Semi-braces and the asymmetric product

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The Yang-Baxter equation

The Yang-Baxter equation is a basic equation of the statistical mechanics that arose from Yang's work in 1967 and Baxter's one in 1972.

If X is a non-empty set, a **set-theoretical solution** of the Yang-Baxter equation is a map $r : X \times X \to X \times X$ that satisfies the **braid equation**, i.e.,

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.

Question: How to determine all set-theoretic solutions of the Yang-Baxter equation?

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Notation and definitions

If $r : X \times X \to X \times X$ is a set theoretical solution of the Yang-Baxter equation, we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a))$$

where λ_a, ρ_b are maps from X into itself.

In particular, r is said to be **left** (right, resp.) **non-degenerate** if λ_a (ρ_a, resp.) is bijective, for every $a \in X$. Moreover, r is **non-degenerate** if r is both left and right non-degenerate.

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Several approaches

- In 1999 Etingof, Schedler and Soloviev, and Gateva-Ivanova and Van den Bergh dealt with the study of the *non-degenerate involutive* solutions, mainly in group theory terms. In the last years Rump, Cedó, Jespers and Okniński provided new results for this class of solutions.
- In 2000 Lu, Yan and Zhu and independently Soloviev started to study *non-degenerate solutions not necessarily involutive*. In 2017 Guarnieri and Vendramin attacked this problem.
- New solutions that are only *left non-degenerate* can be determined through a new structure, the *semi-brace*.

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The semi-brace

Definition (F. Catino, I. Colazzo, and P.S., J. Algebra, 2017)

Let B be a set with two operations $+$ and $\circ$ such that $(B, +)$ is a left cancellative semigroup and $(B, \circ)$ is a group. We say that $(B, +, \circ)$ is a **(left) semi-brace** if

$$a \circ (b + c) = a \circ b + a \circ \left(a^{-} + c\right),$$

holds for all $a, b, c \in B$, where a^{-} is the inverse of a with respect to $\circ$.

We may check that if $(B, +)$ is a group, then $(B, +, \circ)$ is a *skew brace*, the structure introduced by Guarnieri and Vendramin in 2017.

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Examples of semi-braces

1. If $(E, \circ)$ is a group and define the following sum

$$a + b = b,$$

for all $a, b \in E$, then $(E, +, \circ)$ is a semi-brace. We call this semi-brace the **trivial semi-brace**.

2. If $(B, \circ)$ is a group and f is an endomorphism of $(B, \circ)$ such that $f^2 = f$. Set

$$a + b := b \circ f(a),$$

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Properties

If B is a semi-brace we have:

- if 0 is the identity of $(B, \circ)$ and $a \in B$, then $0 + a = a$, i.e., 0 is a left identity and, also, an idempotent of $(B, +)$.
- The additive structure $(B, +)$ is a right group.

We recall that a left cancellative semigroup B is a right group if and only if for all $x, y \in B$ there exists $t \in B$ such that $x + t = y$.

- Denoted by E the set of idempotents of $(B, +)$, since 0 lies in E, we can check that $G := B + 0$ is a group with respect to the sum and

$$B = G + E.$$

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How to obtain a solution through a semi-brace

Theorem (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)

Let B be a semi-brace. Then, the function $r : B \times B \to B \times B$ given by

$$r(a, b) = \left(a \circ \left(a^{-} + b\right), \left(a^{-} + b\right)^{-} \circ b \right)$$

for all $a, b \in B$, is a solution of the Yang-Baxter equation. We call r the ***solution associated to the semi-brace B****. Moreover, r is a left non-degenerate.*

In fact, if B is a semi-brace,

- the function $\lambda_a : B \to B$ defined by $\lambda_a(b) = a \circ \left(a^{-} + b\right)$ is bijective, for every $a \in B$;
- in general, the function $\rho_b : B \to B$ defined by $\rho_b(a) = \left(a^{-} + b\right)^{-} \circ b$ is not bijective if the additive structure $(B, +)$ is not a group.

Therefore, every solution associated to a semi-brace B that is not a skew brace is left non-degenerate and right degenerate.

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Therefore, every solution associated to a semi-brace B that is not a skew brace is left non-degenerate and right degenerate.

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Theorem (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)

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A construction of left semi-braces

In order to find new solutions of the Yang-Baxter equation through left semi-braces we introduce a construction called **asymmetric product** of semi-braces.

In particular, this construction involves classical tools like Schreier's extension of groups (not necessarily abelian) and group actions.

Even if, in the general case, the asymmetric product is more technical, some particular cases are very interesting and allow us to obtain several examples.

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Schreier's extension and cocycles

Let $(H,+),(N,+)$ be groups, $\mathfrak{c}: H \times H \to N$ and $\alpha: H \to \mathrm{Aut}(N)$. Set $n^h := \alpha(h)(n)$ for all $h \in H$, $n \in N$, if

1. $\left(n^{h_1}\right)^{h_2} = -\mathfrak{c}(h_1, h_2) + n^{h_1+h_2} + \mathfrak{c}(h_1, h_2)$,
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is a group known as **Schreier's extension** of N by H (via α and $\mathfrak{c}$). We briefly call the pair $(\alpha, \mathfrak{c})$ a **cocycle** from H into N.

