Conciseness of words of Engel type in residually finite groups

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A word w on n variables is an element of the free group F with free generators $x_1, \dots, x_n$.
Given a group G, we can think of w as a function $w : G^n \mapsto G$.
We denote by G_w the set of w-values and by $w(G)$ the verbal subgroup generated by G_w.
A word w is concise in a class $\mathcal{X}$ of groups if:

$$|G_w| < \infty \implies |w(G)| < \infty \text{ for every group } G \in \mathcal{X}.$$

The word w is concise if it is concise in the class of all groups.

Question (P. Hall $\sim$ 1960)

Is every word concise?

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An important class of commutator words is that of multilinear commutators, also known as outer commutator words.

Multilinear commutators are recursively defined as follows:

Definition

The word $w = x_1$ is a multilinear commutator of weight 1. If u, v are multilinear commutators of weights m and n respectively involving different variables, then $[u, v]$ is a multilinear commutator of weight $m + n$.

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Examples

- the lower central words γ_i defined by: $\gamma_0 = x_1$, $\gamma_i = [\gamma_{i-1}, x_i] = [x_1, \ldots, x_i]$ for $i \geq 1$;
- the derived words δ_i defined by: $\delta_0 = x_1$, $\delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}-1}, \ldots, x_{2^i})]$ for $i \geq 1$.

P. Hall $\sim$ 1960

The lower central words are concise.

Turner-Smith 1964

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J. Wilson 1974

Multilinear commutators are concise.

What about Engel words?
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Abdollahi, Russo 2011 - Fernández-Alcober, M., Traustason 2012

The n-th Engel word is concise for $n \leq 4$.

The proof is based on the fact that n-Engel groups are locally nilpotent if $n \leq 4$ (recall that a group is n-Engel if $e_n(G) = 1$).

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Ivanov 1989

If $n > 10^{10}$ and $p > 5000$ is a prime, the word $[[x^{pn}, y^{pn}]^n, y^{pn}]^n$ is not concise.

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Turner-Smith 1966

Every word is concise in the class of all groups whose quotients are residulally finite.

Question (Segal, Jaikin-Zapirain)

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Acciarri, Shumyatsky 2014

If w is a multilinear commutator and q is a prime-power, then the word w^q is concise in the class of residually finite groups.

A word w is weakly rational if for every finite group G and for every integer e relatively prime to $|G|$, the set G_w is closed under taking e-th powers of its elements.

Guralnick, Shumyatsky 2015

Every weakly rational word is concise in the class of residually finite groups. In particular, so is the word $w = [\dots[x_1^{n_1}, x_2]^{n_2}, \dots, x_k]^{n_k}$.

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What about words of Engel type?

Detomi, M., Shumyatsky 2017

If w is a multilinear commutator and y is a variable not appearing in w then the word $[w,_n y]$ is concise in residually finite groups. In particular, the n-th Engel word is concise in residually finite groups.

A word w is boundedly concise in a class $\mathcal{X}$ of groups if for every integer m there exists a number $\nu = \nu(\mathcal{X}, w, m)$ such that $|G_w| \leq m \implies |w(G)| \leq \nu$ for every group $G \in \mathcal{X}$.

Fernández-Alcober, M. 2010

Let be w a multilinear commutator and G a group. If $|G_w| \leq m$ then $|w(G)| \leq (m-1)^{m-1}$.

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Mann? < 2010

Every word which is concise in the class of all groups is boundedly concise.

The proof uses ultraproducts, and gives no clue on the explicit form of the function $\nu(w, m)$.

Question

Is it true that every word which is concise in the class of residually finite groups is also boundedly concise in that class?

Ultraproducts are not of help because the quotient of a residually finite group need not be residually finite.

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Detomi, M., Shumyatsky 2017

Let γ_k be the k-th lower central word, let y be a variable not appearing in γ_k, and let n and m be positive integers. Then both words $[y, {}_n\gamma_k^m]$ and $[\gamma_k^m, {}_n y]$ are boundedly concise in residually finite groups.

A word w is a law in a group G if $w(G) = 1$.
A word w is said to imply virtual nilpotency if every finitely generated metabelian group where w is a law has a nilpotent subgroup of finite index.

Gruenberg 1953

Engel words imply virtual nilpotency.

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Gruenberg 1953

Engel words imply virtual nilpotency.

Burns, Medvedev 2003

A word w implies virtual nilpotency if and only if it is not a law in the restricted wreath product $C_n \wr C$ for any $n \geq 2$.

