Character Degrees, Class Sizes and Graphs

Silvio Dolfi

Dipartimento di Matematica e Informatica
Università di Firenze

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General notation

Given a finite group G, we denote by $\mathrm{Irr}(G)$ the set of *irreducible complex characters* of G, and by

$$\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(G)\}$$

the *set* of their degrees.

Questions

(a) *What information is encoded in* $\mathrm{cd}(G)$ *?*

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Prime divisors of character degrees

There are connections between the 'arithmetical structure' of $\mathrm{cd}(G)$ and the group structure of G. Two important instances:

Theorem (Ito 1951; Michler 1986)

Let p be prime number.
p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G) \Leftrightarrow$ if G has a normal abelian Sylow p-subgroup.

Theorem (Thompson; 1970)

Let G be a group and p a prime. If every element in $\mathrm{cd}(G) \setminus \{1\}$ is divisible by p, then G has a normal p-complement.

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Theorem (Isaacs, Passman; 1968)

$\mathrm{cd}(G) = \{1, p\}$, *$p$ prime, if and only if*

(a) $\exists A \lhd G$, *A abelian,* $[G : A] = p$; *or*

(b) $[G : \mathbf{Z}(G)] = p^3$

Theorem (Isaacs, Passman; 1968)

If $\mathrm{cd}(G) = \{1, p_1, p_2, \ldots, p_n\}$, *$p_i$ primes, then* $n \leq 2$ *and* $G''' = 1$.

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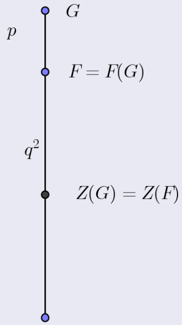

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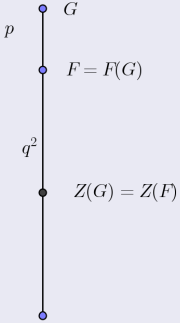

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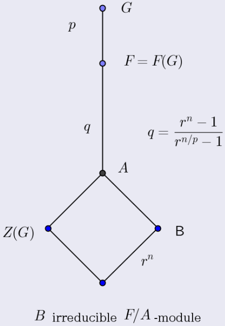

Prime power degrees

Theorem (Manz; 1985)

Assume $\mathrm{cd}(G) = \{1, p_1^{a_1}, p_2^{a_2}, \ldots, p_t^{a_t}\}$, *$p_i$ primes, $a_i > 0$. Let $k = |\{p_i | 1 \leq i \leq t\}|$ (the number of distinct primes).*

(1) *G is solvable if and only if $k \leq 2$ (in this case: $2 \leq dl(G) \leq 5$);*

(2) *G non-solvable if and only if $G \cong S \times A$ with $S \cong PSL(2,4)$ or $PSL(2,8)$ and A is abelian.*

$$\mathrm{cd}(3^2 : \mathrm{GL}(2,3)) = \{1, 2, 3, 4, 8, 16\}$$

$$\mathrm{cd}(\mathrm{PSL}(2,4)) = \{1, 2^2, 3, 5\}$$

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The prime graph

Let X be a non-empty set, $X \subseteq \mathbb{N}$.
For $n \in X$, let $\pi(n)$ be the set of primes dividing n.

$\Delta(X)$ prime graph

- *vertex set:* $\mathrm{V}(\Delta(X)) = \bigcup_{n \in X} \pi(n)$
- *edge set* $\mathrm{E}(\Delta(X)) = \{\{p, q\} : pq \text{ divides some } n \in X\}$

So, the prime $\Delta(X)$ graph on X is the simple undirected graph whose vertices are the primes that divide some number in X, and two (distinct) vertices p, q are adjacent if and only if there exists $x \in X$ such that $pq \mid x$.

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Prime graphs have been considered for the following sets $X(G)$ of invariants:

- $\mathrm{o}(G) = \{o(g) \ : \ g \in G\}$.
- $\mathrm{cd}(G) = \{\chi(1) \ : \ \chi \in \mathrm{Irr}(()G)\}$.
- $\mathrm{cs}(G) = \{|g^G| \ : \ g \in G\}$.

Questions

(I) *Properties of the graphs* $\Delta(X(G))$.

