Identities with derivation of the algebra of upper triangular matrices of size two

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Hopf algebra

Let F be a field, $\operatorname{char} F = 0$, and let H be a Hopf algebra over F with comultiplication $\Delta : H \to H \otimes H$.

Sweedler's notation

For all $h \in H$ $\exists h^i_{(1)}, h^i_{(2)} \in H$ such that

$$\Delta(h) = \sum_i h^i_{(1)} \otimes h^i_{(2)}.$$

This is abbreviated to

$$\Delta(h) = h_{(1)} \otimes h_{(2)}.$$

If $n \geqslant 1$, we write

$$\Delta_n(h) = h_{(1)} \otimes \cdots \otimes h_{(n+1)}.$$

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Hopf algebra

Example

Let L be any F-Lie algebra.

The universal enveloping algebra, $U(L)$, of L becomes a Hopf algebra by defining for all $x \in L$:

- Comultiplication $\Delta : U(L) \to U(L) \otimes U(L)$ as
$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$
- Counite $\epsilon : U(L) \to F$ as
$$\epsilon(x) = 0,$$
- Antipode $S : U(L) \to U(L)$ as
$$S(x) = -x.$$

H-modulo algebra

Definition

An associative algebra A is an H-module algebra or an algebra with an H-action, if A is a left H-module with action

$$h \otimes a \to h \cdot a$$

for all $h \in H, a \in A$, such that

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$$

for all $h \in H$ and $a, b \in A$, where $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

Derivations of an algebra

Let A be an F-associative algebra.

Definition

A derivation of A is a linear map $D : A \to A$ such that

$$D(ab) = D(a)b + aD(b), \qquad \forall a, b \in A.$$

In particular D is an inner derivation induced by $x \in A$ if

$$D(a) = [x, a], \qquad \forall a \in A.$$

The set

$$Der(A) = \{D : A \to A | D \text{ is a derivation}\}$$

is a Lie algebra.

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H-modulo algebra

Example

Let $U(L)$ be the universal enveloping algebra of a F-Lie algebra L.

Let A be an F-associative algebra such that L acts on A as derivation.

The L-action on A can be naturally extended to the following $U(L)$-action

$$x_1 \dots x_n \cdot a = x_1 \cdot (\cdots \cdot (x_n \cdot a) \dots)$$

for all $x_1, \dots, x_n \in L$ and $a \in A$.

A is an $U(L)$-modulo algebra called algebra with derivation.

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H-polynomials

Given a countable set $X = \{x_1, x_2, \ldots\}$ and a basis $B = \{\beta_i | i \in I\}$ of H.

We denote by $F\langle X|H\rangle$ the associative algebra over F freely generated by the set

$$\{x^{\beta_i} = \beta_i(x) | x \in X, \beta_i \in B\}.$$

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If $h = \sum_{i\in I} \alpha_i \beta_i \in H$, $\alpha_i \in F$, where only a finite number of α_i are nonzero, then we put

$$x^h := \sum_{i\in I} \alpha_i x^{\beta_i}.$$

We let H act on $F\langle X|H\rangle$ as follows

$$h(x_{j_1}^{\beta_{i_1}} \dots x_{j_n}^{\beta_{i_n}}) := x_{j_1}^{h_{(1)}\beta_{i_1}} \dots x_{j_n}^{h_{(n)}\beta_{i_n}}$$

where $h \in H$, $\Delta_{n-1}(h) = h_{(1)} \otimes \dots \otimes h_{(n)}$ and $\beta_{i_1}, \dots, \beta_{i_n} \in B$.

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H-polynomials

Let A be an associative algebra with H-action.

Universal property

Any set theoretical map $\varphi : X \to A$ extends uniquely to a homomorphism $\overline{\varphi} : F\langle X|H\rangle \to A$ such that $\overline{\varphi}(f^h) = h \cdot \overline{\varphi}(f)$, for any $f \in F\langle X|H\rangle$ and $h \in H$.

Definition

$F\langle X|H\rangle$ is called the free algebra on X with H-action or free H-modulo algebra, and its elements are called H-polynomials.

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H-identities

Let A be an associative algebra with H-action.

Definition

$f = f(x_1, \dots, x_n) \in F\langle X|H\rangle$ is an H-identity of A, and we write $f \equiv 0$, if

$$f(a_1, \dots, a_n) = 0$$

for all $a_1, \dots, a_n \in A$.

$$Id^H(A) = \{f \in F\langle X|H\rangle | f \equiv 0 \text{ on } A\}$$

is the T^H-ideal of $F\langle X|H\rangle$ of all H-identities of A.

H-identities

Let A be an associative algebra with H-action.

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H-polynomials and H-identities

Example

Le $U(L)$ be the universal enveloping algebra of a F-Lie algebra L.

The free algebra

$$F\langle X|U(L)\rangle = F^d\langle X\rangle$$

is the free algebra with derivation.

If A is an F-associative algebra such that L acts on A as derivation. Then the elements of

$$Id^{U(L)}(A) = Id^d(A)$$

are polynomial identities with derivation of A or differential identities of A.

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Multilinear H-polynomials

The space of multilinear H-polynomials in $x_1, \ldots, x_n$, $n \in \mathbb{N}$,

$$P_n^H = span\{x_{\sigma(1)}^{h_1} \ldots x_{\sigma(n)}^{h_n} | \sigma \in S_n, h_i \in H\},$$

has a natural structure of left S_n-module induced by

$$\sigma(x_i^h) = x_{\sigma(i)}^h, \quad \sigma \in S_n.$$

The space

$$P_n^H(A) = \frac{P_n^H}{P_n^H \cap Id^H(A)}$$

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H-codimension

Definition

The non-negative integer

$$c_n^H(A) := \dim P_n^H(A), \quad n \geq 1,$$

is called the nth H-codimension of A. The sequence $\{c_n^H(A)\}_{n\geq 1}$ is the H-codimension sequence of A.

