Groups in which every non-nilpotent sugroup is self-normalizing

Chiara Nicotera

Lecce
6 settembre 2017

References

- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera, *Groups in which every non-abelian subgroup is self-normalizing*, **Monatsh. Math.** (2017) to appear
- C. Delizia, U. Jezernik, P. Moravec, C. Nicotera , *Groups in which every non-nilpotent subgroup is self-normalizing*, **Ars Mathematica Contemporanea**, to appear

Self-normalizing subgroups

Definition

A subgroup H of a group G is **self-normalizing** if
$N_G(H) = \{g \in G | H^g = H\} = H$

Remark

If every non-trivial subgroup of a group G is self-normalizing, then G is simple and periodic
Moreover if G is locally finite then either $G = \{1\}$ or $|G| = p$ (prime)

Infinite examples

Tarski p-groups

Self-normalizing subgroups

Definition

A subgroup H of a group G is **self-normalizing** if
$N_G(H) = \{g \in G | H^g = H\} = H$

Remark

If every non-trivial subgroup of a group G is self-normalizing, then G is simple and periodic
Moreover if G is locally finite then either $G = \{1\}$ or $|G| = p$ (prime)

Infinite examples

Tarski p-groups

Self-normalizing subgroups

Definition

A subgroup H of a group G is **self-normalizing** if
$N_G(H) = \{g \in G | H^g = H\} = H$

Remark

If every non-trivial subgroup of a group G is self-normalizing, then G is simple and periodic
Moreover if G is locally finite then either $G = \{1\}$ or $|G| = p$ (prime)

Infinite examples

Tarski p-groups

Problem (P. Zaleskii, 2015)

Classify the finite groups in which every non-abelian subgroup is self-normalizing

Proposition

Let G be a finite group in which every **non-abelian** subgroup is self-normalizing. Then G is either soluble or simple

Theorem

Let G be a finite group.

- *If G is a non-abelian simple group, then every non-abelian subgroup of G is self-normalizing iff* $G \simeq \mathrm{Alt}(5)$ *or* $G \simeq \mathrm{PSL}_2(2^{2n+1})$, $n \geq 1$.
- *If G is a soluble non-nilpotent group, then every non-abelian subgroup of G is self-normalizing iff* $G = A \rtimes \langle x \rangle$, *where* $\langle x \rangle$ *is a p-group for some prime p, A is an abelian* p'*-group,* x^p *is central and x acts fixed point freely on A.*
- *If G is a nilpotent group, then every non-abelian subgroup of G is self-normalizing iff G is either abelian or minimal non-abelian p-group for some prime p*

Proposition

Let G be a finite group in which every **non-abelian** subgroup is self-normalizing. Then G is either soluble or simple

Theorem

Let G be a finite group.

- *If G is a non-abelian simple group, then every non-abelian subgroup of G is self-normalizing iff* $G \simeq \mathrm{Alt}(5)$ *or* $G \simeq \mathrm{PSL}_2(2^{2n+1})$, $n \geq 1$.
- *If G is a soluble non-nilpotent group, then every non-abelian subgroup of G is self-normalizing iff* $G = A \rtimes \langle x \rangle$, *where* $\langle x \rangle$ *is a p-group for some prime p, A is an abelian* p'*-group,* x^p *is central and x acts fixed point freely on A.*
- *If G is a nilpotent group, then every non-abelian subgroup of G is self-normalizing iff G is either abelian or minimal non-abelian p-group for some prime p*

Proposition

Let G be a finite group in which every **non-abelian** subgroup is self-normalizing. Then G is either soluble or simple

Theorem

Let G be a finite group.

- *If G is a non-abelian simple group, then every non-abelian subgroup of G is self-normalizing iff $G \simeq \mathrm{Alt}(5)$ or $G \simeq \mathrm{PSL}_2(2^{2n+1})$, $n \geq 1$.*
- *If G is a soluble non-nilpotent group, then every non-abelian subgroup of G is self-normalizing iff $G = A \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, A is an abelian p'-group, x^p is central and x acts fixed point freely on A.*
- *If G is a nilpotent group, then every non-abelian subgroup of G is self-normalizing iff G is either abelian or minimal non-abelian p-group for some prime p*

Proposition

Let G be a finite group in which every **non-abelian** subgroup is self-normalizing. Then G is either soluble or simple

Theorem

Let G be a finite group.