(II) *To what extent the group structure of* G *is reflected on and influenced by the structure of the graph* $\Delta(X(G))$ *?*

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Degree graph and Class graph

Notation:

Character graph $\Delta(G) := \Delta(\mathrm{cd}(G))$

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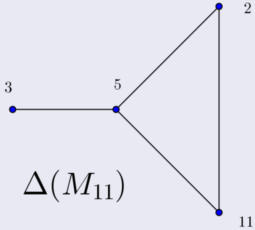

$\Delta(M_{11})$

$$\mathrm{cd}(M_{11}) = \{1, 10, 11, 16, 44, 45, 55\}$$

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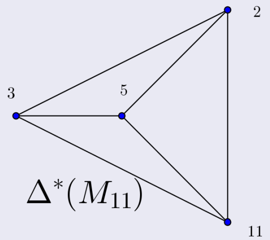
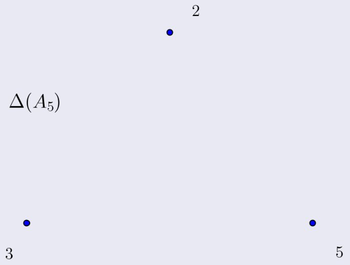


$$\mathrm{cs}(M_{11}) = \{1, 165, 440, 720, 990, 1320, 1584\}$$

$\Delta(A_5)$

Example



$\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$

$\Delta^*(A_5)$

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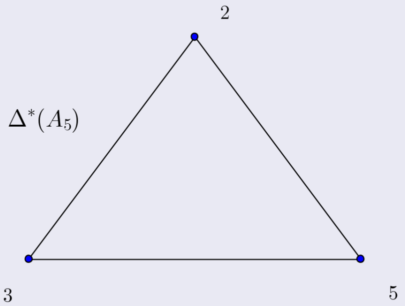

$\mathrm{cs}(A_5) = \{1, 12, 15, 20\}$

Example: $G = \mathrm{PSL}_2(19^4)$

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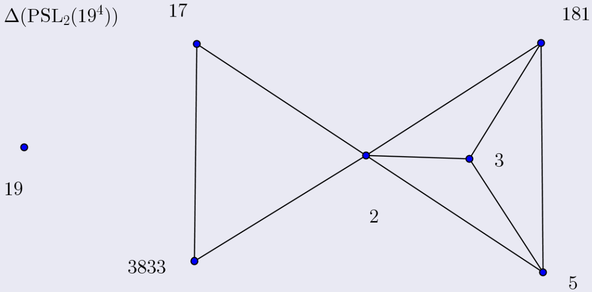

$$\mathrm{cd}(\mathrm{PSL}_2(19^4)) = \{17 \cdot 3833, 2 \cdot 17 \cdot 3833, 2^4 \cdot 3^2 \cdot 5 \cdot 181\}$$

Basic properties

- If $N \triangleleft G$, then both $\Delta(N)$ and $\Delta(G/N)$ are subgraphs of $\Delta(G)$.
- $\Delta(G \times H)$ is the *join* $\Delta(G) * \Delta(H)$.

Vertex set of the character graph $\Delta(G)$:

Theorem (Ito-Michler)

p prime, $P \in \mathrm{Syl}_p(G)$:
$p \notin \mathrm{V}(\Delta(G)) \Leftrightarrow P$ abelian and $P \triangleleft G$

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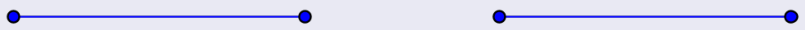
What graph can occur as $\Delta(G)$?

Question

Can this graph be a character graph $\Delta(G)$ for some G?

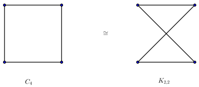


$\Delta(G) \cong C_4$

Theorem (Lewis, Meng; 2012/Lewis, White; 2013)

If $\Delta(G) \cong C_4$, then $G = A \times B$ with $\Delta(A) \cong \Delta(B) \cong \overline{K_2}$ (in particular, G is solvable).

Note: the square C_4 is isomorphic to the complete bipartite graph $K_{2,2}$



Tong-Viet (2013) has classified the groups G such that $\Delta(G)$ contains no subgraph K_3 (no "triangles"). As a consequence:

Theorem (Tong-Viet; 2013)

If $K_{n,m} \cong \Delta(G)$ for some group G, then $n + m \leq 5$.
Precisely, the only instance are: $K_{1,1}$; $K_{1,2}$; $K_{1,3}$ $K_{2,2}$ $K_{2,3}$

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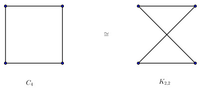

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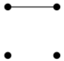
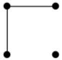
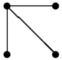
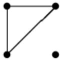
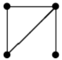
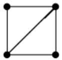
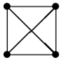
Four vertices, G non-solvable

The possible graphs $\Delta(G)$ on four vertices, for G nonsolvable, are: (Lewis, White; 2013)

2 M.L. Lewis, D.L. White / Journal of Algebra 378 (2013) 1–11

Table 1
Four-vertex graphs occurring as $\Delta(G)$ for nonsolvable G.

