BOVDI UNITS AND FREE PRODUCTS IN INTEGRAL GROUP RINGS OF FINITE GROUPS

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5th of September 2017
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OVERVIEW

1. Motivation

2. Free product of cyclic groups
   1. In matrices
   2. In integral group rings

3. Generalizations

FREE PRODUCT OF CYCLIC GROUPS

$$.^* : \mathbb{Z}G \to \mathbb{Z}G : \sum_{g \in G} a_g g \mapsto \sum_{g \in G} a_g g^{-1}$$

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Theorem (Gonçalves - Passman)

For a finite group G, the unit group $\mathcal{U}(\mathbb{Z}G)$ contains a subgroup isomorphic to $C_p \star C_\infty$ for some prime p if and only if G contains a non-central element of order p.

IN MATRICES

Theorem

Let $n, m \in \mathbb{N}_0$ and denote ζ_n (resp. ζ_m) to be a complex, primitive n-th (m-th) root of unity. Let z_1 and z_2 be complex numbers, with $|z_1 z_2|$ sufficiently large in comparison to ζ_n and ζ_m. Then

$$\begin{bmatrix} 1 & z_1 \\ 0 & \zeta_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ z_2 & \zeta_m \end{bmatrix}$$

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This implies Sanov's theorem.

IN INTEGRAL GROUP RINGS

G a finite group, $g, h \in G$. The elements

$$b_{g,\widetilde{h}} := 1 + (1 - h)g\widetilde{h} \quad \text{and} \quad b_{\widetilde{h},g} := 1 + \widetilde{h}g(1 - h),$$

are called bicyclic units of $\mathbb{Z}G$.

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For intuition, think of these as

$$\begin{bmatrix} 1 & z \\ 0 & \zeta_{o(h)} \end{bmatrix}.$$

IN INTEGRAL GROUP RINGS

Theorem

Let G be a finite, nilpotent group of class 2 and let $g, h \in G$. Assume $o(h) = n \geq 2$. Write $K = C_k \cong \langle h \rangle \cap \langle h \rangle^g$. Then

$$\langle b_{g,\widetilde{h}}(h), b_{g,\widetilde{h}}(h)^* \rangle \cong C_n \star_K C_n.$$

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This is a constructive proof of the result of Gonçalves and Passman in the nilpotent case where the prime p is odd.

BOVDI UNITS AS MORPHISMS

Let $H \leq G$, $\alpha \in \mathbb{Z}G$.

$$b_{\alpha,\widetilde{H}} : H \to \mathcal{U}(\mathbb{Z}G) : h \mapsto h + (1-h)\alpha\widetilde{H},$$

and

$$b_{\widetilde{H},\alpha} : H \to \mathcal{U}(\mathbb{Z}G) : h \mapsto h + \widetilde{H}\alpha(1-h),$$

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Does $H \star H$ (or $H \star C_\infty$) exist in $\mathcal{U}(\mathbb{Z}G)$ and can we construct it explicitly?

BOVDI UNITS BASED ON NON-TRIVIAL UNITS

Let $\beta = (1 + h + \ldots + h^{k-1})^m + \frac{1-k^m}{o(h)}\widetilde{h} \in \mathbb{Z}G$ be a Bass unit based on $h \in H \leq G$, $\alpha \in \mathbb{Z}G$.

$$\beta + (1 - h)\alpha\widetilde{H} \text{ is again a unit of } \mathbb{Z}G.$$

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Theorem

Let $h \in H \leq G$, $\alpha \in \mathbb{Z}G$ and β_1, β_2 both Bass units based on h. If $(1 - h)\alpha\widetilde{H} \neq 0$ and β_1 and β_2 are "sufficiently different", then the set

$$\{\beta_1 + (1 - h)\alpha\widetilde{H}, \beta_2 + (1 - h)\alpha\widetilde{H}\}$$

generates a free monoid, which is contained in a solvable group.

Thank you for
your attention!