Product varieties of groups and varieties generated by wreath products

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See the material related to this talk at: **https://goo.gl/V2SRWB**

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For an outline of the theory of varieties of groups we refer to the classic monography of Hanna Neumann "Varieties of groups", 1967.

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In particular, the product $\mathfrak{A}\mathfrak{A} = \mathfrak{A}^2$ is noting but the variety of metabelian groups, and the variety of soluble groups of length n is the product $\mathfrak{A} \cdots \mathfrak{A} = \mathfrak{A}^n$ of n copies of $\mathfrak{A}$.

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This product operation has a key role in theory of varieties of groups, and as it is remarked by A.L. Shmel'kin: *"The most part of non-trivial results of the general theory of varieties of groups concerns study of products of varieties"* (1965).

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[1] *On varieties of groups generated by wreath products of abelian groups*, Abelian groups, rings and modules (Perth, Australia, 2000), Contemp. Math., 273, Amer. Math. Soc., Providence, RI (2001), 223–238.

[2] *On wreath products of finitely generated abelian groups*, Advances in Group Theory, Proc. Internat. Research Bimester dedicated to the memory of Reinhold Baer, (Napoli, Italy, May-June, 2002), Aracne, Roma, 2003, 13–24.

[3] *Metabelian varieties of groups and wreath products of abelian groups*, J. Algebra, 2007 (313), 2, 455–485.

[4] *Varieties Generated by Wreath Products of Abelian and Nilpotent Groups*, Algebra and Logic, 54 (2015), 1, 70-73.

[5] *The criterion of Shmel'kin and varieties generated by wreath products of finite groups*, Algebra and Logic, 56, 2, (2017), 164–175.

[6] *On K_p-series and varieties generated by wreath products of p-groups*, Int. J. Algebra Comput., accepted for publication, see ArXiv:1505.06293.

[7] *A classification theorem for varieties generated by wreath products of groups*, submitted, see ArXiv:1607.02464.

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I am pleased to mention that one of the early stages of that research (article **[2]** above) was presented at the first *Advances in Group Theory and Applications* Conference:

Research Bimester dedicated to the memory of Reinhold Baer in Napoli, May-June, 2002.

Some photos from that conference can be found in AGTA site, in **Group Theory Archivum** section and also in drive https://goo.gl/V2SRWB that I mentioned.

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If p^u is the highest degree of p dividing n, then the point (b) of Theorem 1 requires that the above decomposition of B contains infinitely many copies of the summand C_{p^u}.

Example 2. By point (a) of Theorem 1 the equality $(*)$ holds for any abelian B, as soon as

$$A = C \quad \text{or, say,} \quad A = C_2 \times C_3 \times C_4 \times \cdots$$

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So $(*)$ does not hold for $B = C_2^{1000} \times C_3^{1000} \times C_5^{1000}$ because 1000 is not "hight enough" power for C_2 or for C_3.

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Example 4. Take A to be a finite nilpotent group of nilpotency class 2, say, the dihedral group D_4 of order 8, or the quaternions group Q_8.

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Example 4. Take A to be a finite nilpotent group of nilpotency class 2, say, the dihedral group D_4 of order 8, or the quaternions group Q_8.

If B contains an element of 2'nd oder, i.e., if it contains the subgroup C_2, then $(*)$ does *not* hold.

The case of wreath products of finite groups

In **[5]** we gave a classification for all cases when $(*)$ holds for *finite* groups A and B:

Theorem 3. *For finite non-trivial groups A and B the equality $(*)$ holds if and only if:*

a) *the exponents of group A and B are coprime;*

b) *A is a nilpotent group, B is an abelian group;*

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Now, let us take $B = C_3^t \times C_5^l$, and find those values t, l for which $(*)$ holds. Since the class of A is 2, we must have $t \geq 2$ and $l \geq 2$.

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3. if p divides m but not n, then is has no impact on feasibility of $(*)$.

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We have $c = 2$, the exponent of A is $4 = 2^2$, the exponent of B is $n = 30 = 2 \cdot 3 \cdot 5$, and $d = 15 = 3 \cdot 5$ with

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So $(*)$ would hold only if B contained the summand

$$C_2^{\infty} \times C_3^2 \times C_5^2.$$

i.e., we have $k = \infty$, $t \geq 2$, $l \geq 2$.