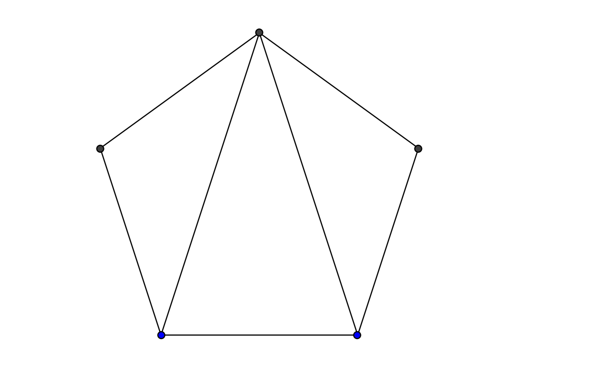
Graph $\Delta(G)$	Group G	cd(G)
	$\mathrm{PSL}_2(16)$	$\{1, 3\cdot 5, 2^4, 17\}$
	$\mathrm{PSL}_2(25)$	$\{1, 13, 2^3\cdot 3, 5^2, 2\cdot 13\}$
	$A_5 \times p^{1+2}$	$\{1, 3, 2^2, 5, p, 3p, 2^2p, 5p\}$
	$\mathrm{PSL}_2(31)$	$\{1, 3\cdot 5, 2\cdot 3\cdot 5, 31, 2^5\}$
	M_{11}	$\{1, 2\cdot 5, 11, 2^4, 2^2\cdot 11, 3^2\cdot 5, 5\cdot 11\}$
	A_8	$\{1, 7, 2\cdot 7, 2^2\cdot 5, 3\cdot 7, 2^2\cdot 7, 5\cdot 7, 3^2\cdot 5, 2^3\cdot 7, 2^6, 2\cdot 5\cdot 7\}$
	A_7	$\{1, 2\cdot 3, 2\cdot 5, 2\cdot 7, 3\cdot 5, 3\cdot 7, 5\cdot 7\}$

An unknown graph

It is still open, for the following graph $K_5 - e$, the question whether it is the character degree graph of any group:

Problem

Does there exist G such that $\Delta(G)$ is the following:



$\Delta(G)$: G solvable; non-adjacent vertics

Theorem (J. Zhang; 1996)

Assume G solvable. If $p, q \in \mathrm{V}(\Delta(G))$ are not adjacent in $\Delta(G)$ then $l_p(G) \leq 2$ and $l_q(G) \leq 2$.
If $l_p(G) + l_q(G) = 4$, then G has a normal section isomorphic to $(C_3 \times C_3) \rtimes \mathrm{GL}(2,3)$.

Number of Connected Components

Theorem (Manz, Staszewski, Willems; 1988)

- $\mathrm{n}(\Delta(G)) \leq 3$
- $\mathrm{n}(\Delta(G)) \leq 2$ *if* G *is solvable*

The groups G with disconnected graph $\Delta(G)$ have been classified

- *G solvable: (Zhang; 2000/Palfy; 2001/Lewis; 2001).*
- *any G: (Lewis, White; 2003).*

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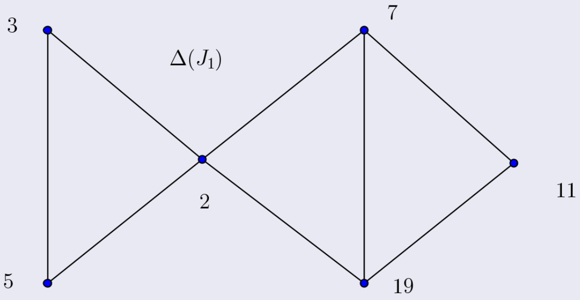
Example

A non-solvable example: J_1



$\mathrm{cd}(J_1) = \{1, 56, 76, 77, 120, 133, 209\}$

Palfy's Three Vertex Theorem

We denote by $\mathrm{V}(G)$ the vertex set of $\Delta(G)$.

Theorem (Palfy; 1998)

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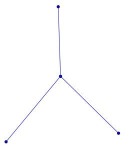

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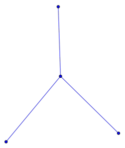

But the above graph, $K_{1,3}$ is the character graph of $A_5 \times 7^{1+2}$, for instance.

Independence number

The independence number $\alpha(\Delta)$ of a graph Δ is the largest cardinality of an set of pairwise non-adjacent vertices (independent set).

Theorem

- *(Palfy; 1998) For G solvable, $\alpha(\Delta(G)) \leq 2$.*
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Corollary

- *If G is solvable, then $\Delta(G)$ has at most two connected components.*
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By Lewis-White(2013), hence $\Delta(G) \ncong P_2 + P_2$ for every finte group G.

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If G is solvable and $|\mathrm{V}(G)| \leq 5$, then $\mathrm{diam}(\Delta)(G) \leq 2$.

