Engel elements in the first Grigorchuk group

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(joint work with G. Fernández-Alcober and A. Tortora)

Advances in Group Theory and Applications

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Lecce

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Outline

Engel elements

- Let G be a group. We say that $g \in G$ is a *right Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[g, {}_n x] = 1$, where

$$[g,x] = g^{-1}g^{x} \text{ and } [g, {}_n x] = [[g, x, \overset{n-1}{\ldots}, x], x] \text{ if } n > 1.$$

- If n can be chosen independently of x, we say that g is a *bounded right Engel element*.
- Similarly g is (bounded) left Engel if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, {}_n g] = 1$ ($\exists n = n(g) \geq 1$ such that $[x, {}_n g] = 1$).
- The group G is said to be an Engel group if all its elements are right Engel or left Engel.

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Notation:

- $L(G) = \{$left Engel elements of $G\}$
- $R(G) = \{$right Engel elements of $G\}$
- $\bar{L}(G) = \{$bounded left Engel elements of $G\}$
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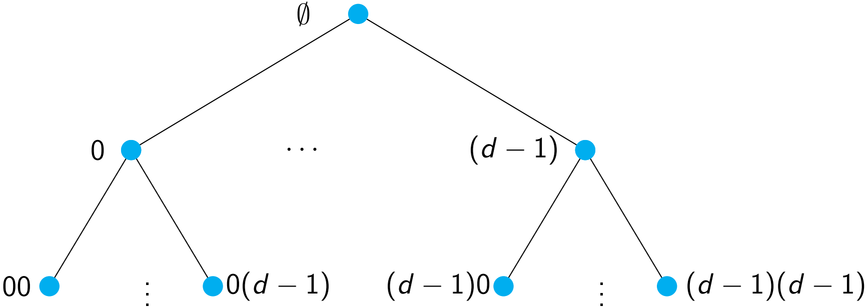
Automorphisms of a d-adic tree

A tree is a connected graph with no cycles.



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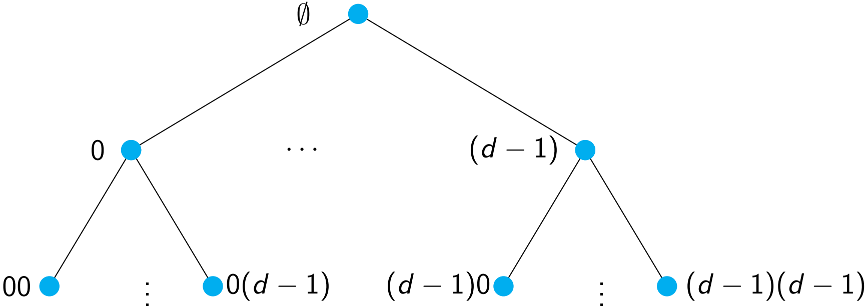

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- A *vertex* is a word in X^*, i.e. the set of all words in the alphabet $X = \{0, \ldots, d-1\}$. Moreover X^n is the set of all words of length n.
- An automorphism is a bijective map from X^* to X^* which preserves incidence.
- The set of all of these automorphisms is a group

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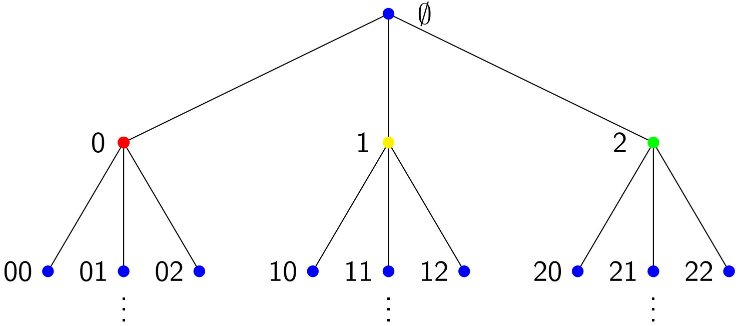

