Basic definition

Definition

If S is a subset of a group $(G, \cdot)$, write

$$S^2 = SS := \{xy \mid x, y \in S\}.$$

S^2 is also called the square of S.

If G is an additive group, then we put

$$2S = S + S := \{x + y \mid x, y \in S\}.$$

$2S$ is also called the double of S.

Doubling problems

General question

Find information about finite subsets S of a group G which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

where α and β **are real numbers**.

Problems of this kind are called "doubling problems" in Additive Number Theory.

The coefficient α is called the **doubling coefficient of S**.

Small doubling problems

Problem

Find the precise description of those finite subsets S of a group G which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ **small**.

Problems of this kind are called

inverse problems of small doubling type.

The additive group of integers

Remark

Let S be a finite set of integers with k elements. Then

$$|2S| \geq 2k - 1.$$

Remark

Let S be a finite set of integers with k elements. Then

$$|2S| = 2k - 1$$

if and only if there exist integers a and q such that $q > 0$ and

$$S = \{a, a+q, a+2q, \ldots, a+(k-1)q\}$$

i.e. S is an arithmetic progression of length k.

The additive group of integers

$3k - 4$ Theorem
(G.A. Freiman, Izv. Vyss. Ucebn. Zaved. Matematika,1959)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that $|2S| \leq 3k - 4$.
Then there exist integers a and q such that $q > 0$ and

$$S \subseteq \{a, a + q, a + 2q, \ldots, a + (2k - 4)q\} \ .$$

The additive group of integers

Freiman studied also the cases

$$|2S| \leq 3|S| - 3$$

and

$$|2S| \leq 3|S| - 2.$$

Now let G be a **torsion-free** group.

Small doubling problems with doubling coefficient 2

Theorem (J.H.B. Kemperman, *Indag. Mat.*, 1956)

If S is a non-empty finite subset of a torsion-free group, then we have $$|S^2| \geq 2|S| - 1.$$

Theorem (G.A. Freiman, B.M. Schein, *Proc. Amer. Math. Soc.*, 1991)

If S is a finite subset of a torsion-free group, $|S| = k \geq 2$,

$$|S^2| = 2|S| - 1$$

if and only if

$S = \{a, aq, \cdots, aq^{k-1}\}$, and either $aq = qa$ or $aqa^{-1} = q^{-1}$.

Small doubling problems with doubling coefficient 3

Problem 1 - I

Let G be any torsion-free group, S a finite subset of G, $|S| = k$, and

$$|S^2| \leq 3|S| - 4.$$

What is the structure of S ?

Is S contained in a geometric progression of length at most $2|S| - 3$?

Problem 1 - II

Let G be any torsion-free group, S a finite subset of G, $|S| = k$, and

$$|S^2| \leq 3|S| - \beta, \text{ where } \beta = 2, 3.$$

What is the structure of S ?

Small doubling problems have been studied

in abelian groups

by many authors:

Y.O. Hamidoune, B. Green, M. Kneser,

A.S. Lladó, A. Plagne, P.P. Palfy,

I.Z. Ruzsa, O. Serra, Y.V. Stanchescu, . . .

Small doubling problems with doubling coefficient 3

Problem 2 - I

Let G be any torsion-free group, S a finite subset of G, $|S| = k$, and

$$|S^2| \leq 3|S| - 4.$$

What about the structure of $\langle S \rangle$?

Problem 2 - II

Let G be any torsion-free group, S a finite subset of G, $|S| = k$, and

$$|S^2| \leq 3|S| - \beta, \text{ where } \beta = 2, 3.$$

What about the structure of $\langle S \rangle$?

In a series of papers with

Gregory Freiman, Marcel Herzog, Patrizia Longobardi,

Yonutz Stanchescu, Alain Plagne, Derek Robinson

we solved these problems

in the class of

orderable groups.