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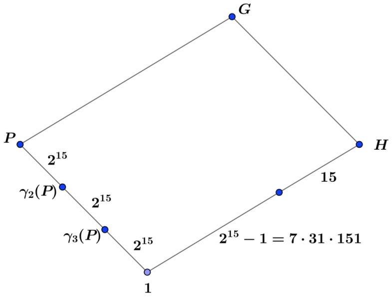

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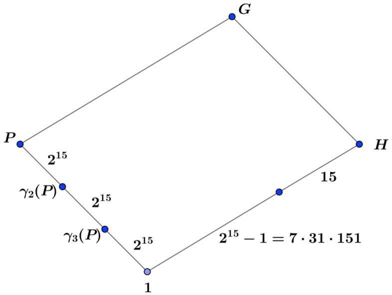

$\Delta(G/P)$

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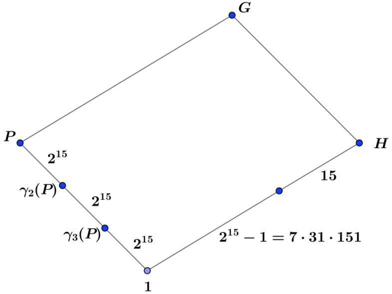

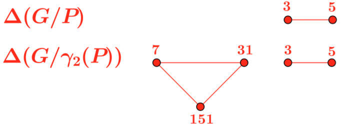

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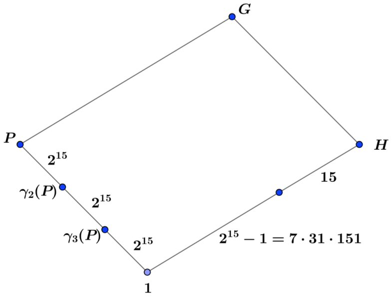

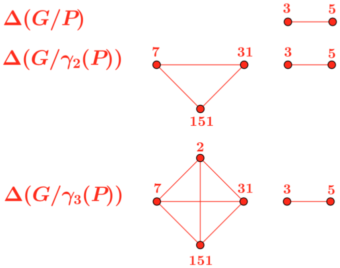

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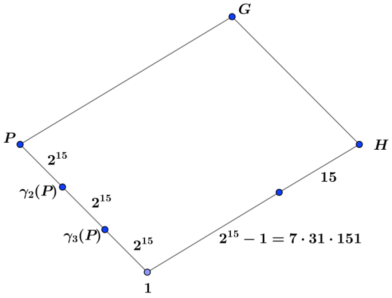

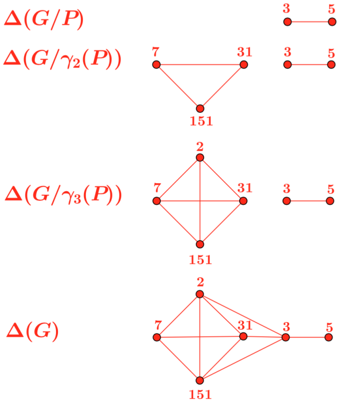

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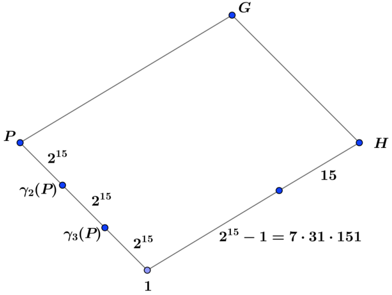
$$cd(G) = \{1,\ 3,\ 5,\ 3 \cdot 5,\ 7 \cdot 31 \cdot 151,\ 2^{12} \cdot 31 \cdot 151,$$
$$2^a \cdot 7 \cdot 31 \cdot 151 \ \ (a \in 7, 12, 13),$$
$$2^b \cdot 3 \cdot 31 \cdot 151 \ \ (b \in 12, 15)\}$$



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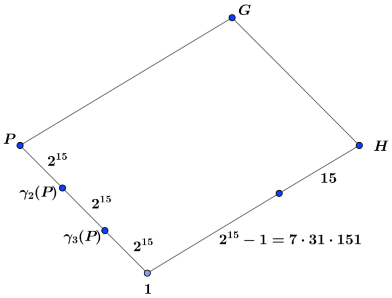
Some questions



$$|G| = 2^{45}\cdot(2^{15}-1)\cdot 15$$

(a) Is this example "minimal"?

(b) Let G be a solvable group such that $\Delta(G)$ is connected with diameter three. What can we say about the structure of G? For instance, what about $\mathrm{h}(G)$?

Some questions by Lewis

(c) For G as above, is it true that there exists a normal subgroup N of G with

$$\mathrm{Vert}(\Delta(G/N)) = \mathrm{Vert}(\Delta(G))$$

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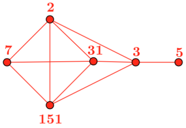
$\Delta(G)$



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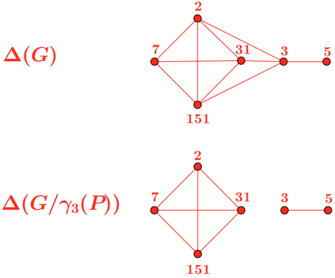

