UNIVERSITÀ DEL SALENTO

A new family of dynamical extensions of left cycle sets

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- Some links between left cycle sets and Group Theory;
- A new dynamical extension of left cycle sets.

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Set-theoretic solutions of the Yang-Baxter equation

Definition

A *set-theoretic solution of the Yang-Baxter equation* on a set X is a pair (X, r), where the map $r : X \times X \to X \times X$ is such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.

Problem (Drinfield, 1992)

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A set-theoretic solution of the Yang-Baxter equation $r : X \times X \to X \times X, \quad (x, y) \to (\lambda_x(y), \rho_y(x))$ is called:

1) *involutive* if $r^2 = id_{X\times X}$;
2) *non-degenerate* if $\lambda_x, \rho_x \in Sym(X)$ for every $x \in X$;
3) *square-free* if $r(x, x) = (x, x)$ for every $x \in X$.

Convention: From now on, by a solution I mean a non-degenerate involutive set-theoretic solution.

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Non-degenerate left cycle sets

The solutions of the Yang-Baxter equation are in bijective corrispondence with **non-degenerate left cycle sets**, where

Definition (Rump, 2004)

A pair $(X, \cdot)$ is said a *non-degenerate left cycle set* if X is a non-empty set, and $\cdot$ a binary operation on X such that

1) $(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z)$ for all $x, y, z \in X$;
2) the left multiplication $\sigma_x : X \longrightarrow X,\ y \longmapsto x\cdot y$ is bijective for every $x \in X$;
3) $\mathfrak{q} : X \longrightarrow X,\ x \longmapsto x\cdot x$ is bijective.

Furthermore, we will call $(X, \cdot)$ *square-free* if $\mathfrak{q} = id_X$.

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Non-degenerate left cycle sets

Example

If X is the set $\{1,2,3\}$ and $\cdot$ the binary operation on X given by $\sigma_1 = \sigma_2 := id_X$ and $\sigma_3 := (12)$, then the pair $(X,\cdot)$ is a non-degenerate left cycle set. Its multiplication table is

$\cdot$	1	2	3
1	1	2	3
2	1	2	3
3	2	1	3

Left cycle sets and Yang-Baxter equation

Theorem (Rump, 2004)

If (X, r) is a solution, where $r(x, y) := (\lambda_x(y), \rho_y(x))$ then the pair $(X, \cdot)$, where $\cdot$ is given by $x \cdot y := \lambda_x^{-1}(y)$ for every $x, y \in X$, is a left cycle set called the **associated left cycle set**.

Vice versa if $(X, \cdot)$ is a non-degenerate left cycle set and σ_x its left multiplication then, the pair (X, r), where $r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) \cdot x)$ for every $x, y \in X$, is a solution and we call it the **associated solution**.

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The retraction of a left cycle sets

Definition (Etingov, Schedler, Soloviev, 1999)

Let X be a non-degenerate left cycle set and $\sim$ the relation on X given by

$$x \sim y :\Leftrightarrow \sigma_x = \sigma_y.$$

Then $\sim$ is a congruence called the **retract relation**. of X and $X/\sim$ is a non-degenerate left cycle set.

Definition

A left cycle set $(X, \cdot)$ is said **irretractable** if $X = X/\sim$, otherwise X is called **retractable**.

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An example of retractable left cycle set

Example

Let X be a non-empty set and $\alpha \in Sym(X)$. Let $\cdot$ be the binary operation on X given by

$$x \cdot y := \alpha(y)$$

for all $x, y \in X$.
Then $(X, \cdot)$ is a retractable left cycle set: indeed, $\sigma_x = \alpha$ for every $x \in X$.

An example of irretractable left cycle set

Example

Let $X := \{1, 2, 3, 4\}$ and $\cdot$ be the operation on X given by

$$i \cdot j := \sigma_i(j)$$

for all $i, j \in X$, where $\sigma_i \in Sym(X)$ for all $i \in X$ and they are given by:

$$\sigma_1 := (34) \quad \sigma_2 := (1423) \quad \sigma_3 := (1324) \quad \sigma_4 := (12)$$

Then $(X, \cdot)$ is an irretractable left cycle set.

A conjecture about left cycle sets

In 2004 Gateva-Ivanova posed the following conjecture:

Conjecture (Gateva-Ivanova, 2004)

Every square-free left cycle set X such that $2 \leq |X| < \infty$ is retractable.

