Dynamical extension of indecomposable left cycle sets

Giuseppina Pinto

Cycle sets and the Yang-Baxter equation

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Definition (Rump, 2005)

Let X be a non-empty set and $\cdot$ a binary operation on X.
The pair $(X, \cdot)$ is said a *left cycle set* if for all $x, y, z \in X$ it holds

$$(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z) \quad (1)$$

and the left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x \cdot y$ is bijective, for every $x \in X$.
We call $(X, \cdot)$ *non-degenerate* if the squaring map $\mathfrak{q} : X \longrightarrow X,\ x \longmapsto x \cdot x$ is bijective.

We denote by $\mathcal{G}(X)$ the subgroup of $Sym(X)$ on X generated by $\{\sigma_x | x \in X\}$.
We call $\mathcal{G}(X)$ the *permutation group* of X.

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This new algebraic structure was introduced by Rump in order to find all set-theoretic solutions of the Yang-Baxter equation.

Definition

Let X be a non-empty set, $r : X \times X \to X \times X$ a map and write

$$r(x, y) = (\lambda_x(y), \gamma_y(x))$$

for all $x, y \in X$.
Then (X, r) is said to be a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation if and only if the following properties hold:

(1) $r^2 = id_{X^2}$; (r is involutive)

(2) $\lambda_x, \gamma_y \in Sym_X$ for all $x \in X$; (r is non degenerate)

(3) $r_1 r_2 r_1 = r_2 r_1 r_2$

where $r_1 = r \times id_X$ and $r_2 = id_X \times r$.

Convention: From now on, by a solution I mean a non-degenerate involutive set-theoretic solution.

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Rump (2005) showed that there is a bijective correspondence between the solutions of the Yang-Baxter equation and the left non degenerate cycle sets.

Problem (Drinfield, 1992)

Find all set-theoretic solution of the Yang-Baxter equation.

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A possible tool to classify the solutions of the Yang-Baxter equation is to study their decomposability and analyze the potential $r-$ *invariant* subsets.

Definition

Let (X, r) be a solution then a subset Y of X is said $r-$ *invariant* if $r(Y \times Y) \subseteq Y \times Y$.

In terms of left cycle sets, recall that $\mathcal{G}(X)$ is the permutation group of X.

Definition

A subset Y of X is called *invariant* when $\sigma_x(Y) \subseteq Y$ for every $x \in Y$ and $\mathcal{G}(X)-$ *invariant* when $\sigma_x(Y) \subseteq Y$ for every $x \in X$.

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Definition (P. Etingof, T.Schedler, A.Soloviev, 1999)

A solution (X, r) is said to be *decomposable* if there exists a non trivial partition $X = Y \cup Z$ with $r(Y \times Y) \subseteq Y \times Y$ and $r(Z \times Z) \subseteq Z \times Z$, such that the restrictions of r to $Y \times Y$ and $Z \times Z$ are again non-degenerate.
Otherwise it is said *indecomposable*.

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A non-degenerate left cycle set X is said to be *decomposable* if it is an union of two disjoint subset $X = Y \cup Z$, such that $\sigma_x(Y) \subseteq Y$ for all $x \in Y$ and $\sigma_x(Z) \subseteq (Z)$, for all $x \in Z$.
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Example

Let $X := \{1, 2, 3, 4\}$ and $\cdot$ the binary operation defined as follow

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	2	3	4
3	1	2	3	4
4	1	2	3	4

Then $(X, \cdot)$ is decomposable left cycle set.

Example

Let $X := \{1, 2, 3, 4\}$ and $\cdot$ the binary operation defined as follow

$\cdot$	1	2	3	4
1	4	3	2	1
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In literature it is known that

Proposition (P. Etingof, T.Schedler, A.Soloviev, 1999)

A non-degenerate left cycle set X is indecomposable if and only if the associated permutation group $\mathcal{G}(X)$ is transitive on X.

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If X is finite and $|X| = p$ with p a prime number. Every indecomposable non-degenerate left cycle set , up to isomorphism, is defined by $x \cdot y := \alpha(y)$ where $\alpha := (12 \dots p) \in Sym_X$.

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Moreover:

- In 2005, Rump proved that every square-free non-degenerate left cycle set is decomposable.
  (Recall that a non-degenerate left cycle set is *square-free* if $\mathfrak{q} : X \to X, x \mapsto x \cdot x$ is such that $\mathfrak{q} = id_X$.)
- In 2012, Chouraqui characterize indecomposable left cycle sets using the associated monoid.
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Vendramin in 2016, inspired by the theory of rack and quandle, characterized the finite left cycle sets in terms of dynamical extension.

Definition

Let I be a left cycle set, S a non-empty set and $\alpha : I \times I \times S \longrightarrow Sym(S)$, $(i,j,s) \mapsto \alpha_{i,j}(s,-)$. Then α is said *dynamical cocycle of I* if and only if

$$\alpha_{i \cdot j, i \cdot k}(\alpha_{i,j}(r,s), \alpha_{i,k}(r,t)) = \alpha_{j \cdot i, j \cdot k}(\alpha_{j,i}(s,r), \alpha_{j,k}(s,t)).$$

Then, if α is a dynamical cocycle then $S \times_\alpha I := (S \times I, \cdot)$ is a left cycle set where

$$(s,i) \cdot (t,j) := (\alpha_{i,j}(s,t), i \cdot j). \quad (2)$$

$(S \times_\alpha I)$ is called the **dynamical extension of I by α**.