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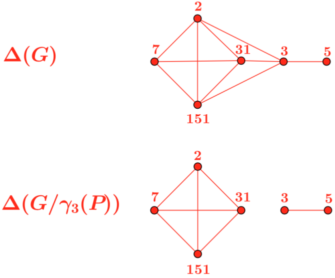
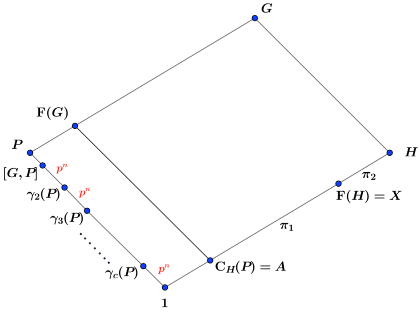
and with $\Delta(G/N)$ disconnected?
Can $\mathrm{Vert}(\Delta(G))$ be partitioned into two subsets π_1 and π_2, both inducing complete subgraphs of $\Delta(G)$, such that $|\pi_1| \geq 2^{|\pi_2|}$?

(In Lewis' example:)



If $\Delta(G)$ is connected with diameter three then... [Casolo, D., Pacifici, Sanus (2016) ; Sass (2016)]



(a) There exists a prime p such that $G = PH$, with P a normal nonabelian Sylow p-subgroup of G and H a p-complement.

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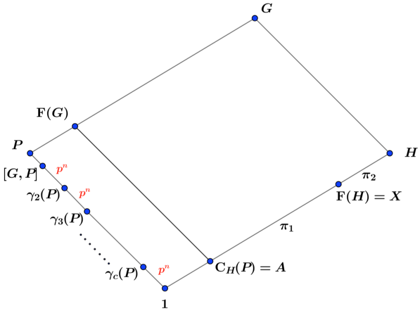

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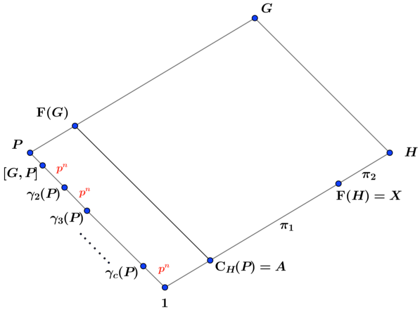

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(c)
$$h(G) = 3$$

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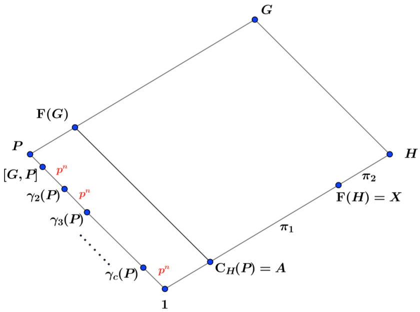

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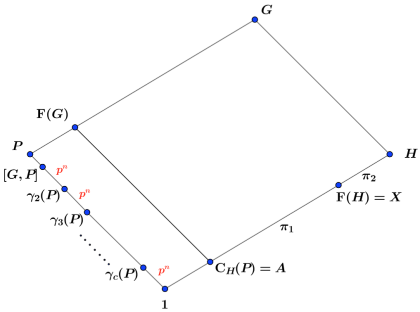

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$$\Gamma(p^n) = \{x \mapsto ax^\sigma \mid a, x \in \mathbb{K}, a \neq 0, \sigma \in \mathrm{Gal}(\mathbb{K})\} \quad \text{with} \quad \mathbb{K} = \mathrm{GF}(p^n)$$

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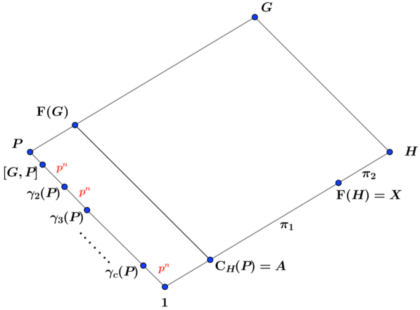
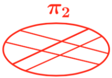


$\Delta(G/P)$



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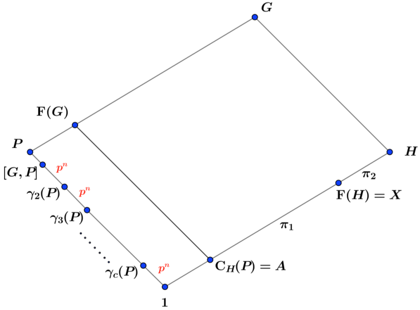

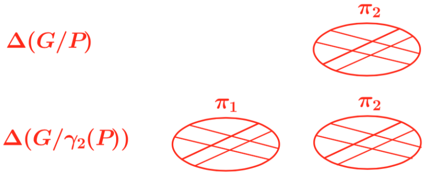

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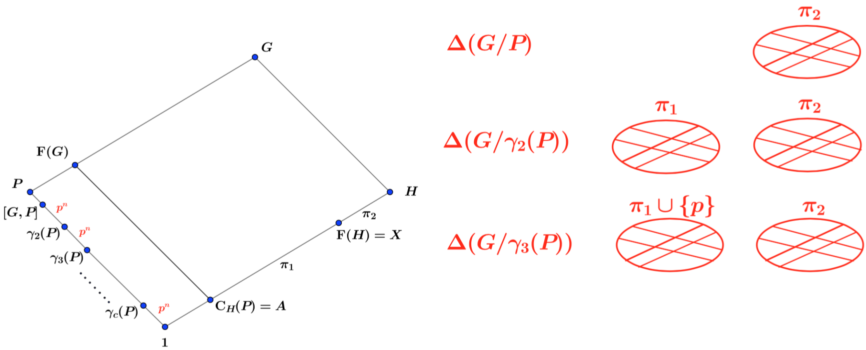

