A Criterion for Metanilpotency of a Finite Group

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joint work with
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Background

In finite group theory the recovery of global information about a group from local information plays an important role.

In particular, one can derive global information from knowledge of the order of group's elements.

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In this direction, in 2014 **B. Baumslag** and **J. Wiegold** proved the following criterion for nilpotency of a finite group.

Theorem (B. Baumslag, J. Wiegold)

Let G be a finite group in which $|ab| = |a||b|$ whenever the elements a, b have coprime orders. Then G is nilpotent.

Observe that here the symbol $|x|$ stands for the order of the element x in a group G.

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Theorem (R. Bastos, P. Shumyatsky)

Let G be a finite group in which $|ab| = |a||b|$ whenever the elements a, b are commutators of coprime orders. Then G' is nilpotent.

Recall that an element x of a group G is a commutator if there exist elements g, h in G such that $x = [g, h] = g^{-1}h^{-1}gh$. The subgroup generated by all commutators is called commutator subgroup, and it is denoted by G'.

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In view of the above theorem one might suspect that a similar phenomenon holds for other group-words.

A group-word $w = w(x_1, \ldots, x_s)$ is a nontrivial element of the free group $F = F(x_1, \ldots, x_s)$ on free generators $x_1, \ldots, x_s$.

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Given a group-word w, the subgroup of G generated by the w-values is called the verbal subgroup of G corresponding to the word w, and it is usually denoted by $w(G)$.

Question

Let w be a commutator word and G a finite group with the property that if a and b are w-values of coprime order, then $|ab| = |a||b|$. Is then the verbal subgroup $w(G)$ nilpotent?

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Asking a similar question for noncommutator words would not be interesting.

Counter-example

Choose a nonabelian finite simple group, say of exponent e, and the word x^n, where n is a divisor of e such that e/n is prime.

Nevertheless, even in the general case of commutator words the answer to our question is negative!

Indeed, **M. Kassabov** and **N. Nikolov** showed that for any $n \geq 7$ the alternating group A_n admits a commutator word all of whose nontrivial values have order 3.

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The present talk is intended to present an extension of the theorem of **R. Bastos** and **P. Shumyatsky** as follows.

Given an integer $k \geq 1$, the word $\gamma_k = \gamma_k(x_1, \dots, x_k)$ is defined inductively by the formulae

$$\gamma_1 = x_1, \qquad \text{and} \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \dots, x_k] \quad \text{for } k \geq 2.$$

The subgroup of a group G generated by all γ_k-values is denoted by $\gamma_k(G)$. Of course, this is the familiar kth term of the lower central series of G, and observe that if $k = 2$ we have $\gamma_k(G) = G'$.

In the sequel the values of the word γ_k in G will be called γ_k-commutators.

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The kth term of the lower central series of a finite group G is nilpotent if and only if $|ab| = |a||b|$ for any γ_k-commutators $a, b \in G$ of coprime orders.

Recall that a group G is called metanilpotent if there is a normal nilpotent subgroup N such that G/N is nilpotent.

The following corollary is immediate.

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A finite group G is metanilpotent if and only if there exists a positive integer k such that $|ab| = |a||b|$ for any γ_k-commutators $a, b \in G$ of coprime orders.

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Open Question

What about the nilpotency of the kth term $G^{(k)}$ of the derived series?

The counter-example constructed by **M. Kassabov** and **N. Nikolov** uses commutator words in which an element appears together with a power of it.

Therefore, we suspect that the theorem holds in the case of multilinear commutator words, that is words having a form of a multilinear Lie monomial like for example $[[x_1,[x_2,x_3]],[x_4,x_5]]$.

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Notations and Definitions

During the rest of this talk all groups are finite.

If π is a set of primes, we denote by π' the set of all primes that do not belong to π.

For a group G we denote by $\pi(G)$ the set of primes dividing the order of G. The maximal normal π-subgroup of G is denoted by $O_\pi(G)$.

The Fitting subgroup of G is the subgroup generated by all the normal nilpotent subgroups of G, and it is denoted by $F(G)$.

A subgroup H of G is a tower of height h if H can be written as a product $H = P_1 \cdots P_h$, where

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Some Lemmas

Lemma (A)

Let p be a prime and G a metanilpotent group. Suppose that x is a p-element in G such that $[O_{p'}(F(G)), x] = 1$. Then $x \in F(G)$.

Lemma (B)

Let k be a positive integer and G a group such that $G = G'$. Let $q \in \pi(G)$. Then G is generated by γ_k-commutators of p-power order for every prime $p \in \pi(G) \setminus \{q\}$.

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Lemma (C)

Let G be a group in which $|ab| = |a||b|$ for any γ_k-commutators $a, b \in G$ of coprime orders. Let x be a γ_k-commutator and N be a subgroup of a group G normalized by x. If $(|N|, |x|) = 1$, then $[N, x] = 1$.

Proof.

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Theorem

The kth term of the lower central series of a group G is nilpotent if and only if $|ab| = |a||b|$ for any γ_k-commutators $a, b \in G$ of coprime orders.

Proof. It is clear that if $\gamma_k(G)$ is nilpotent, then $|ab| = |a||b|$ for any γ_k-commutators $a, b \in G$ of coprime orders.

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Sketch of the proof

Soluble case:

- Consider the Fitting height of G, that is the smallest positive integer h such that G possesses a series

$$1 = F_0 \lhd F_1 \lhd \cdots \lhd F_h = G$$

  with $F_{i+1}/F_i = F(G/F_i)$ for $i = 0, \dots, h-1$.

- If $h = 1$, then G is nilpotent and the result is immediate.

- Assume $h = 2$. If $G/F(G)$ has nilpotency class at most $k-1$, then $\gamma_k(G) \leq F(G)$ is nilpotent.
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- Argue by contradiction, and assume that G is a counterexample of minimal order. Therefore $G = G'$.
- Let R be the soluble radical of G, that is the largest normal soluble subgroup of G. Then it follows that G/R is nonabelian simple.
- Prove that R coincides with the center $Z(G)$ of G, so G is quasisimple .
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Thank you for the attention!