Finite multipermutation solutions of the Yang–Baxter equation

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Solutions of the YBE

Definition

Let X be a non-empty set. A set-theoretic solution of the Yang-Baxter equation on X is a bijective map $r\colon X\times X\to X\times X$ such that

$$r_1r_2r_1=r_2r_1r_2,$$

where $r_1=r\times \mathrm{id}_X$ and $r_2=\mathrm{id}_X\times r$ are maps from $X\times X\times X$ to itself.

We write $r(x,y)=(\sigma_x(y),\gamma_y(x))$, for all $x,y\in X$.
The map r is involutive if $r^2=\mathrm{id}_{X^2}$.
We say that r is non-degenerate if the maps $\sigma_x,\gamma_x\colon X\to X$ are bijective, for all $x\in X$.
Convention. By a solution of the YBE we mean an involutive non-degenerate set-theoretic solution of the Yang-Baxter equation.

Solutions of the YBE

Let (X, r) be a solution of the YBE. Etingof, Schedler and Soloviev introduced two groups associated to (X, r), its structure group

$$G(X, r) = \langle X \mid xy = \sigma_x(y)\gamma_y(x),\ \forall x, y \in X\rangle,$$

where $r(x, y) = (\sigma_x(y), \gamma_y(x))$, and its permutation group

$$\mathcal{G}(X, r) = \langle \sigma_x \mid x \in X\rangle \leq \mathrm{Sym}_X\,.$$

Furthermore, the map $x \mapsto \sigma_x$ $(x \in X)$ extends to a morphism of groups $\rho\colon G(X, r) \to \mathcal{G}(X, r)$.

Braces and the Yang-Baxter equation

In 2007 Rump introduced braces as a generalization of radical rings to study solutions of the YBE. The following definition is equivalent to the original definition of Rump.

Definition

A left brace is a set B with two binary operations, $+$ and $\cdot$, such that $(B,+)$ is an abelian group, $(B,\cdot)$ is a group, and for every $a,b,c \in B$,

$$a \cdot (b+c) + a = a \cdot b + a \cdot c.$$

Note that in a left brace B, $1 = 0$ (taking $a = 1$ and $b = c = 0$ in the above formula).
In any left brace B there is an action $\lambda\colon (B,\cdot) \to \mathrm{Aut}(B,+)$ defined by $\lambda(a) = \lambda_a$ and $\lambda_a(b) = ab - a$, for $a,b \in B$.

Braces and the Yang-Baxter equation

Rump proved that each left brace B produces a solution of the YBE: $r_B\colon B\times B\to B\times B$, $\quad r_B(a,b)=(\lambda_a(b),\lambda^{-1}_{\lambda_a(b)}(a))$.

Definition

An ideal I of a left brace B is a normal subgroup I of the multiplicative group of B such that $\lambda_a(y)\in I$ for all $a\in B$ and $y\in I$.

It is easy to check that every ideal I of a left brace B also is a subgroup of the additive group of B. Note that

$$a-b=bb^{-1}a-b=\lambda_b(b^{-1}a),$$

thus $a-b\in I$ if and only if $b^{-1}a\in I$. Therefore the natural sum and multiplication on B/I define a natural structure of left brace, the quotient left brace of B modulo I.

Braces and the Yang-Baxter equation

The socle of a left brace B is defined as the set

$$\mathrm{Soc}(B) = \{a \in B : \lambda_a = \mathrm{id}\} = \{a \in B : a + b = ab \text{ for all } b \in B\}.$$

The socle of B is an ideal of B.
Let (X, r) be a solution of the YBE.
It is known that there exists a unique left brace structure over the structure group $G(X, r)$ such that the additive group of $G(X, r)$ is isomorphic to $\mathbb{Z}^{(X)}$, and $\lambda_x(y) = \sigma_x(y)$, for all $x, y \in X$. Then the kernel of the map $\rho : G(X, r) \to \mathcal{G}(X, r)$ is

$$\mathrm{Ker}(\rho) = \mathrm{Soc}(G(X, r)).$$

Therefore $\mathcal{G}(X, r)$ inherits a structure of left brace, such that ρ is a homomorphism of left braces.