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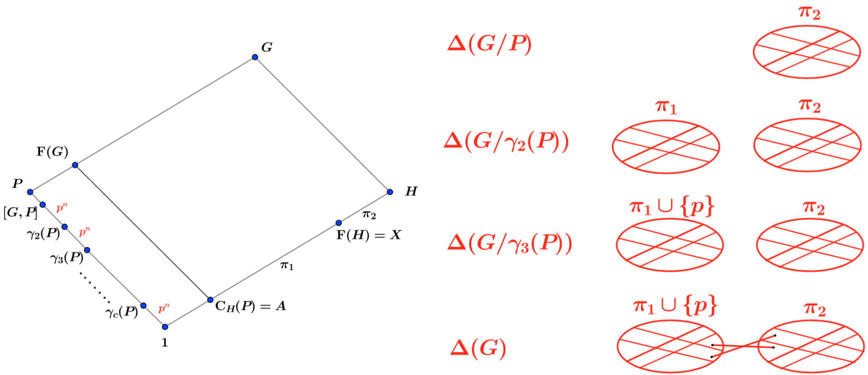

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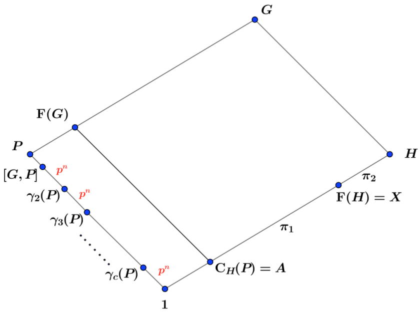

Finally, setting $d = |H/X|$, we have that $|X/A|$ is divisible by $(p^n - 1)/(p^{n/d} - 1)$. Since c must be at least 3, we get

$$|G| \geq p^{3n} \cdot \frac{p^n - 1}{p^{n/d} - 1} \cdot d \geq$$

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If $|\pi| \geq 3$, then by Palfy's "Three Vertex Theorem" the subgraph $\Delta(G)[\pi]$ induced by π in $\Delta(G)$ contains at least one edge.

Also, if $|\pi| \geq 6$, by elementary Ramsey Theory $\Delta(G)[\pi]$ contains at least a K_3.

Question

Does $|\pi| = 5$ imply $K_3 \leq \Delta(G)$?

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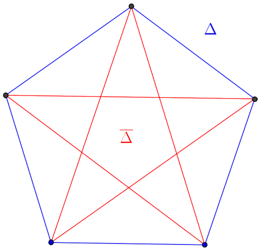
The complement graph

Let Δ be a graph. The *complement* of Δ is the graph $\overline{\Delta}$ whose vertices are those of Δ, and two vertices are adjacent in $\overline{\Delta}$ if and only if they are non-adjacent in Δ.



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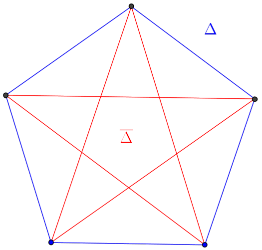

The question on the previous slide is equivalent to the following:

Question

Can C_5 be a subgraph of $\overline{\Delta(G)}$, for G solvable ?

For G solvable, $\overline{\Delta(G)}$ is bipartite!

Pálfy's "Three Vertex Theorem" can be rephrased as follows:

Theorem (Pálfy; 1998)

Let G be a solvable group. Then the graph $\overline{\Delta(G)}$ does not contain any cycle of length 3.

Theorem (Akhlaghi, Casolo, D., Khedri, Pacifici; 2018?)

Let G be a solvable group. Then the graph $\overline{\Delta(G)}$ does not contain any cycle of odd length.

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Some consequences

But the graphs containing no cycles of odd length are precisely the bipartite graphs. Therefore the previous theorem asserts that, for any solvable group G, the graph $\overline{\Delta(G)}$ is bipartite. As an immediate consequence:

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Let G be a solvable group. Then the set $\mathrm{V}(G)$ of the vertices of $\Delta(G)$ is covered by two subsets, each inducing a complete subgraph in $\Delta(G)$. In particular, for every subset $\mathcal{S}$ of $\mathrm{V}(G)$, at least half the vertices in $\mathcal{S}$ are pairwise adjacent in $\Delta(G)$.

