variety of finite basic rank is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(d_1, \ldots, d_n)$ and

$$\mathrm{Id}(UT(d_1, \ldots, d_n)) = \mathrm{Id}(M_{d_1}) \cdots \mathrm{Id}(M_{d_n})$$

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- Giambruno-Zaicev [Adv. Math. **174** (2003), 310–323] proved that a variety of finite basic rank is minimal if, and only if, it is generated by an upper block triangular matrix algebra $UT(d_1, \dots, d_n)$ and

$$\mathrm{Id}(UT(d_1,\dots,d_n)) = \mathrm{Id}(M_{d_1}) \cdots \mathrm{Id}(M_{d_n})$$

- Minimal superalgebras in which all the summands of the semisimple part are simple graded simple generate minimal supervarieties, but in general they do not generate the same supervariety not even if they have the same graded components $A_1, \dots, A_n$. Hence we cannot hope that the T_2-ideal of superidentities of these minimal superalgebras is factorable.

The case with two graded simple summands

Theorem [Di Vincenzo-S., 2012]

Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$. Then $\mathrm{Id}_{\mathbb{Z}_2}(A) = \mathrm{Id}_{\mathbb{Z}_2}(A_1) \cdot \mathrm{Id}_{\mathbb{Z}_2}(A_2)$ if one of the following conditions is satisfied:

- at least one between A_1 and A_2 is non-simple as algebra;
- A_1 and A_2 are both simple $\mathbb{Z}_2$-simple and there exists $1 \leq i \leq 2$ such that $A_i = M_{k_i,k_i}$.

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Theorem [Di Vincenzo-S., 2012]

Let $A = A_{ss} + J$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2$. Then the supervariety generated by A is minimal of graded exponent $\dim_F(A_1 \oplus A_2)$.

On the structure of minimal superalgebras

- It has been shown that a minimal superalgebra $A = (A_1 \oplus \ldots \oplus A_n) + J$ has the following vector space decomposition:
$$A = \bigoplus_{1 \leq i \leq j \leq n} A_{ij},$$
where $A_{11} := A_1, \ldots, A_{nn} := A_n$ and, for all $i < j$,
$$A_{ij} := A_i w_{i,i+1} A_{i+1} \cdots A_{j-1} w_{j-1,j} A_j.$$
Moreover $J = \oplus_{i<j} A_{ij}$ and $A_{ij} A_{kl} = \delta_{jk} A_{il}$, where δ_{jk} is the Kronecker delta.
- For every $1 \leq k < l \leq n$ set
$$A^{(k,l)} := \bigoplus_{k \leq i \leq j \leq l} A_{ij},$$
which is a minimal superalgebra as well.

A factorization property

Theorem [Di Vincenzo-da Silva-S., 2016]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \ldots, A_h$ are non-simple graded simple and $A_{h+1}, \ldots, A_n$ are simple graded simple algebras, then

$$\mathrm{Id}_{\mathbb{Z}_2}(A) = \mathrm{Id}_{\mathbb{Z}_2}(A_1) \cdots \mathrm{Id}_{\mathbb{Z}_2}(A_h) \cdot \mathrm{Id}_{\mathbb{Z}_2}(A^{(h+1,n)}).$$

On the other hand, if $h < n$ and $A_1, \ldots, A_h$ are simple graded simple and $A_{h+1}, \ldots, A_n$ are non-simple graded simple algebras, then

$$\mathrm{Id}_{\mathbb{Z}_2}(A) = \mathrm{Id}_{\mathbb{Z}_2}(A^{(1,h)}) \cdot \mathrm{Id}_{\mathbb{Z}_2}(A_{h+1}) \cdots \mathrm{Id}_{\mathbb{Z}_2}(A_n).$$

Another positive result

Theorem [Di Vincenzo-da Silva-S., 2016]

Let $A = A_{ss} + J$ be a minimal superalgebra. If $A_{ss} = A_1 \oplus \cdots \oplus A_n$ and there exists $1 \leq h \leq n$ such that $A_1, \ldots, A_h$ are non-simple graded simple and $A_{h+1}, \ldots, A_n$ are simple graded simple algebras (or vice versa), then the supervariety generated by A is minimal of graded exponent $\dim_F(A_{ss})$.