H-cocharacter

Definition

The character, $\chi_n^H(A)$, of $P_n^H(A)$ is called the nth H-cocharacter of A.

The nth H-cocharacter of A can be decompose as

$$\chi_n^H(A) = \sum_{\lambda \vdash n} m_\lambda^H \chi_\lambda,$$

where λ is partition of n, χ_λ is the irreducible S_n-character associated to λ, and $m_\lambda^H \geq 0$ is the corresponding multiplicity.

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H-codimension and H-cocharacter

Example

Let L be any F-Lie algebra and let A be an F-associative algebra such that L acts on A as derivation.

$$c_n^{U(L)}(A) = c_n^d(A)$$

is the nth differential codimension of A, and

$$\chi_n^{U(L)}(A) = \chi_n^d(A)$$

is the nth differential cocharacter of A.

Derivations of UT_2

Let $B = \{e_{11} + e_{22}, e_{11} - e_{22}, e_{12}\}$ be a basis of UT_2.
Let

$$\varepsilon(a) = \frac{1}{2}[e_{11} - e_{22}, a]$$

and

$$\delta(a) = \frac{1}{2}[e_{12}, a],$$

for all $a \in UT_2$.

If $a = \alpha(e_{11} + e_{22}) + \beta(e_{11} - e_{22}) + \gamma e_{12} \in UT_2$, then

$$\varepsilon(a) = \gamma e_{12}$$

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$$\delta(a) = -\beta e_{12}.$$

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Derivations of UT_2

Theorem (Coelho,Polcino Milies,1993)

Any derivation of the algebra of $n \times n$ upper triangular matrices over a field F, $UT_n(F)$, is inner.

$L = Der(UT_2)$ is the non-abelian Lie algebra of dimension 2 with basis $\{\varepsilon, \delta\}$ such that

$$[\varepsilon, \delta] = \delta.$$

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Differential identities of UT_2

Theorem

Let $UT_2^d(F)$ be the algebra of 2×2 upper triangular matrices over F with $L = Der(UT_2)$-action. Then

- $Id^d(UT_2) = \langle [x,y]^{\varepsilon} - [x,y], [x,y]^{\delta}, x^{\alpha}yz^{\beta} \rangle_{T^d}$ *where* $\alpha, \beta \in \{\varepsilon, \delta\}$.
- $c_n^d(UT_2) = 2^{n-1}(n+2)$.

Differential identities of UT_2

Notation

- $F\langle X\rangle$=the free associative algebra on a countable set X over F
- $Id(UT_2)$=T-ideal of all (ordinary) polynomials identities of UT_2
- P_n=space of multilinear (ordinary) polynomials of degree n in $x_1, \ldots, x_n$ variables
- $P_n(UT_2) = \dfrac{P_n}{P_n \cap Id(UT_2)} = \dfrac{P_n}{P_n \cap Id^d(UT_2)}$

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Differential identities of UT_2

Notation

- E=Lie algebra over F with basis $\{\varepsilon\}$
- $U(E)$=universal enveloping algebra of E
- $P_n^\varepsilon = P_n^{U(E)}$=space of multilinear ε-polynomials of degree n in $x_1, \dots, x_n$ variables
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Differential identities of UT_2

Notation

- Δ=Lie algebra over F with basis $\{\delta\}$
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- $P_n^\delta = P_n^{U(\Delta)}$=space of multilinear δ-polynomials of degree n in $x_1, \ldots, x_n$ variables
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Differential identities of UT_2

Corollary

$$P_n^d(UT_2) \cong P_n(UT_2) \oplus P_n^{\varepsilon}(UT_2) \oplus P_n^{\delta}(UT_2)$$

as S_n-module.

Differential PI-exponent of UT_2

Theorem (Gordienko, Kochtov, 2014)

If A is an algebra with derivations satisfying a non trivial differential identity, then there exists

$$\mathrm{Exp}^d(A) := \lim_{n\to\infty} (c_n^d(A))^{\frac{1}{n}} \in \mathbb{Z}_+.$$

It is called differential PI-exponent of A.

Corollary

$\mathrm{Exp}^d(UT_2) = 2.$

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Corollary

$\mathrm{Exp}^d(UT_2) = 2.$

Cocharacter of UT_2

The nth differential cocharacter of UT_2

$$\chi_n^d(UT_2) = \sum_{\lambda \vdash n} m_\lambda^d \chi_\lambda$$

is the character of $P_n^d(UT_2)$.

The nth ordinary cocharacter of UT_2

$$\chi_n(UT_2) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$$

is the character of $P_n(UT_2)$.

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Ordinary cocharacter of UT_2

Theorem

Let $\chi_n(UT_2) = \sum_{\lambda\vdash n} m_\lambda \chi_\lambda$ be the nth (ordinary) cocharacter of UT_2. Then

1. $m_{(n)} = 1$;
2. $m_\lambda = q + 1$ *if* $\lambda = (p + q, p)$;
3. $m_\lambda = q + 1$ *if* $\lambda = (p + q, p, 1)$;
4. $m_\lambda = 0$ *in all other case.*

Differential cocharacter of UT_2

Theorem

Let $\chi_n^d(UT_2) = \sum_{\lambda \vdash n} m_\lambda^d \chi_\lambda$ be the nth differential cocharacter of UT_2. Then

1. $m_{(n)}^d = 2n + 1$;
2. $m_\lambda^d = 3(q+1)$ *if* $\lambda = (p+q, p)$;
3. $m_\lambda^d = q + 1$ *if* $\lambda = (p+q, p, 1)$;
4. $m_\lambda^d = 0$ *in all other case.*

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