Hence: for G solvable, $\pi \subseteq \mathrm{V}(\Delta(G))$;

- $|\pi| \geq 7$ implies $K_4 \leq \Delta(G)[\pi]$;
- $|\pi| \geq 9$ implies $K_5 \leq \Delta(G)[\pi]$;...

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Huppert's $\rho - \sigma$ conjecture

Let

$$\rho(G) = \bigcup_{n \in \mathrm{cd}(G)} \pi(n)$$

and

$$\sigma(G) = \max\{|\pi(n)| \;:\; n \in \mathrm{cd}(G)\}$$

Conjecture ($\rho - \sigma$ conjecture)

- *If G is solvable, then $|\rho(G)| \leq 2\sigma(G)$*
- *In general, $|\rho(G)| \leq 3\sigma(G)$*

The conjecture has been verified for simple groups (Alvis-Barry; 1991), groups G with $\sigma(G) = 1$ (Manz; 1985), $\sigma(G) = 2$ (Gluck; 1991) and with square-free character degrees (Gluck; 1991).

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$\mathrm{SL}(2,3)$ *and A_5 show that these bounds would be best-possible*

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Let G be a group and $\pi \subseteq \mathrm{V}(\Delta(G))$. If $|\pi| \geq 4$, then at least two vertices of π are adjacent in $\Delta(G)$ (i.e. $\alpha(\Delta(G)) \leq 3$).

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Classify the groups G with $\alpha(\Delta(G)) = 3$.

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Let $\pi \subseteq \mathrm{V}(G)$ with $|\pi| = 3$. The subgraph of $\Delta(G)$ induced by π is empty if and only if $\mathbf{O}^{\pi'}(G) = S \times A$, where A is abelian and $S \cong \mathrm{SL}(p^a)$ or $S \cong \mathrm{PSL}(p^a)$, p is a prime, a is a positive integer, and $\pi = \{p, q, r\}$, $q, r \neq 2$, q divides $p^a + 1$ and r divides $p^a - 1$.

Observe that:

- The π-parts of the character degrees of G and $S \triangleleft G$ are the same
- p^a is neither 4 nor a Mersenne nor a Fermat prime.

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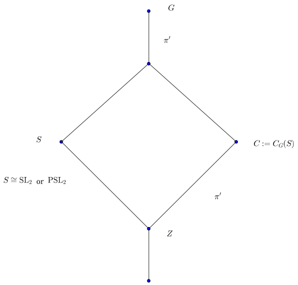

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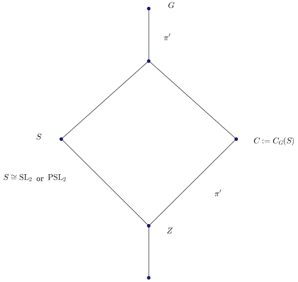

Applications

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For every G, $\alpha(\Delta(G)) \leq 3$.

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G non-solvable; G has only prime power degrees if and only if $G \cong S \times A$ with $S \cong PSL(2,4)$ or $PSL(2,8)$ and A is abelian.

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$\Delta(G)$ has three connected components if and only if $G \cong S \times A$ with $S \cong PSL(2, 2^a)$ and A is abelian.

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Tools: orbit results

Let V be a faithful G-module and let q be a prime divisor of $|G|$.
We say that:

- (G,V) satisfies $\mathcal{N}_q$ if for every non-trivial $v \in V$ there exists a $Q \in \mathrm{Syl}_q(G)$ such that $Q \lhd \mathbf{C}_G(v)$.
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Examples:

(a) Let $V = V(2,3)$; then $(\mathrm{SL}_2(3), V)$ satisfies $\mathcal{C}_q$ and $(\mathrm{GL}_2(3), V)$ satisfies $\mathcal{N}_q$.

(b) Let $|V| = r^n$, r prime, $q \mid n$ and $q \nmid r^n - 1$. Let $\Gamma(V)$ be the semilinear group on V, i.e.

$$\Gamma(V) = \Gamma(r^n) = \{x \mapsto ax^\sigma \mid a, x \in \mathbb{K}, a \neq 0, \sigma \in \mathrm{Gal}(\mathbb{K})\}$$

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Why

The condition $\mathcal{N}_q$ arises naturally. Consider the following situation: $(*)$: $G = V \rtimes H$, p, q non-adjacent vertices of $\Delta(G)$, V minimal normal in G, $\mathbf{C}_H(V) = 1$, and $P \in \mathrm{Syl}_p(H)$, $P \lhd H$.

So, for every $1 \neq \lambda \in \mathrm{Irr}(V)$, p divides $[G : \mathbf{C}_G(\lambda)]$. Since λ extends to $\mathbf{C}_G(\lambda)$, by Gallagher's theorem we have

$$\{\chi(1) : \chi \in \mathrm{Irr}(G|\lambda)\} = \{\beta(1)[G : \mathbf{C}_G(\lambda)] : \beta \in \mathrm{Irr}(\mathbf{C}_G(\lambda)/V)\}.$$

Since p and q are non-adjacent in $\Delta(G)$, it follows that $\mathbf{C}_H(\lambda) = \mathbf{C}_G(\lambda)/V$ contains a Sylow q-subgroup Q of H and that Q is abelian and normal in $\mathbf{C}_H(\lambda)$.