Retraction

Let (X, r) be a solution of the YBE. Consider the equivalence relation on X given by $x \sim y$ if and only if $\sigma_x = \sigma_y$. The *retraction* of (X, r) is defined as the solution $\mathrm{Ret}(X, r) = (X/\sim, \bar{r})$, where

$$\bar{r}([x], [y]) = ([\sigma_x(y)], [\gamma_y(x)]),$$

for all $x, y \in X$. One defines recursively $\mathrm{Ret}^{m+1}(X, r) = \mathrm{Ret}(\mathrm{Ret}^m(X, r))$ for all m.

Definition

A solution (X, r) of the YBE is said to be a multipermutation solution of level m if m is the minimal positive integer such that $\mathrm{Ret}^m(X, r)$ has only one element. In this case, we write $\mathrm{mpl}(X, r) = m$. A solution (X, r) of the YBE is said to be irretractable if $\mathrm{Ret}(X, r) = (X, r)$.

Braces

Remark

Let B be a left brace. Using the operation

$$a * b = ab - a - b = (\lambda_a - \mathrm{id})(b), \quad a, b \in B,$$

Rump introduced the series

$$B = B^{(1)} \supseteq B^{(2)} \supseteq B^{(3)} \supseteq \cdots ,$$

where $B^{(m+1)} = B^{(m)} * B$ is the additive group generated by

$$\{(\lambda_a - \mathrm{id})(b) : a \in B^{(m)},\ b \in B\}$$

for all $m \geq 1$. Rump proved that each $B^{(m)}$ is an ideal of B.

Braces and multipermutation solutions

Theorem (Gateva-Ivanova)

Let (X, r) be a solution of the YBE. Let $G = G(X, r)$. Then (G, r_G) is a multipermutation solution if and only if (X, r) is a multipermutation solution.

Theorem (C., Gateva-Ivanova, Smoktunowicz)

Let B be a nonzero left brace and let (B, r) be its associated solution of the YBE. Then $\mathrm{mpl}(B, r) = m < \infty$ if and only if $B^{(m+1)} = 0$ and $B^{(m)} \neq 0$.

Bieberbach groups

Definition

A Bieberbach group is a finitely generated torsion-free abelian-by-finite group.

Theorem (Gateva-Ivanova, Van den Bergh)

Let (X, r) be a finite solution of the YBE. Then the structure group $G(X, r)$ is a Bieberbach group.

Poly-$\mathbb{Z}$ groups

Recall that a group G is said to be *poly-$\mathbb{Z}$* if it has a subnormal series

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that each quotient G_i/G_{i-1} is isomorphic to $\mathbb{Z}$.

Theorem (Jespers, Okniński)

Let (X, r) be a finite multipermutation solution of the YBE. Then the structure group $G(X, r)$ is poly-$\mathbb{Z}$

Left orderable groups

A group G is said to be *left orderable* if there is a total order $<$ on G such that for any $x, y, z \in G$, $x < y$ implies $zx < zy$.

Theorem (Farkas)

Let G be a Bieberbach group. Then the following statements are equivalent.

1. *Every nontrivial subgroup of G has a nontrivial center.*
2. *G is a poly-$\mathbb{Z}$ group.*
3. *G is left orderable.*

Corollary

Let (X, r) be a finite solution of the YBE. Then $G(X, r)$ is poly-$\mathbb{Z}$ if and only if it is left orderable.

Main result

Theorem (Bachiller, C., Vendramin)

Let (X, r) be a finite non-degenerate involutive set-theoretic solution of the Yang–Baxter equation. Then the following statements are equivalent:

1. *(X, r) is a multipermutation solution.*
2. *$G(X, r)$ is left orderable.*
3. *$G(X, r)$ is poly-$\mathbb{Z}$.*

Proof

Proof. Let $G = G(X, r)$ and $\mathcal{G} = \mathcal{G}(X, r)$. By the above Corollary, we have the equivalence between (2) and (3). The Theorem of Jespers and Okniński is the implication $(1) \implies (3)$.

Let us prove $(2) \implies (1)$.