Bibliography

- L. Babai, V.T. Sós, Sidon sets in groups and induced subgraphs of Cayley graphs, *Europ. J. Combinatorics*, **6** (1985), 101-114.
- Ja.G. Berkovich, G.A. Freiman, C.E. Praeger, Small squaring and cubing properties for finite groups, *Bull. Austral. Math. Soc.*, **44** (1991), 429-450.
- O. Bogopolski, Abstract commensurators of solvable Baumslag-Solitar groups, *Comm. Algebra*, **40(7) 44** (2012), 2494-2502.
- K.J. Böröczky, P.P. Palfy, O. Serra, O the cardinality of sumsets in torsion-free groups, *Bull. London Math. Soc.*, (2012), 1034-1041.
- R. Botto Mura, A. Rhemtulla, *Orderable Groups*, Lecture notes in pure and applied mathematics, **27** Marcel Dekker (1977).

Bibliography

- L.V. Brailovsky, G.A. Freiman, On a product of finite subsets in a torsion-free group, *J. Algebra* **130** (1990), 462-476.
- E. Breuillard and B. J. Green, Approximate groups, I: the torsion-free nilpotent case, *J. Inst. Math. Jussieu* **10** (2011), no. 1, 37-57.
- E. Breuillard, B.J. Green and T.C. Tao, The structure of approximate groups, *Publ. Math. IHES.* **116** (2012), 115-221.
- E. Breuillard, B.J. Green and T.C. Tao, A nilpotent Freiman dimension lemma, *European J. Combin.* **34** (8) (2013), 1287-1292.
- G.A. Freiman, On the addition of finite sets, *Izv. Vyss. Ucebn. Zaved. Matematika,* **6** (13) (1959), 202-213.
- G.A. Freiman, Inverse problems of additive number theory IV. On the addition of finite sets II, *(Russian) Elabuž. Gos. Ped. Inst. Učen. Zap.,* **8** (1960), 72-116.

Bibliography

- G.A. Freiman, Groups and the inverse problems of additive number theory, Number-theoretic studies in the Markov spectrum and in the structural theory of set addition (Russian), *Kalinin. Gos. Univ., Moscow* (1973), 175-183.
- G. A. Freiman, On two and three-element subsets of groups, *Aeq. Math.*, **22** (1981), 140-152.
- G.A. Freiman, *Foundations of a structural theory of set addition*, Translations of mathematical monographs, **v. 37**, American Mathematical Society, Providence, Rhode Island, 1973.
- G.A. Freiman, Structure Theory of Set Addition, *Astérisque*, **258** (1999), 1-33.
- G.A. Freiman, On finite subsets of nonabelian groups with small doubling, *Proc. of the Amer. Math. Soc.* **140**, no. 9 (2012), 2997-3002.

Bibliography

- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Small doubling in ordered groups, *J. Austral. Math. Soc.*, **96** (2014), no. 3, 316-325.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Direct and inverse problems in Additive Number Theory and in non-abelian group theory, *European J. Combin.*, **40** (2014), 42-54.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, A small doubling structure theorem in a Baumslag-Solitar group, *European J. Combin.*, **44** (2015), 106-124.
- G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Inverse problems in Additive Number Theory and in Non-Abelian Group Theory, *arXiv*:1303.3053 (2013), preprint, 1-31.

Bibliography

G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y.V. Stanchescu, Some inverse problems in group theory, *Note di Matematica*, **34** (2014), 89-104.

G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, D.J.S. Robinson, Y.V. Stanchescu, On the structure of subsets of an orderable group, with some small doubling properties, *J. Algebra*, **445** (2016), 307-326.

G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, Y.V. Stanchescu, Small doubling in ordered groups: generators and structures, *Groups Geom. Dyn.*, **11** (2017), 585-612.

G.A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Small doubling in ordered nilpotent group of class 2, *European Journal of Combinatorics,* (2017) http://dx.doi.org/10.1016/j.ejc.2017.07.006., to appear.