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Cocycles

We extend the definition of cocycle between two groups to one between additive structures of two semi-braces B_1 and B_2.

Set $B_1 = H + E$ and $B_2 = N + F$, where E is the set of idempotents of B_1, F is the set of idempotents of B_2, 0_{B_1} is the identity of $(B_1, \circ)$, $H := B_1 + 0_{B_1}$, 0_{B_2} is the identity of $(B_2, \circ)$, $N := B_2 + 0_{B_2}$.

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The main result

Theorem (F. Catino, I. Colazzo, P.S., J. Algebra, 2017)

Let $B_1 = H + E, B_2 = N + F$ be semi-braces, $(\alpha, \mathfrak{c})$ a cocycle from H into N and $\beta : B_2 \to \mathrm{Aut}(B_1)$ a homomorphism from the group $(B_2, \circ)$ into the group of automorphisms of the semi-brace B_1. Set ${}^{(n+f)}(h+e) := \beta(n+f)(h+e)$ for all $h+e \in B_1$, $n+f \in B_2$, if

$$\begin{aligned}&\mathfrak{c}\left((h_1+e_1)\circ {}^{(n_1+f_1)}h_2+0, \lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0\right)\\&+\left((n_1+f_1)\circ n_2+0\right)^{\lambda_{(h_1+e_1)}\left({}^{(n_1+f_1)}h_3\right)+0}\\&+(n_1+f_1)\circ\left(\mathfrak{c}\left({}^{(n_1+f_1)^-}\left(\rho_{e_1}\left(h_1^-\right)\right), h_3\right)+\left(\rho_{f_1}\left(n_1^-\right)\right)^{h_3}\right)\\&=(n_1+f_1)\circ\left(\mathfrak{c}\left(h_2,h_3\right)+{n_2}^{h_3}\right)\end{aligned}$$

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The semidirect product of semi-braces

Corollary

Let B_1 and B_2 be semi-braces. Let $\beta : B_2 \to \mathrm{Aut}(B_1)$ be a homomorphism from the group $(B_2, \circ)$ into the group of automorphism of the semi-brace $(B_1, +, \circ)$. Define on $B_1 \times B_2$

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*Then $(B_1 \times B_2, +, \circ)$ is a semi-brace. We call this structure the **semidirect product of B_1 and B_2 (via β)** and we denote it by $B_1 \rtimes B_2$.*

In this case $\alpha : H \to \mathrm{Aut}(N)$ is such that $\alpha(h) = \mathrm{id}_N$, for every $h \in H$, and $\mathfrak{c} : H \times H \to N$ is such that $\mathfrak{c}(h_1, h_2) = 0$, for all $h_1, h_2 \in H$.

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Example

Let G be a group. Set $x + y = x \circ y$, for all $x, y \in G$, then the structure $B_1 = (G, +, \circ)$ is semi-brace that we call *zero semi-brace*.
If B_2 is the trivial semi-brace with G as multiplicative structure, and $\beta : B_2 \to \mathrm{Aut}(B_1)$ a group homomorphism. The semidirect product of $B_1 \rtimes B_2$ by β is given by

$$(h_1, f_1) + (h_2, f_2) = (h_1 + h_2, f_2)$$

and

$$(h_1, f_1) \circ (h_2, f_2) = (h_1 +^{f_1} h_2, f_2)$$

for all $h_1, h_2 \in B_1$ and $f_1, f_2 \in B_2$.

Let us note that this semi-brace is not a trivial semi-brace neither a zero semi-brace.

Example

Let G be a group. Set $x + y = x \circ y$, for all $x, y \in G$, then the structure $B_1 = (G, +, \circ)$ is semi-brace that we call *zero semi-brace*.

If B_2 is the trivial semi-brace with G as multiplicative structure, and $\beta : B_2 \to \mathrm{Aut}(B_1)$ a group homomorphism. The semidirect product of $B_1 \rtimes B_2$ by β is given by

$$(h_1, f_1) + (h_2, f_2) = (h_1 + h_2, f_2)$$

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Consequences

Corollary

Let H, N be skew braces, $(\alpha, \mathfrak{c})$ a cocycle from H into N and $\beta : N \to \mathrm{Aut}(H)$ a homomorphism from the group $(N, \circ)$ into the group of automorphisms of the skew brace H such that

$$n_1 \circ \mathfrak{c}(h_2, h_3) - \left(n_1 \circ (-n_2)^{h_3}\right) = \mathfrak{c}(h_1 \circ^{n_1} h_2, -h_1 + h_1 \circ^{n_1} h_3) + (n_1 \circ n_2 - n_1)^{-h_1 + h_1 \circ^{n_1} h_3} - \mathfrak{c}(h_1, -h_1 + h_1 \circ^{n_1} h_3),$$

holds for all $n_1, n_2 \in N$, $h_1, h_2, h_3 \in H$. Then the sum and the multiplication over the cartesian product $H \times N$ respectively given by

$$(h_1, n_1) + (h_2, n_2) := \left(h_1 + h_2, \mathfrak{c}(h_1, h_2) + {n_1}^{h_2} + n_2\right)$$

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Thanks for your attention!