- *If G is a non-abelian simple group, then every non-abelian subgroup of G is self-normalizing iff $G \simeq \mathrm{Alt}(5)$ or $G \simeq \mathrm{PSL}_2(2^{2n+1})$, $n \geq 1$.*
- *If G is a soluble non-nilpotent group, then every non-abelian subgroup of G is self-normalizing iff $G = A \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, A is an abelian p'-group, x^p is central and x acts fixed point freely on A.*
- *If G is a nilpotent group, then every non-abelian subgroup of G is self-normalizing iff G is either abelian or minimal non-abelian p-group for some prime p*

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$*-groups*

The class $\mathfrak{X}$ *also contains:*

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ *is subgroup and quotient closed*

If $G \in \mathfrak{X}$ *is product of two proper normal subgroups, then it is nilpotent.*

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Every nilpotent group lies in the class $\mathfrak{X}$

Groups in which every non-abelian subgroup is self-normalizing are $\mathfrak{X}$-groups

The class $\mathfrak{X}$ also contains:

1. **minimal non-nilpotent groups** *(non-nilpotent groups in which every proper subgroup is nilpotent)*
2. **groups in which every non-trivial subgroup is self-normalizing**

The class $\mathfrak{X}$ is subgroup and quotient closed

If $G \in \mathfrak{X}$ is product of two proper normal subgroups, then it is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Remark 1

Let G be a $\mathfrak{X}$-group; then

- either $G = G'$
- or G' is nilpotent (and so G is soluble)

Remark 2

Let G be a $\mathfrak{X}$-group and $F := F(G)$ be the Fitting subgroup of G; then $G = F$ or F is nilpotent.

The class $\mathfrak{X}$

Groups in which every **non-nilpotent** subgroup is self-normalizing

Remark 1

Let G be a $\mathfrak{X}$-group; then

- either $G = G'$
- or G' is nilpotent (and so G is soluble)

Remark 2

Let G be a $\mathfrak{X}$-group and $F := F(G)$ be the Fitting subgroup of G; then $G = F$ or F is nilpotent.

Soluble $\mathfrak{X}$-groups

Soluble groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a soluble $\mathfrak{X}$-group and $F := F(G)$ be the Fitting subgroup of G. Then:

1. $G' \leq F$;
2. *G is a Fitting group or G/F has prime order;*
3. *if G/G' is finitely generated then G is a Fitting group or G/G' is cyclic of prime-power order;*
4. *if G is non-nilpotent then G/G' is a locally cyclic p-group for some prime p and $G' = \gamma_3(G)$.*

Soluble $\mathfrak{X}$-groups

Soluble groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a soluble $\mathfrak{X}$-group and $F := F(G)$ be the Fitting subgroup of G. Then:

1. $G' \leq F$;
2. *G is a Fitting group or G/F has prime order;*
3. *if G/G' is finitely generated then G is a Fitting group or G/G' is cyclic of prime-power order;*
4. *if G is non-nilpotent then G/G' is a locally cyclic p-group for some prime p and $G' = \gamma_3(G)$.*

Soluble $\mathfrak{X}$-groups

Proposition

Every infinite polycyclic $\mathfrak{X}$-group is nilpotent

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{X}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{X}$-group iff

- $G = H \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group and x^p acts trivially on H;
- put $\rho_x : h \in H \to h^{-x}h \in H$, for every $\langle x \rangle$-invariant subgroup K of H either there exists $n \geq 1$ such that $\rho_x^n(K) = 1$, or $\langle \rho_x(K) \rangle = K$.

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{X}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Soluble $\mathfrak{X}$-groups

Proposition

Every infinite polycyclic $\mathfrak{X}$-group is nilpotent

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{X}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{X}$-group iff

- *$G = H \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group and x^p acts trivially on H;*
- *put $\rho_x : h \in H \to h^{-x}h \in H$, for every $\langle x \rangle$-invariant subgroup K of H either there exists $n \geq 1$ such that $\rho_x^n(K) = 1$, or $\langle \rho_x(K) \rangle = K$.*

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{X}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Soluble $\mathfrak{X}$-groups

Proposition

Every infinite polycyclic $\mathfrak{X}$-group is nilpotent

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{X}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{X}$-group iff

- $G = H \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group and x^p acts trivially on H;
- put $\rho_x : h \in H \to h^{-x}h \in H$, for every $\langle x \rangle$-invariant subgroup K of H either there exists $n \geq 1$ such that $\rho_x^n(K) = 1$, or $\langle \rho_x(K) \rangle = K$.