A crucial example

Consider $R := UT_6$ endowed with the $\mathbb{Z}_2$-grading induced by the automorphism ϕ (of order 2) defined on E_{ij} by

$$\phi(E_{ij}) := E_{\rho(i),\rho(j)}, \qquad \rho := (12)(34)(56).$$

Take the subalgebra A of R having as a linear basis the set

$$\mathcal{B}_A := \{E_{11}+E_{22}, E_{33}, E_{44}, E_{55}+E_{66}, E_{13}, E_{24}, E_{35}, E_{46}, E_{15}, E_{26}\},$$

which is homogeneous. Its Wedderburn-Malcev decomposition is $A_{ss} = A_1 \oplus A_2 \oplus A_3$, where

- $A_1 = \langle E_{11} + E_{22}\rangle \cong F$ ($e_1 := E_{11} + E_{22}$);
- $A_2 = \langle E_{33}, E_{44}\rangle \cong F \oplus tF$, where $t^2 = 1$, with grading (F, tF) ($e_2 := E_{33} + E_{44}$);
- $A_3 = \langle E_{55} + E_{66}\rangle \cong F$ ($e_3 := E_{55} + E_{66}$)

and Jacobson radical $J(A) = \langle E_{13}, E_{24}, E_{35}, E_{46}, E_{15}, E_{26}\rangle$ is generated as an ideal by the homogeneous elements

$$w_{12} := E_{13} + E_{24}, \qquad w_{23} := E_{35} + E_{46}.$$

Consider $S := UT_4$ endowed with the $\mathbb{Z}_2$-grading induced by the automorphism ψ defined by

$$\psi(E_{ij}) := E_{\sigma(i),\sigma(j)}, \qquad \sigma := (23).$$

Take the subalgebra B of S having as a linear basis the set

$$\mathcal{B}_B := \{E_{11}, E_{22}, E_{33}, E_{44}, E_{12}, , E_{13}, E_{24}, E_{34}, E_{14}\}.$$

B is a minimal superalgebra with graded simple summands of B_{ss}, where

- $B_1 = \langle E_{11}\rangle \cong F$ $(e_1 := E_{11})$;
- $B_2 = \langle E_{22}, E_{33}\rangle \cong F \oplus tF$, where $t^2 = 1$, with grading (F, tF) $(e_2 := E_{22} + E_{33})$;
- $B_3 = \langle E_{44}\rangle \cong F$ $(e_3 := E_{44})$

and Jacobson radical $J(B) = \langle E_{12}, E_{13}, E_{24}, E_{34}, E_{14}\rangle$ generated as an ideal by the homogeneous elements

$$w_{12} := E_{12} + E_{13}, \qquad w_{23} := E_{24} + E_{34}.$$

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- The graded polynomial $g := [y_1, y_2]z_3[y_4, y_5]$ lies in $\mathrm{Id}_{\mathbb{Z}_2}(B)$ but $[e_1, w_{12}](E_{33} - E_{44})[w_{23}, e_3] \neq 0_A$ Hence

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$$\mathrm{Id}_{\mathbb{Z}_2}(A) \neq \mathrm{Id}_{\mathbb{Z}_2}(B)$$

- $\mathrm{var}^{\mathbb{Z}_2}(A)$ is not minimal.

The case with three graded simple addends

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- Di Vincenzo-da Silva-Spinelli [Math. Z., in press] have completely described the isomorphism types of minimal superalgebras whose maximal semisimple homogeneous subalgebra is the sum of three graded simple algebras.
- A minimal superalgebra A belonging to this class generates a minimal supervariety of fixed graded exponent except for one case

Theorem [Di Vincenzo-da Silva-S., 2017]

Let $A = A_{ss} + J(A)$ be a minimal superalgebra such that $A_{ss} = A_1 \oplus A_2 \oplus A_3$ with

$$A_1 = M_{k,l}, \qquad A_2 = M_m(F \oplus tF) \quad \text{and} \quad A_3 = M_{r,s}.$$

(a) If A_{13} is irreducible as an (A_1, A_3)-bimodule, then A generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$;

(b) if A_{13} is not irreducible as an (A_1, A_3)-bimodule, then A generates a minimal supervariety of superexponent $\dim_F(A_1 \oplus A_2 \oplus A_3)$ if, and only if, either $k = l$ or $r = s$.

Minimal G-graded algebras

Definition

Let F be an algebraically closed field. A G-graded algebra A is called *minimal* if it is finite-dimensional and $A = A_{ss} + J(A)$ where

(*i*) $A_{ss} = A_1 \oplus \cdots \oplus A_n$, with $A_1, \ldots, A_n$ G-simple algebras;

(*ii*) there exist homogeneous elements $w_{12}, \ldots, w_{n-1,n} \in J(A)$ and minimal homogeneous idempotents $e_1 \in A_1, \ldots, e_n \in A_n$ such that

$$e_i w_{i,i+1} = w_{i,i+1} e_{i+1} = w_{i,i+1} \qquad 1 \leq i \leq n-1$$

and

$$w_{12} w_{23} \cdots w_{n-1,n} \neq 0_A;$$

(*iii*) $w_{12}, \ldots, w_{n-1,n}$ generate $J(A)$ as a two-sided ideal of A.