Burns, Medvedev 2003

The words of the form $w = uv^{-1}$, where u and v are positive words (i.e. they do not involve any of the inverses of the variables) imply virtual nilpotency.

Detomi, M., Shumyatsky 2017

Words implying virtual nilpotency are boundedly concise in residually finite groups.

In particular Englel words are boundedly concise in residually finite groups, and also some of their generalizations, like words of the form: $[x^{n_1}, y^{n_2}, \ldots, y^{n_k}]$.

Proposition

Words implying virtual nilpotency are boundedly concise in residually finite groups.

Proof. We combine two ideas:

- (Turner Smith) Every word is concise in the class of residually nilpotent groups;
- The class $\mathcal{X}$ of groups having a normal subgroup of finite index at most t which is nilpotent of class at most c is is closed by forming ultraproducts of its members;

and we show that:

Proposition

Let c, t be positive integers and let $\mathcal{X}$ be the class of groups having a normal subgroup of finite index at most t which is nilpotent of class at most c. Then every word is boundedly concise in $\mathcal{X}$.

Let $w = w(x_1, \ldots, x_k)$ be a word implying virtual nilpotency of finitely generated metabelian groups, and let G be a finite group such that $|G_w| \leq m$.
We want to show that $|w(G)|$ is bounded.
G contains a subgroup H that can be generated by at most mk elements such that $G_w = H_w$.
We can replace G with H so now G can be generated by at most mk elements.
The group G acts on G_w by conjugation so $G/C_G(w(G))$ embeds in $Sym(m)$. Thus $N = C_G(w(G))$ has m-bounded index in G and so N has a (k, m)-bounded number of generators.
Since w is a law in the quotient $N/w(N)$, by a theorem of Burns and Medvedev $N/w(N)$ has a normal nilpotent subgroup $M/w(N)$ such that the nilpotency class of $M/w(N)$ and the exponent of N/M are bounded in terms of w only.

So the group N/M has bounded exponent and bounded number of generators. The solution of the restricted Burnside problem now tells us that N/M has (k, m)-bounded order and therefore M has (k, m)-bounded index in G.
Observe that $w(N)$ is contained in the center of N. As $M/w(N)$ has bounded nilpotency class, the same holds for M.
So the group G belongs to the class $\mathcal{X}$ of groups having a normal subgroup of finite index at most t and nilpotency class at most c, for some (m, w)-bounded c and t.
An application of the previous proposition completes the proof.

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An ad hoc argument shows that $[\delta_2,_n y]$ is boundedly concise in residually finite groups.

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The solution of the restricted Burnside problem implies that

Proposition

- The class of locally nilpotent n-Engel groups is a variety.
- If G is a finite n-Engel d-generated group, then G is nilpotent with (d, n)-bounded nilpotency class.

The results needed in our proofs are of this type, in particular we use:

Shumyatsky, Tortora, Tota 2016

Let m, n be positive integers, and w a multilinear commutator word. The class of all groups G in which the w^m-values are n-Engel and the verbal subgroup $w^m(G)$ is locally nilpotent is a variety.

This gives as a consequence

Proposition 1

Given positive integers m, n and a multilinear commutator word w, let G be a finite group in which the w^m-values are n-Engel. Suppose that a subgroup H can be generated by d elements which are w^m-values. Then H is nilpotent with (d, m, n, w)-bounded class.

Conciseness in profinite groups

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.
In profinite groups, if w is a word the verbal subgroup $w(G)$ is the topological closure of $\langle G_w \rangle$, and might differ from $\langle G_w \rangle$.

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In profinite groups, if w is a word the verbal subgroup $w(G)$ is the topological closure of $\langle G_w \rangle$, and might differ from $\langle G_w \rangle$.
The following question is still open:

Question (Jaikin-Zapirain)

Is every word concise in the class of profinite groups?

We suggest that in profinite groups the definition of conciseness might be relaxed:

Conjecture

Assume that the word w has only countably many values in a profinite group G. Then the verbal subgroup $w(G)$ is finite.

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The above conjecture holds true whenever w is multilinear commutator.

Detomi, M., Shumyatsky 2016

Let w be a multilinear commutator and G a profinite group having only countably many w-values. Then the verbal subgroup $w(G)$ is finite.

P. Hall proved that non-commutator words are concise in the class of all (abstract) groups, but the above conjecture is wide open also for non-commutator words. Only the case $w = x^2$ is settled.

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We were also able to confirm the conjecture for the commutator word $w = [x^2, y]$.

Detomi, M., Shumyatsky 2016

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Contributions are wellcome!!