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The structure group of a left cycle set

Definition

Let $(X, \cdot)$ be a non-degenerate left cycle set. Put $t_{x,y} := \sigma_x^{-1}(y)$ and $z_{x,y} := \sigma_x^{-1}(y) \cdot x$ for every $x, y \in X$. Then the group

$$G(X) := < X| \ \ xy = t_{x,y} z_{x,y} \text{ for every x,y} \in X >$$

is called the **structure group** of the left cycle set $(X, \cdot)$.

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Theorem

Let X be a non-degenerate left cycle set. Then the structure group $G(X)$ is

- **Solvable** *(Etingov, Schedler, Soloviev);*
- **Bieberbach** *(Gateva-Ivanova, Van den Bergh);*
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Finding all the non-degenerate left cycle sets.

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Construction of non-trivial examples of groups having interesting properties.

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Irretractable left cycle sets and poly-$\mathbb{Z}$ groups

Theorem (Bachiller, Cedó, Vendramin, 2017)

Let X be an irretractable non-degenerate left cycle set. Then the structure group $G(X)$ is not a poly-$\mathbb{Z}$ group.

where

Definition

A group G is called a poly-$\mathbb{Z}$ group if G has a subnormal series

$$<1> \lhd G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that $G_{i+1}/G_i \cong \mathbb{Z}$ for every $i \in \{0, \ldots, n-1\}$.

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Irretractable left cycle sets and poly-$\mathbb{Z}$ groups

The poly-$\mathbb{Z}$ groups verify an important conjecture due to Kaplansky:

Conjecture (Kaplansky)

Let K be a field and G a torsion-free group. Then the group ring $K[G]$ does not contain any non-trivial units.

New irretractable left cycle sets allow us to construct new non-trivial examples of torsion-free groups that are not poly-$\mathbb{Z}$ groups. In particular, one can use these groups for testing Kaplansky's conjecture.

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A new construction of left cycle sets

Definition

Let X be a left cycle set, S a non-empty set and $\alpha : X \times X \times S \longrightarrow Sym(S)$, $(i,j,s) \mapsto \alpha_{i,j}(s,-)$. Then α is said **dynamical cocycle** of X if and only if

$$\alpha_{i\cdot j, i\cdot k}(\alpha_{i,j}(r,s), \alpha_{i,k}(r,t)) = \alpha_{j\cdot i, j\cdot k}(\alpha_{j,i}(s,r), \alpha_{j,k}(s,t)).$$

for every $i,j,k \in X$, $s,t \in S$.

Proposition (Vendramin, 2015)

If α is a dynamical cocycle then $S \times_\alpha X := (S \times X, \cdot)$ is a left cycle set, where

$$(s,i)\cdot(t,j) := (\alpha_{ij}(s,t), i\cdot j),$$

and we will call $S \times_\alpha X$ **dynamical extension** *of X by α.*

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A counterexample to the Gateva-Ivanova's conjecture

In 2015 Vendramin found a counterexample to the Gateva-Ivanova Strong Conjecture.

Example (Vendramin, 2015)

Let $X := \{a, b\}$ be the left cycle set given by $x \cdot y = y$ for every $x, y \in X$ and let $S := \{1, 2, 3, 4\}$ be a set. Let $\alpha : X \times X \times S \longrightarrow Sym(S)$ given by:

$$\alpha(a, a, 1) = \alpha(a, a, 2) = \alpha(b, b, 1) = \alpha(b, b, 2) := (34);$$
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Then $X \times_\alpha S$ is a square-free irretractable left cycle set of cardinality 8.

This is the square-free irretractable left cycle set of minimal cardinality.

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In 2016 Bachiller, Cedó, Jespers and Okniński found a new dynamical extension of left cycle sets which allows to construct new (irretractable) left cycle sets.

An example of dynamical extension

Theorem (Bachiller, Cedó, Jespers and Okniński, 2016)

Let A, B be non-trivial **abelian groups**, *$S := A \times B$ and let I be the left cycle set given by $i \cdot j := j$ for every $i, j \in I$ a set with $|I| > 1$. Let $\varphi_1 : A \longrightarrow B$ be a function such that $\varphi_1(-a) = \varphi_1(a)$ for all $a \in A$ and let $\varphi_2 : B \longrightarrow A$ be a* **homomorphism**. *Let $\alpha : I \times I \times S \longrightarrow Sym(S)$ given by*

$$\alpha(i, j, (a, b))(c, d) := \begin{cases} (c, d - \varphi_1(a - c)), & \text{if } i = j \\ (c - \varphi_2(b), d), & \text{if } i \neq j \end{cases}$$

for all $i, j \in I$, $a, c \in A$ and $b, d \in B$. Then $X(A, B, I) := S \times_\alpha I$ is a left cycle set and it is irretractable whenever $\varphi_1^{-1}(\{0\}) = \{0\}$ and φ_2 is injective.