$\mathbb{Z}_p$-simple algebras

Let $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and

$$D = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ a_p & a_1 & \ddots & & a_{p-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_3 & & \ddots & \ddots & a_2 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix}, \text{ where } a_1, a_2, \ldots, a_p \in F,$$

with its natural grading induced by the p-tuple $(1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1})$.

Proposition

Let F be an algebraically closed field and $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ a group of prime order p. If A is a finite-dimensional G-simple algebra, then it is isomorphic to one of the following G-graded algebras:

(i) M_n with an elementary grading;

(ii) $D \otimes M_r$ with the grading induced by the trivial grading on M_r and the natural one on D. In other words, in this case, A is isomorphic to the homogeneous subalgebra $M_r(D)$ of M_{pr} with the grading induced by the (pr)-tuple

$$(\underbrace{\underbrace{1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}}, \ldots, \underbrace{1_G, \epsilon, \epsilon^2, \ldots, \epsilon^{p-1}}}_{r \text{ times}}).$$

The graded algebra $UT_{\mathbb{Z}_p}(A_1, \ldots, A_m)$

- Assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and F is algebraically closed.

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- Assume that $G = \langle \epsilon \rangle \cong \mathbb{Z}_p$ and F is algebraically closed.
- Given an m-tuple $(A_1, \dots, A_m)$ of G-simple algebras, let

$$\Gamma_0 := \{i \,|\, A_i \text{ is simple graded simple}\}, \quad \Gamma_1 := [1, m] \setminus \Gamma_0.$$

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$$\Gamma_0 := \{i \mid A_i \text{ is simple graded simple}\}, \quad \Gamma_1 := [1, m] \setminus \Gamma_0.$$

- For every $k \in [1, m]$, let us denote the *size* of A_k by

$$s_k := \begin{cases} n_k & \text{if } k \in \Gamma_0 \text{ and } A_k \cong M_{n_k}, \\ pn_k & \text{if } k \in \Gamma_1 \text{ and } A_k \cong M_{n_k}(D) \subseteq M_{pn_k} \end{cases}$$

and set $\nu_0 := 0$, $\nu_k := \sum_{1=1}^{k} s_i$ and $\mathrm{Bl}_k := [\nu_{k-1} + 1, \nu_k]$

- Let

  $$UT(A_1, \ldots, A_m) := \{(a_{ij}) \in UT(s_1, \ldots, s_m) | a_{kk} \in M_{n_k}(D),\ k \in \Gamma_1\}$$

  and $\alpha_k : [1, s_k] \longrightarrow G$ be the map inducing the elementary grading on A_k. In particular, if $k \in \Gamma_1$,

  $$(\alpha_k(1), \ldots, \alpha_k(s_k)) := (\underbrace{\underbrace{1_G, \epsilon, \ldots, \epsilon^{p-1}}, \ldots, \underbrace{1_G, \epsilon, \ldots, \epsilon^{p-1}}}_{n_k \text{ times}}).$$

- Let

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- Let us define the maps

  $$\alpha : [1, \nu_m] \longrightarrow G, \qquad i \longmapsto \alpha_k(i - \nu_{k-1})$$

  and, for any m-tuple $\tilde{g} := (g_1, \ldots, g_m) \in G^m$,

  $$\alpha_{\tilde{g}} : [1, \nu_m] \longrightarrow G, \qquad i \longmapsto g_k \alpha(i),$$

  where $k \in [1, m]$ is the (unique) integer such that $i \in \mathrm{Bl}_k$.

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  where $k \in [1, m]$ is the (unique) integer such that $i \in \mathrm{Bl}_k$.
- Let us denote any such G-graded algebra by $UT_G(A_1, \ldots, A_m)$.

A characterization theorem

Theorem [Di Vincenzo-da Silva-S., 2017]

Let F be a field of characteristic zero and G a group of prime order p. A variety of G-graded PI-algebras of finite basic rank is minimal of G-exponent d if, and only if, it is generated by a G-graded algebra $UT_G(A_1, \ldots, A_m)$ satisfying $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.

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