## Normalizers of residuals of finite groups <sup>1</sup>

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<sup>1</sup>Joint work with Adolfo Ballester-Bolinches and Sergey Kamornikov 💿 🚊 🗠 🔍







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### Introduction

All our groups will be finite.

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It was motivated by a recent paper of Gong and Isaacs[3] published in Archiv der Mathematik.

### Theorem (Gong and Isaacs, 2017)

If a subgroup M of a group G normalises the nilpotent (respectively soluble) residual of every non-subnormal subgroup of G, then M normalises the nilpotent (respectively soluble) residual of every subgroup of G.

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### Introduction

At the end of the Gong and Isaacs's paper, they left unanswered the question.

#### Question

Whether or not the supersoluble analog of the result is true.

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Our main aim here is to show that surprinsingly the results of Gong and Isaacs are not accidental and can be obtained owing to a general completeness property of all subgroup-closed saturated formations containing all nilpotent groups. Therefore the answer to Gong and Isaacs' particular question is affirmative.

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### Notation:

- A class of groups \$\vec{v}\$ is called a formation if it is closed under taking epimorphic images and subdirect products.
- The 𝔅-residual subgroup G<sup>𝔅</sup> of G, the smallest normal subgroup N with quotient G/N ∈ 𝔅.
- A formation  $\mathfrak{F}$  is *saturated* if it is closed under taking Frattini extensions.
- A formation  $\mathfrak{F}$  is *subgroup-closed* if it is closed under taking subgroups.

Typical examples of formations are abelian, nilpotent, supersoluble and soluble groups. The last three ones are saturated but the first one not. All of ones are subgroup-closed.

- A formation \$\vec{v}\$ is of full characteristic if \$\vec{v}\$ contains every cyclic group of prime order.
- The formations containing all abelian groups are of full characteristic.

#### Theorem (Doerk and Hawkes, 1992)

If  $\mathfrak{F}$  is a saturated formation, then  $\mathfrak{F}$  is of full characteristic if and only if  $\mathfrak{F}$  contains of all nilpotent groups.

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## Introduction

### We prove:

#### Theorem (Main Theorem)

Let  $\mathfrak{F}$  be a subgroup-closed saturated formation of full characteristic. Assume that M is a subgroup of a group G normalising  $H^{\mathfrak{F}}$  for every non-subnormal subgroup H of G. Then M normalises  $H^{\mathfrak{F}}$  for all subgroups H of G.

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## Introduction

In particular, we have:

### Corollary

Let  $\mathfrak{F}$  be a subgroup-closed saturated formation of full characteristic. If  $H^{\mathfrak{F}}$  is normal in G for every non-subnormal subgroup H of G, then  $H^{\mathfrak{F}}$  is normal in G for all subgroups H of G.

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 Main Theorem does not hold for formations without full characteristic, Note that as Gong and Isaacs observed in their paper.

### Example

Let *G* be a finite *p*-group which has a non-normal subgroup *H*, *p* a prime, and suppose that  $\mathfrak{F}$  is a class of all *q*-groups, where *q* is a prime and  $q \neq p$ . Observed that  $\mathfrak{F}$  is a subgroup-closed saturated formation but not of full characteristic and *G* normalising  $X^{\mathfrak{F}}$  for every non-subnormal subgroup *X* of *G*. But *G* does not normalise  $H^{\mathfrak{F}} = H$ .

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 Main Theorem does not hold for the formation of all abelian groups as Gong, Zhao and Guo shows in [4]. Actually there exists many such example. and here we give another simple example.

#### Example

Let  $G = (\langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \times \langle a_4 \rangle) \rtimes \langle b \rangle \cong Z_2 \wr Z_4$ , where  $a_i^b = a_{(i+1)mod 4}$ , and  $H = (\langle a_1 \rangle \times \langle a_3 \rangle) \rtimes \langle b^2 \rangle$ . Note that every subgroup of *G* is subnormal. Observed that  $H' = \langle a_1 a_3 \rangle \not \supseteq G$ .

 Thus the hypothesis "ℑ is saturated" in Main Theorem is necessary..

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In the following, I will give you an sketch of the proof. It is based on an detailed analysis of a minimal counterexample using lot of results from formation theory and the structure of critical groups associated to formations.