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Example (*Rump's semidirect product*)

Let I and S be left cycle sets and suppose that I acts on S, which means that there is a map $I \times S \longrightarrow S$, $(i, s) \longmapsto {}^{i}s$ such that

1) ${}^{i}(s \cdot t) = {}^{i}s \cdot {}^{i}t$
2) ${}^{(i \cdot j)i}s = {}^{(j \cdot i)j}s$
3) The map $S \to S$, $s \mapsto {}^{i}s$ is bijective

for all $i, j \in I$, $s, t \in S$. Put $\alpha : I \times I \times S \longrightarrow Sym(S)$

$$\alpha(i, j, s)(t) := {}^{(i \cdot j)}s \cdot {}^{(j \cdot i)}t.$$

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It's remarkable to see that this left cycle set coincides with Rump's semidirect product.

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Theorem (Vendramin, 2016)

Let X, I be a finite left cycle sets and $p : X \to I$ a covering map. Then, there exist a non-empty set S and a dynamical cocycle α such that $X \cong S \times_\alpha I$.

Now we give a characterization of indecomposable left cycle sets written in terms of dynamical extensions.

Definition

Let I be a left cycle set, S a non-empty set and $\alpha : I \times I \times S \to Sym(S)$ a cocycle. For every $i \in I$, let H_i be the subset of the permutation group $\mathcal{G}(S \times_\alpha I)$ given by

$$H_i := \{h \in \mathcal{G}(S \times_\alpha I) \quad | \quad h(S \times \{i\}) \subseteq S \times \{i\}\}.$$

It's easy to see that H_i is a subgroup of $\mathcal{G}(S \times_\alpha I)$ for every $i \in I$.

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$$H_i := \{h \in \mathcal{G}(S \times_\alpha I) \quad | \quad h(S \times \{i\}) \subseteq S \times \{i\}\}.$$

It's easy to see that H_i is a subgroup of $\mathcal{G}(S \times_\alpha I)$ for every $i \in I$.

Now we give a characterization of indecomposable left cycle sets written in terms of dynamical extensions.

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It is well known that if I is a left cycle set then it is indecomposable if and only if its permutation group $\mathcal{G}(I)$ acts transitively on I.

Theorem (M. Castelli, F. Catino, G.P., in preparation)

Let I be a left cycle set, S a non-empty set and $\alpha : I \times I \times S \to Sym(S)$ a dynamical cocycle. Then $S \times_\alpha I$ is indecomposable if and only if I is indecomposable and there exists $i \in I$ such that H_i is transitive on $S \times \{i\}$.

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Example

Let $I := \mathbb{Z}/2\mathbb{Z}$ be the indecomposable left cycle set defined by $x \cdot y := y + 1$ and let $S := \{1, 2, 3\}$ and $\alpha : I \times I \times S \longrightarrow Sym(S)$ given by $\alpha_{(i,j,s)} := (1\ 2\ 3)$ for all $i, j \in I$ and $s \in S$. Then the left cycle set $S \times_\alpha I$ is indecomposable: indeed I is indecomposable, $H_1 = H_2 = H_3 =< (1\ 3\ 2) >$ and H_1 is transitive on $S \times \{1\}$.

The previous results suggest a characterization of all finite retractable indecomposable left cycle set.

We recall that if X is a left cycle set we can define an equivalence relation

$$x \sim y :\leftrightarrow \sigma_x = \sigma_y$$

Then, $Ret(X) := X/\sim$ is a left cycle sets if and only if X is non degenerate.

$$Ret^n(X) := Ret^{(n-1)}(Ret(X))$$

A left cycle set is called *irretractable* if $Ret(X) \cong X$ and *retractable* otherwise.

Moreover a left cycle set is called *multipermutational of level n* if n is the minimal integer such that $|Ret^n(X)| = 1$.

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Proposition (M. Castelli, F. Castelli. G.P., in preparation)

Let I be a finite retractable left cycle set. Then I is indecomposable if and only if $Ret(I)$ is indecomposable and there exist a set S, a dynamical cocycle α and $x \in Ret(I)$ such that

$$I \cong S \times_{\alpha} Ret(I)$$

and H_x is transitive on $S \times \{x\}$.

Observation: the canonical epimorphism $f : I \to Ret(I)$ is a covering map, unlike the decomposable case.

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In the following proposition we give a characterization of finite left cycle set of level k starting from a finite left cycle set of level $k-1$ in terms of dynamical extensions of left cycle sets.

Recall that a left cycle set X is called *multipermutation of level* m, if m is the minimal non-negative integer such that $Ret^m(X)$ has cardinality one.

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Proposition

Let X be a finite left cycle set. Then X is an indecomposable left cycle set of level k if and only if there exist a left cycle set I, a set S and a dynamical cocycle α such that $X \cong S \times_\alpha I$ and the following properties hold:

1) *I is an indecomposable left cycle set of level $k-1$;*
2) *$\alpha(i,j,s) = \alpha(i,j,t)$ for every $s,t \in S$, $i,j \in I$;*
3) *$\alpha(i,j,s) \neq \alpha(i',j,t)$ for every $i \neq i'$, $i,i',j \in I$, $s,t \in S$;*
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Example

Let $I := (\mathbb{Z}/2\mathbb{Z}, \cdot\,)$ be the left cycle set given by $x \cdot y := y + 1$ and $S := \mathbb{Z}/2\mathbb{Z}$. Let $\alpha : I \times I \times S \longrightarrow Sym(S)$ be the function given by

$$\alpha(0,0,0) = \alpha(0,0,1) = \alpha(1,1,0) = \alpha(1,1,1) := t_1$$

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Then α is a cocycle and $S \times_\alpha I$ is an indecomposable left cycle set and $mpl(S \times_\alpha I) = 2$.

The last result suggests a way to study finite retractable indecomposable left cycle sets:

Inductively study finite left cycle set of level k finding all the dynamical extensions of the left cycle sets of level $k-1$.

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THANKS FOR YOUR ATTENTION!