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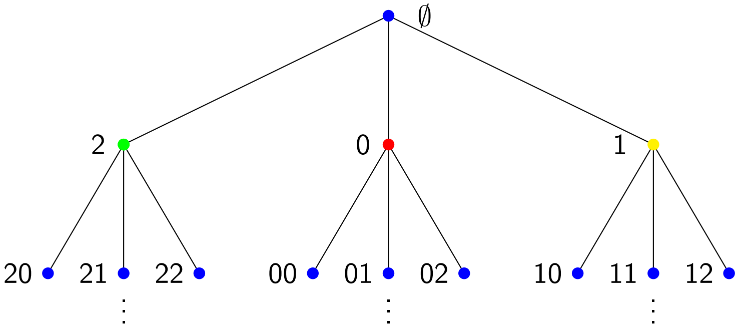

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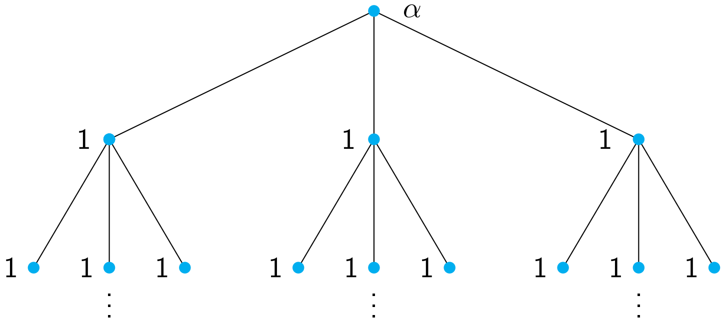
Let $\alpha = (012)$. The portrait of this automorphism is



A subgroup of Aut $\mathcal{T}$

If u is a vertex of $\mathcal{T}$, the *stabilizer* of u is:

$$\mathrm{st}(u) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u\}.$$

We can generalize and define stabilizers of levels:

$$\mathrm{st}(n) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u \;\forall u \in X^n\}.$$

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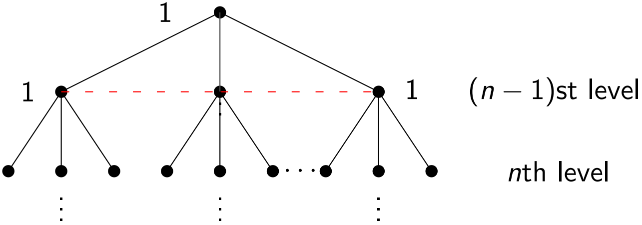

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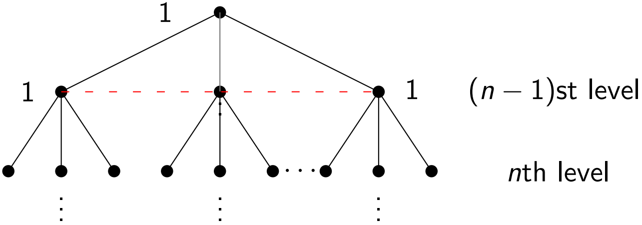

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Facts

If $n \in \mathbb{N}$, we can define a map

$$\psi_n : \mathrm{st}(n) \longrightarrow \operatorname{Aut}\mathcal{T} \times \overset{d^n}{\cdots} \times \operatorname{Aut}\mathcal{T}.$$

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The Grigorchuk group

Let us consider $\mathcal{T}(2)$.