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{X}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Soluble $\mathfrak{X}$-groups

Proposition

Every infinite polycyclic $\mathfrak{X}$-group is nilpotent

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{X}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{X}$-group iff

- *$G = H \rtimes \langle x \rangle$, where $\langle x \rangle$ is a p-group for some prime p, H is a nilpotent p'-group and x^p acts trivially on H;*
- *put $\rho_x : h \in H \to h^{-x}h \in H$, for every $\langle x \rangle$-invariant subgroup K of H either there exists $n \geq 1$ such that $\rho_x^n(K) = 1$, or $\langle \rho_x(K) \rangle = K$.*

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{X}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Soluble $\mathfrak{X}$-groups

Proposition

Every infinite polycyclic $\mathfrak{X}$-group is nilpotent

Theorem

Let G be a soluble non-periodic group; then G is a $\mathfrak{X}$-group iff G is nilpotent

Theorem

Let G be a periodic soluble group, and suppose G is not locally nilpotent; then G is a $\mathfrak{X}$-group iff

- $G = H \rtimes \langle x \rangle$, *where* $\langle x \rangle$ *is a* p*-group for some prime* p, H *is a nilpotent* p'*-group and* x^p *acts trivially on* H;
- *put* $\rho_x : h \in H \to h^{-x}h \in H$, *for every* $\langle x \rangle$*-invariant subgroup* K *of* H *either there exists* $n \geq 1$ *such that* $\rho_x^n(K) = 1$, *or* $\langle \rho_x(K) \rangle = K$.

Theorem

Let G be a soluble, locally nilpotent group; then G is a $\mathfrak{X}$-group iff either it is nilpotent or it is minimal non-nilpotent and it is a p-group for some prime p.

Perfect $\mathfrak{X}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

If G is a perfect $\mathfrak{X}$-group and $F := F(G)$ is its Fitting subgroup, then G/F is a non-abelian simple group.

Lemma

Let G be a finite simple group. Then G is a $\mathfrak{X}$-group iff all of its maximal subgroups are $\mathfrak{X}$-groups.

Proposition

Let G be a finite non-abelian simple group; then G is a $\mathfrak{X}$-group iff $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Perfect $\mathfrak{X}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

If G is a perfect $\mathfrak{X}$-group and $F := F(G)$ is its Fitting subgroup, then G/F is a non-abelian simple group.

Lemma

Let G be a finite simple group. Then G is a $\mathfrak{X}$-group iff all of its maximal subgroups are $\mathfrak{X}$-groups.

Proposition

Let G be a finite non-abelian simple group; then G is a $\mathfrak{X}$-group iff $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Perfect $\mathfrak{X}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

If G is a perfect $\mathfrak{X}$-group and $F := F(G)$ is its Fitting subgroup, then G/F is a non-abelian simple group.

Lemma

Let G be a finite simple group. Then G is a $\mathfrak{X}$-group iff all of its maximal subgroups are $\mathfrak{X}$-groups.

Proposition

Let G be a finite non-abelian simple group; then G is a $\mathfrak{X}$-group iff $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Perfect $\mathfrak{X}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

If G is a perfect $\mathfrak{X}$-group and $F := F(G)$ is its Fitting subgroup, then G/F is a non-abelian simple group.

Lemma

Let G be a finite simple group. Then G is a $\mathfrak{X}$-group iff all of its maximal subgroups are $\mathfrak{X}$-groups.

Proposition

Let G be a finite non-abelian simple group; then G is a $\mathfrak{X}$-group iff $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Perfect $\mathfrak{X}$-groups

Perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Theorem

Let G be a finite perfect group; then G is a $\mathfrak{X}$-group iff

- *either $G \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime*
- *or $G \simeq SL_2(5)$*

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.

Infinite perfect $\mathfrak{X}$-groups

Infinite perfect groups in which every **non-nilpotent** subgroup is self-normalizing

Lemma

Let G be a perfect $\mathfrak{X}$-group; then G is simple iff its Fitting subgroup $F = \{1\}$.

Lemma

Let G be an infinite perfect $\mathfrak{X}$-group; then $G \neq F$.

Infinite perfect $\mathfrak{X}$-groups are either simple or non-simple and non-Fitting.

Open question

There exist infinite perfect $\mathfrak{X}$-groups which are not simple?

If such a group G exists and it is locally graded and finitely generated, then G/F is still locally graded and hence has to be finite. Therefore $G/F \simeq PSL_2(2^n)$, where $2^n - 1$ is a prime.