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A new dynamical extension of left cycle sets

A larger family of irretractable left cycle set is obtained by M. C., Francesco Catino and Giuseppina Pinto.

Let I be a left non-degenerate cycle set, let A, B be non-empty sets, $\gamma : B \longrightarrow Sym(A)$, $\beta : A \times A \times I \longrightarrow Sym(B)$, $S := A \times B$ and $\alpha : I \times I \times S \longrightarrow Sym(S)$ given by

$$\alpha(i, j, (a, b))(c, d) := \begin{cases} (c, \beta_{(a,c,i)}(d)), & \text{if } i = j \\ (\gamma_b(c), d), & \text{if } i \neq j \end{cases}$$

for all $i, j \in I$, $a, c \in A$ and $b, d \in B$.

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Theorem (M. C., F. Catino, G. Pinto, 2017)

If $\gamma : B \longrightarrow Sym(A)$ and $\beta : A \times A \times I \longrightarrow Sym(B)$ are such that

1) $\gamma_b \gamma_d = \gamma_d \gamma_b$,
2) $\beta_{(a,c,i)} = \beta_{(\gamma_b(a),\gamma_b(c),j\cdot i)}$,
3) $\gamma_{\beta_{(a,c,i)}(d)} \gamma_b = \gamma_{\beta_{(c,a,i)}(b)} \gamma_d$,
4) $\beta_{(a,c,i\cdot i)} \beta_{(a',c,i)} = \beta_{(a',c,i\cdot i)} \beta_{(a,c,i)}$

hold for all $a, a', c \in A$, $b, d \in B$ and $i, j \in I$, $i \neq j$, then α is a dynamical cocycle and so $X(A, B, I, \beta, \gamma) := S \times_\alpha I$ is a non-degenerate left cycle set.

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Theorem (M. C., F. Catino, G. Pinto, 2017)

$X(A, B, I, \beta, \gamma)$ is irretractable if and only if γ is injective and they hold:

1) *For every $i \in I$, $a, c \in A$, $a \neq c$ there exists $e \in A$ such that $\beta_{(a,e,i)} \neq \beta_{(c,e,i)}$*
2) *If $id_A \in \gamma(B)$, for all $a, c \in A$, $i, j \in I$ such that $i \neq j$ and $\sigma_i = \sigma_j$ exists $e \in A$ such that $\beta_{(a,e,i)} \neq id_B$ or $\beta_{(c,e,j)} \neq id_B$.*

Observation

Let I be the left cycle set given by $x \cdot y = y$ for all $x, y \in I$, $\varphi_1 : A \longrightarrow B$ a function such that $\varphi_1(-a) = \varphi_1(a)$ for every $a \in A$ and let $\varphi_2 : B \longrightarrow A$ be an homomorphism. Put

$$\beta_{(a,c,i)} := t_{-\varphi_1(a-c)} \quad \text{and} \quad \gamma_b := t_{-\varphi_2(b)}$$

for all $a, c \in A$, $b \in B$ and $i \in I$, where t_v is the translation by v. Then $X(A, B, I, \beta, \gamma)$ is the non-degenerate left cycle set $X(A, B, I)$ obtained by Bachiller, Cedó, Jespers and Okniński.

A family of irretractable left cycle sets

Example (M.C., F. Catino, G. Pinto, 2017)

Let $k>1$, $I:=\{1,2\}$ the left cycle set given by $x\cdot y=y$ for every $x,y\in I$, $A=B:=\mathbb{Z}/2k\mathbb{Z}$ and $\delta:=(1\dots 2k)$. Put

$$\beta_{(a,a,i)}:=id_A \qquad \beta_{(a,b,1)}:=\delta$$

$$\beta_{(a,b,2)}:=\delta^2 \qquad \gamma_a:=t_{-a-1}$$

for all $i\in I$ and $a\in A$, $b\in B$, $a\neq b$ where $t_a:A\to A$, $t_a(x):=x+a$ for all $a\in A$.
Then $X(A,B,I,\beta,\gamma)$ is an irretractable square-free left cycle set different from those obtained by Bachiller, Cedó, Jespers and Okniński.

A family of irretractable left cycle sets

Example (M.C., F. Catino, G. Pinto, 2017)

Let $k > 1$, $I := \{1, 2\}$ the left cycle set given by $x \cdot y = y$ for every $x, y \in I$, $A = B := \mathbb{Z}/2k\mathbb{Z}$ and $\delta := (1 \ldots 2k)$. Put

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Thanks!