$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (01),\ \psi(b) = (a, c),\ \psi(c) = (a, d)\ \psi(d) = (1, b)$$

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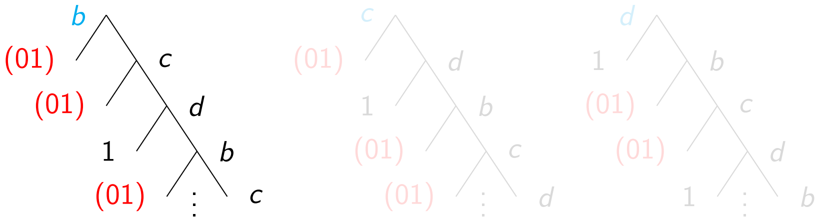

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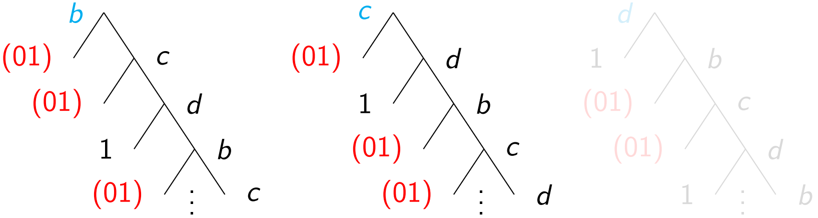

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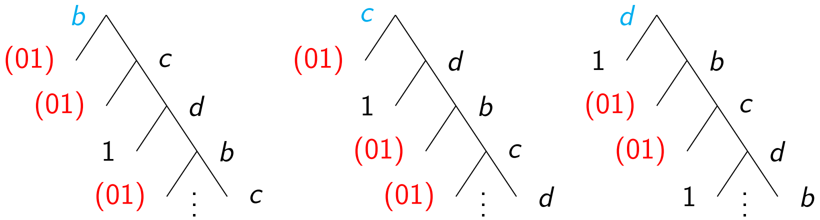

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$$\Gamma = \langle a \rangle \ltimes \mathrm{st}_\Gamma(1)$$

Γ has the following properties:

- It is finitely generated
- It is a 2-group
- It is infinite
- $\psi : \mathrm{st}_\Gamma(1) \longrightarrow \Gamma \times \Gamma$
  It is regular branch over $K = \langle [a, b] \rangle^\Gamma$ (i.e. $\psi(K) \supseteq K \times K$)

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The set $L(G)$: Bludov's example

- Let $K = \langle g \rangle$, where $o(g) = 4$. We denote by $G = \Gamma \wr K$. There exists

$$h = (1, ab, ca, d) \in (\Gamma \times \Gamma \times \Gamma \times \Gamma)$$

  such that $[h,_n g] \neq 1$ for any $n \geq 1$.
- As a consequence, $H = \Gamma \wr D_8$ is not an Engel group.
- Remark: Every involution in a 2-group is a left Engel element.
- Then, $L(H)$ is not a subgroup.

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The set $L(\Gamma)$

Theorem (Bartholdi, 2015)

The Grigorchuk group is not Engel.

The proof uses GAP.

Main result

Theorem (Fernández-Alcober, N, Tortora)

We have

$$\bar{L}(\Gamma) = R(\Gamma) = \bar{R}(\Gamma) = \{1\}.$$

Right Engel elements in Γ

Key facts used during the proof

- Γ is regular branch over K;
- K contains an element that is not left Engel.

Generalization

The previous result used only a few properties of Γ. It can be widely generalized:

Theorem (Fernández-Alcober, N, Tortora)

Let $G \leq \mathrm{Aut}\,\mathcal{T}$ be a regular branch group (over a subgroup K). Suppose that K contains at least an element that is not left Engel. Then $R(G) = \{1\}$.

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The GGS groups

Let us consider $\mathcal{T}(p)$, for p an odd prime.

$$G = \langle a, b \rangle$$

- $a = (1 \dots p)$.
- $\mathbf{e} = (e_1, \dots, e_{p-1}) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$ is a nonzero vector,

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Right Engel elements in G

Theorem (Fernández-Alcober, N, Tortora)

Let G be a nontorsion GGS group with nonconstant defining vector $\mathbf{e}$. Then, $R(G) = \{1\}$.

Open problems

- What about the other GGS groups?

Regarding the Grigorchuk group:

- Can we find a GAP-free proof of the fact that the only left Engel elements in Γ are the involutions?

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Grazie per l'attenzione! :)

Eskerrik asko!

CS ;)