For that purpose let us assume that (X, r) is not a multipermutation solution. By the Theorem of Gateva-Ivanova, the solution (G, r_G) is not a multipermutation solution. This implies that the solution $(\mathcal{G}, r_{\mathcal{G}})$ is not a multipermutation solution. By the Theorem of C., Gateva-Ivanova and Smoktunowicz, one obtains that $G^{(m)} \neq \{0\}$ and $\mathcal{G}^{(m)} \neq \{0\}$ for all m. Since $\mathcal{G}$ is finite, there exists m such that $\mathcal{G}^{(m+1)} = \mathcal{G}^{(m)} \neq 0$.

Proof

By the Theorem of Farkas, to prove that G is not left orderable it suffices to prove that the non-trivial subgroup $H = G^{(m+1)}$ of $(G, \cdot)$ has trivial center.
Let $z \in Z(H)$. Recall that

$$G/\mathrm{Soc}(G) \cong \mathcal{G} \leq \mathsf{Sym}_X .$$

Since $\mathrm{Soc}(G)$ has finite index in G and G is torsion free, without loss of generality we may assume that $z \in \mathrm{Soc}(G)$. Notice that if $h \in H$, then

$$\lambda_h(z) = hz - h = zh - h = z + h - h = z. \tag{1}$$

Proof

Let $X_1, \dots, X_s$ be the orbits of X under the action of $\mathcal{G}^{(m)}$. These orbits are the orbits of X under the action of $G^{(m)}$ through the map λ. Note that

$$\begin{aligned} H = G^{(m+1)} &= \langle (\lambda_a - \mathrm{id})(b) : a \in G^{(m)},\, b \in G \rangle_+ \\ &= \langle (\lambda_a - \mathrm{id})(x) : a \in G^{(m)},\, x \in X \rangle_+ \\ &= \langle y - x : x, y \in X_i,\, 1 \le i \le s \rangle_+ . \end{aligned} \tag{2}$$

The second equality follows from the fact that $(G, +)$ is generated by X and λ_a is an automorphism of $(G, +)$. The third equality is obtained using that $\lambda_a(x) \in X$ for all $x \in X$ and all $a \in G$.

Proof

Since $(G,+)$ is the free abelian group with basis X, the element z can be uniquely written as

$$z = z_1 + \cdots + z_s,$$

where each $z_i \in \langle X_i \rangle_+$. From the uniqueness of the decomposition of z and (1) one obtains that $\lambda_h(z_i) = z_i$ for all $i \in \{1, \ldots, s\}$ and $h \in H$. Now write each z_i as

$$z_i = \sum_{t \in X_i} n_t t,$$

where each $n_x \in \mathbb{Z}$. By (2), we have that $\sum_{t \in X_i} n_t = 0$. This decomposition is unique since $(G,+)$ is the free abelian group with basis X. Let $x, y \in X_i$ be such that $x \neq y$. Then there exists $g \in G^{(m)}$ such that $\lambda_g(x) = y$.

Proof

From $\mathcal{G}^{(m+1)} = \mathcal{G}^{(m)}$ it follows

$$G^{(m)} = G^{(m+1)} + (\mathrm{Soc}(G) \cap G^{(m)}) = H + (\mathrm{Soc}(G) \cap G^{(m)}).$$

Thus $g = g_1 + g_2$, where $g_1 \in H$ and $g_2 \in \mathrm{Soc}(G) \cap G^{(m)}$. Since $g_2 \in \mathrm{Soc}(G)$, $g = g_2 g_1$. Therefore

$$y = \lambda_g(x) = \lambda_{g_2 g_1}(x) = \lambda_{g_2}\lambda_{g_1}(x) = \lambda_{g_1}(x).$$

Since $z_i = \lambda_{g_1}(z_i) = \sum_{t \in X_i} n_t \lambda_{g_1}(t)$, we conclude that $n_x = n_y$. Since $\sum_{t \in X_i} n_t = 0$, it follows that $n_t = 0$ for all $t \in X_i$ and all $i \in \{1, \ldots, s\}$. Therefore $z = 0 = 1$ and the result follows.

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Thank you for your attention!