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The methods we developed to study the equality $(*)$ can further be adapted to find subvarieties in some product varieties. Namely, if $A \in \mathfrak{U}$, $B \in \mathfrak{V}$, then $A \,\mathrm{Wr}\, B$ is in the product $\mathfrak{U}\mathfrak{V}$, and so $\mathrm{var}(A \,\mathrm{Wr}\, B)$ is a subvariety in $\mathfrak{U}\mathfrak{V}$.

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Thus, if we classify "a plenty" of pairs of such groups A, B for which we get distincts subvarieties $\operatorname{var}(A \operatorname{Wr} B)$, we will have discovered "a plenty" of subvarieties in $\mathfrak{U}\mathfrak{V}$.

One motivation for this approach is that the information on subvariety structures of $\mathfrak{U}\mathfrak{V}$ is incomplete even when $\mathfrak{U}$ and $\mathfrak{V}$ are such small varieties as the abelian varieties $\mathfrak{A}_m$ and $\mathfrak{A}_n$ respectively.

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R.A. Bryce classified the subvarieties of $\mathfrak{A}_m\mathfrak{A}_n$ where m and n are *nearly prime*, i.e., if a prime p divides m, then p^2 does not divide n (1970).

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As this brief summary shows, one of the cases when the subvariety structure of $\mathfrak{U}\mathfrak{V}$ is less known is the case when $\mathfrak{U}$ and $\mathfrak{V}$ have non-coprime exponents divisible by high powers p^u of many prime numbers p.

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Thus, it may be intersing to to classify the varieties generated by $A \operatorname{Wr} B$ for distinct pairs of groups $A \in \mathfrak{U}$, $B \in \mathfrak{V}$ (especially in cases when exponents of A and B have many common prime divisors).

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Interestingly, the second question has a positive answer for any A_1, A_2, B, and we concentrate on the *first question* $(**)$ which allows us to get infinitely many subvarieties by altering the group B.

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Theorem 9 below will answer the above question for any nilpotent variety $\mathfrak{U}$, abelian variety $\mathfrak{V}$, both of finite exponent m and n respectively such that any prime divisor of n also divides m.

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If the cardinality of factors of order p^u in this decomposion is $\mathfrak{m}_{p^u}$, we write their direct product as $C_{p^u}^{\mathfrak{m}_{p^u}}$.

Then $B(p)$ is a product of some summands of that type:

$$B(p) = C_{p^{u_1}}^{\mathfrak{m}_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{\mathfrak{m}_{p^{u_r}}} \tag{1}$$

where we may suppose $u_1 \geq \cdots \geq u_r$.

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The cardinal numbers $\mathfrak{m}_{p^{u_1}}, \ldots, \mathfrak{m}_{p^{u_r}}$ are invariants of $B(p)$ in the sense that they characterize $B(p)$ uniquely (see L. Fuchs' textbook "Abelian Groups" from where we adopted the symbol $\mathfrak{m}_{p^u}$ and the above notation).

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Otherwise at least one of them will be infinite, and we can choose the *first* one of such infinite invariants $\mathfrak{m}_{p^{u_i}}$.

Example 7. Consider the group:

$$B = C_{3^5}^{6} \times C_{3^3}^{\aleph_0} \times C_{3^2}^{5} \times C_{3}^{\aleph} \times C_{5^3}^{4} \times C_{5^2}.$$

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For the 3-component $B(3)$ of B we have $u_1 = 5$, $\mathfrak{m}_{3^{u_1}} = 6$; $u_2 = 3$, $\mathfrak{m}_{3^{u_2}} = \aleph_0$; $u_3 = 2$, $\mathfrak{m}_{3^{u_3}} = 5$; $u_4 = 1$, $\mathfrak{m}_{3^{u_4}} = \aleph$.

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For the 5-component $B(5)$ we have $u_1 = 3$, $\mathfrak{m}_{5^{u_1}} = 4$; $u_2 = 2$, $\mathfrak{m}_{5^{u_2}} = 1$. So $B(5)$ has *no* infinite factor.