Bibliography

- G.A. Freiman, B.M. Schein, Interconnections between the structure theory of set addition and rewritability in groups, *Proc. Amer. Math. Soc.* **113** (1991), 899-910.
- A. Geroldinger, I.Z. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Birkäuser Verlag, Basel - Boston - Berlin, 2009.
- A.M.W. Glass, *Partially ordered groups*, World Scientific Publishing Co., Series in Algebra, **v. 7**, 1999.
- B. Green, What is ... an approximate group?, *Notices Amer. Math. Soc.* **59** (2012), no. 5, 655-656.
- B. Green, I. Z. Ruzsa, Freiman's theorem in an arbitrary abelian group, *J. London Math. Soc.* **75** (2007), no. 1, 163-175.

Bibliography

- Y.O. Hamidoune, A.S. Lladó, O. Serra, On subsets with small product in torsion-free groups, *Combinatorica* (1998), no. 4, 529-540.
- Y.O. Hamidoune, A. Plagne, A generalization of Freiman's 3k–3 theorem, *Acta Arith.* **103** (2002), no. 2, 147-156.
- M. Herzog, P. Longobardi, M. Maj, On a combinatorial problem in group theory, *Israel J. Math* **82** (1993), 329-340.
- M.I. Kargapolov, Completely orderable groups, *Algebra i Logica (2)* **1** (1962), 16-21.
- J.H.B. Kemperman, On complexes in a semigroup, *Indag. Mat.* **18** (1956), 247-254.
- V.F. Lev, P.Y. Smeliansky, On addition of two distinct sets of integers, *Acta Arith.* **70** (1995), no. 1, 85-91.

Bibliography

- P. Longobardi, M. Maj, The classification of groups with the small squaring property on 3-sets, *Bull. Austral. Math. Soc.*, **46, 2** (1992), 263-269.
- A.I. Mal'cev, On ordered groups, *Izv. Akad. Nauk. SSSR Ser. Mat.* **13** (1948), 473-482.
- M.B. Nathanson, *Additive number theory - Inverse problems and geometry of sumsets*, Springer, New York, 1996.
- M.B. Nathanson, Inverse problems for linear forms over finite sets of integers, *J. Ramanujan Math. Soc.* **23** (2008), no. 2, 151-165.
- B.H. Neumann, On ordered groups, *Amer. J. Math.* **71** (1949), 1-18.
- A. Plagne, S. Tringali, Sums of dilates in ordered groups, *Comm. Algebra* **44** (2016), 5223-5236.

Bibliography

- Z. Ruzsa, Generalized arithmetic progressions and sumsets, *Acta Math. Hungar.* **65(4)** (1994), 379-388.
- I.Z. Ruzsa, An analog of Freiman's theorem in groups, *Astérisque* **258** (1999), 323-326.
- T. Sanders, The structure theory of set addition revisited, *Bull. Amer. Math. Soc.* **50(1)** (2013), 93-127.
- J.F. Sample, A. Shalev, Combinatorial Conditions in Residually Finite Groups, I, *J. Algebra* **157** (1993), no. 1, 43-50.
- A. Shalev, Combinatorial Conditions in Residually Finite Groups, II, *J. Algebra* **157** (1993), no. 1, 51-62.
- Y.V. Stanchescu, On addition of two distinct sets of integers, *Acta Arith.* **75** (1996), no. 2, 191-194.

Bibliography

- Y.V. Stanchescu, On the structure of sets with small doubling property on the plane, I, *Acta Arith.* **83** (1998), no. 2, 127-141.
- Y.V. Stanchescu, The structure of d-dimensional sets with small sumset, *J. Number Theory* **130** (2010), no. 2, 289-303.
- T.C. Tao, Product set estimates for non-commutative groups, *Combinatorica* **28(5)** (2008), 547-594.
- M.Tointon, Freiman's theorem in an arbitrary nilpotent group, *Proc. London Math. Soc.* **109** (2014), 318-352.