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 Let X be a class of groups, a group G is called X-critical if G is not in X but all proper subgroups of G are in X.

A description of the soluble  $\mathfrak{X}$ -critical groups for a subgroup saturated formation  $\mathfrak{X}$  can be found in

### Theorem [2, Theorem VII.6.18]

Let  $\mathfrak{F}$  be a saturated formation and let *G* be a solvable group such that *G* does not belong to  $\mathfrak{F}$  but all its maximal subgroups belong to  $\mathfrak{F}$ . Then  $G^{\mathfrak{F}}$  is a *p*-group for some prime *p* and  $G^{\mathfrak{F}}/\Phi(G^{\mathfrak{F}})$  is an  $\mathfrak{F}$ -eccentric chief factor of *G*.

#### Theorem [1, Theorem 1]

Let  $\mathfrak{F}$  be a saturated formation and let G be a group such that G does not belong to  $\mathfrak{F}$  but all its proper subgroups belong to  $\mathfrak{F}$ . If  $G^{\mathfrak{F}}$  is soluble, then  $G^{\mathfrak{F}}/(G^{\mathfrak{F}})'$  is an  $\mathfrak{F}$ -eccentric chief factor of G.

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Let  $\mathfrak{F}$  be a subgroup-closed saturated formation and *G* be a group. An ordered sequence of groups (G, S, M) is called a *normalising* 3-*tuple*, where

- S is a subnormal subgroup of G.
- *M* is a subgroup of *G* normalising the  $\mathfrak{F}$ -residual of every non-subnormal subgroup of *G*.

We denote by  $\mathfrak{H}$  the class of all normalising 3-tuples associated to  $\mathfrak{F}.$ 

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Assume that the theorem is false. Consider  $(G, S, M) \in \mathfrak{H}$  with |G| + |S| + |G : M| minimal such that M does not normalise  $S^{\mathfrak{F}}$ . We have the following Statements:

- Let K be the intersection of N<sub>G</sub>(H<sup>𝔅</sup>) for all non-subnormal subgroups H of G.
- M = K and G = SK.
- S is contained in exactly one maximal normal subgroup of G, H say, and H normalises S<sup>3</sup>.
- Let  $T = \langle S^{\mathfrak{F}} \rangle^{G}$ .  $\Phi(T) = Core_{G}(S^{\mathfrak{F}}) = 1$  and  $Soc(G) \subseteq T$ .
- *T* is a elementary abelian *p*-group for some prime *p*.
- S is an *ξ*-critical group and S<sup>*ξ*</sup> is a minimal normal subgroup of S.

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The key Statement:

• (\*) Let X, Y be subgroups of G such that  $S \le X \lhd Y \le H$ . Suppose that  $S^{\mathfrak{F}} = X^{\mathfrak{F}}$  and  $Y/X \in \mathfrak{F}$ . Then  $Y^{\mathfrak{F}} = S^{\mathfrak{F}}$ .

Then we will get the finial contradiction.

Consider a piece of a composition series of H passing through S:

$$S = J_0 \lhd J_1 \lhd ... \lhd J_k = H$$

• Assume  $J_i/J_{i-1}$  is cyclic for i = 0, 1, ..., k.

By (\*),  $S^{\mathfrak{F}} = J_1^{\mathfrak{F}} = ... = J_k^{\mathfrak{F}} = H^{\mathfrak{F}} \trianglelefteq G$ , contrary to assumption.

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 Assume that J<sub>t</sub>/J<sub>t-1</sub> is a non-abelian simple group for some t.

Choose such *t* as small as possible, that is,  $J_i/J_{i-1}$  is cyclic for all  $0 \le i \le t - 1$ . Let  $J_t^*/J_{t-1}$  be a cyclic subgroup of  $J_t/J_{t-1}$ . Then  $J_t^*$  is a proper subgroup of  $J_t$  which is not subnormal in *G* and  $(J_t^*)^{\mathfrak{F}}$  is normalized by *K*.

Then we apply Statement  $(\star)$  to the series

$$S = J_0 \lhd ... \lhd J_{t-1} \lhd J_t^* \leq H$$

and conclude that  $S^{\mathfrak{F}} = J_0^{\mathfrak{F}} = ... = (J_t^*)^{\mathfrak{F}}$  is normalized by *K*. This is a contradiction and the proof is complete.

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