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Solvable groups

Theorem (Zhang; Wolf;1998)

If H is solvable and (H, V) satisfies $\mathcal{N}_q$, then either

(a) $H \leq \Gamma(V)$; *or*

(b) $q = 3$, $|V| = 3^2$ *and* $H \cong \mathrm{SL}_2(3)$ *or* $\mathrm{GL}_2(3)$.

Non-solvable groups: Casolo's Theorem

For non-solvable groups we have this beautiful result (btw: its proof doesn't use the classification!).

Theorem (Casolo; 2010)

If (H, V) satisfies $\mathcal{C}_q$ and $q \neq char(V)$, then $H \leq \Gamma(V)$.

In order to use it in all reduction cases, we had to extend it slightly:

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Disconnected Conjugacy Class Graphs

Theorem (Kazarin; 1981/ Bertram, Herzog, Mann; 1991)

$\mathrm{n}(\Delta^*(G)) \leq 2$

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$\mathrm{cs}(G) = \{1, |\bar{K}|, |\bar{H}|\}$

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Non-adjacent vertices in $\Delta^*(G)$

Theorem (Ito; 1956/ D.; 1998/ Casolo, D.; 2009)

If p, q are non-adjacent vertices of $\Delta^*(G)$, then G is $\{p,q\}$-solvable $l_{\{p,q\}}(G) = 1$ and G has abelian Sylow p-subgroups and q-subgroups.

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Example

$$\mathrm{cs}(C_{35} \rtimes C_6) = \{1, 2, 6, 7, 14, 35\}$$

Independence number for $\Delta^*(G)$

Independence number α = largest size of a set of mutually non-adjacent vertices.

Also: $\alpha(\Delta) \leq 2$ implies $\mathrm{diam}(\Delta) \leq 3$, $\mathrm{n}(\Delta) \leq 2$ and that disconnected graphs have complete components.

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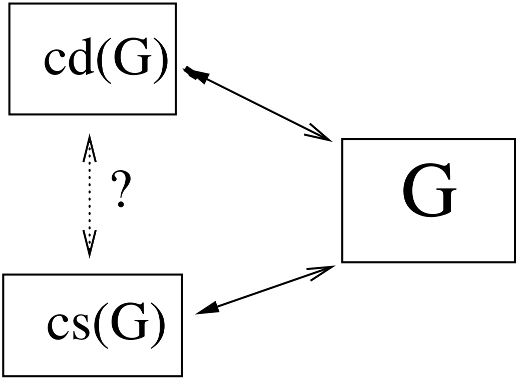
If $\mathbf{F}(G) = 1$, *then* $\Delta^*(G)$ *is complete.*

A direct connection ?



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$$\mathrm{cd}(C_{p^n} \wr C_p) = \{1, p\} \qquad \mathrm{cs}(C_{p^n} \wr C_p) = \{1, p, p^{n(p-1)}\}$$

Theorem (Isaacs, Keller, Meierfrankenfeld, Moreto; 2006)

Let p be a prime and $\chi \in \mathrm{Irr}(G)$. If χ is primitive, then there exists a $x \in G$ such that $\chi(1)_p$ divides $(|x^G|_p)^3$.

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Does there exist, for every primitive $\chi \in \mathrm{Irr}(G)$, an element $x \in G$ such that $\chi(1)$ divides $|x^G|$?

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If p and q are distinct primes such that pq divides some $n \in \mathrm{cd}(G)$, then there exists some $m \in \mathrm{cs}(G)$ such that pq divides m.
Equivalently: $\Delta(G)$ is a subgraph of $\Delta^*(G)$.

Problem

Let $p_1, p_2, \ldots, p_n$ be distinct primes such that $p_1 \cdot p_2 \cdots p_n$ divides some $n \in \mathrm{cd}(G)$. Does there always exist some $m \in \mathrm{cs}(G)$ such that $p_1 \cdot p_2 \cdots p_n$ divides m ?

$\Delta(G)$ and $\Delta^*(G)$

Theorem (Casolo, D.; 2009)

If p and q are distinct primes such that pq divides some $n \in \mathrm{cd}(G)$, then there exists some $m \in \mathrm{cs}(G)$ such that pq divides m.
Equivalently: $\Delta(G)$ is a subgraph of $\Delta^*(G)$.

Problem

Let $p_1, p_2, \ldots, p_n$ be distinct primes such that $p_1 \cdot p_2 \cdots p_n$ divides some $n \in \mathrm{cd}(G)$. Does there always exist some $m \in \mathrm{cs}(G)$ such that $p_1 \cdot p_2 \cdots p_n$ divides m ?