Example 7. Consider the group:

$$B = C_{3^5}^{6} \times C_{3^3}^{\aleph_0} \times C_{3^2}^{5} \times C_{3}^{\aleph} \times C_{5^3}^{4} \times C_{5^2}.$$

For the 3-component $B(3)$ of B we have $u_1 = 5$, $\mathfrak{m}_{3^{u_1}} = 6$; $u_2 = 3$, $\mathfrak{m}_{3^{u_2}} = \aleph_0$; $u_3 = 2$, $\mathfrak{m}_{3^{u_3}} = 5$; $u_4 = 1$, $\mathfrak{m}_{3^{u_4}} = \aleph$.

The *first* infinite factor of $B(3)$ is $C_{3^3}^{\aleph_0}$ although the factor $C_3^{\aleph}$ is of higher cardinality.

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Now suppose two abelian p-groups B_1, B_2 of finite exponent are given, and define a specific equivalence relation $\equiv$ between B_1 and B_2 using their decompositions.

$$B_1 = C_{p^{u_1}}^{\mathfrak{m}_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{\mathfrak{m}_{p^{u_r}}} \tag{2}$$

$$B_2 = C_{p^{v_1}}^{\mathfrak{m}_{p^{v_1}}} \times \cdots \times C_{p^{v_s}}^{\mathfrak{m}_{p^{v_s}}} \; : \tag{3}$$

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i) if B_1, B_2 are *finite*, then $B_1 \equiv B_2$ iff B_1 and B_2 are isomorphic, i.e., $r = s$ and $u_i = v_i$, $\mathfrak{m}_{p^{u_i}} = \mathfrak{m}_{p^{v_i}}$ for $i = 1, \ldots, r$;

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Notice that in point (ii) above we do not require isomorphism of $C_{p^{u_k}}^{\mathfrak{m}_{p^{u_k}}}$ and $C_{p^{v_k}}^{\mathfrak{m}_{p^{v_k}}}$. We just need them both to be direct products of *infinitely* many copies of the *same* cycle $C_{p^{u_k}}$.

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The factors coming after the k'th factor in (2) and (3) have no role for $\equiv$.

Example 8. We will get groups equivalent to the group

$$B = C_{3^5}^6 \times C_{3^3}^{\aleph_0} \times C_{3^2}^5 \times C_3^{\aleph} \ \times \ C_{5^3}^4 \times C_{5^2}.$$

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Theorem 9. *Let A be a non-trivial nilpotent group of exponent m, and let B_1, B_2 be non-trivial abelian groups of exponent n such that any prime divisor of n also divides m.*

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Notice that the primes that divide m but not n have no impact on equality $(**)$.

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In particular, in classification of subvarieties of $\mathfrak{A}_p^2$ L.G. Kovács and M.F. Newman use the fact that those subvarieties either differ in class or in exponent mainly (1971).

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In particular, in classification of subvarieties of $\mathfrak{A}_p^2$ L.G. Kovács and M.F. Newman use the fact that those subvarieties either differ in class or in exponent mainly (1971).

So it would be interesting to get examples of $\mathrm{var}(A \,\mathrm{Wr}\, B_1)$ and $\mathrm{var}(A \,\mathrm{Wr}\, B_2)$ which have the *same class and exponent*, but are distinct.

Example 11. Take the groups $A = C_3,\ B_1 = C_{3^2}^2,\ B_2 = C_{3^2} \times C_3^4$.

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However, $\mathrm{var}(A \,\mathrm{Wr}\, B_1)$ and $\mathrm{var}(A \,\mathrm{Wr}\, B_2)$ are distinct by Theorem 9, since $B_1 \not\equiv B_2$.

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However, by Theorem 9 they generate distinct varieties of groups, since , as $B_1 \not\equiv B_2$. One can also show that $A \,\mathrm{Wr}\, B_2$ does belong to the variety $\mathfrak{N}_4\mathfrak{B}_2$ which does not contain the group $A \,\mathrm{Wr}\, B_1$.

Example 13. Let A be the quaternions group Q_8, and let

$$B_1 = C_{2^5}^2 \times C_{2^4}^{\aleph_0} \times C_2, \qquad B_2 = C_{2^5}^2 \times C_{2^4}^{\aleph} \times C_{2^3}^{\aleph} \times C_{2^2}^{\aleph} \times C_2^{\aleph}.$$

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The second group seems to be much larger, at least, it has higher cardinality.

Nevertheless, $B_1 \equiv B_2$ and by Theorem 9 $A \operatorname{Wr} B_1$ and of $A \operatorname{Wr} B_2$ generate the same variety.

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