Groups that have the same holomorph as a finite perfect group

Andrea Caranti[1] & Francesca Dalla Volta[2]

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[1]Dipartimento di Matematica
Università degli Studi di Trento

[2]Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano–Bicocca

Four questions

It's just opposite

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The opposite group of G is isomorphic to G via

$$\begin{aligned} \mathrm{inv} : G &\to (G, \circ) \\ x &\mapsto x^{-1} \end{aligned}$$

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Is $(G, \circ)$ isomorphic to G? Yes, via

$$\begin{aligned} H \times K &\to (H \times K, \circ) \\ (x, y) &\mapsto (x^{-1}, y) \end{aligned}$$

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The answer is possibly not obvious.

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Let $(G, \circ)$ be obtained from G by replacing H_1 with its opposite.

Are the H_i still characteristic in $(G, \circ)$?

(Multiple) holomorphs

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More generally, if $N \leq S(G)$ is a regular subgroup, then

$$N_{S(G)}(N)$$

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the multiple (double) holomorph of G, and the group

$$T(G) = N_{S(G)}(\mathrm{Hol}(G))/\,\mathrm{Hol}(G)$$

acts regularly on the set

$$\mathcal{H}(G) = \left\{\, N \leq S(G) : N \text{ is regular, } N_{S(G)}(N) = \mathrm{Hol}(G) \text{ and } N \cong G \,\right\}$$

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The latter appears to be easier to compute.

Describing the regular (normal) subgroups of the holomorph

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Skip ring setting, please

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Here the condition $\mathrm{Aut}(G) \le \mathrm{Aut}(G, \circ)$ translates into the study of the commutative rings $(G, +, \cdot)$ such that every automorphism of the group $(G, +)$ is also an automorphism of the ring $(G, +, \cdot)$.

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Commutators and perfect groups

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The regular normal subgroups of the holomorph of a perfect group

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Examples

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- Fix an isomorphism $\zeta : L_1 \to L_2$, and define $z_2 = z_1^{\zeta} \in L_2$. Then *every* isomorphism $L_1 \to L_2$ takes z_1 to z_2.
- Choose the amalgamation so that $z_2 = z_1^{-1}$.
- Then M and the L_i are characteristic in G: an automorphism of G
  - takes M to M

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  - if it takes L_1 to L_2, then it takes z_1 to $z_2 = z_1^{-1}$, i.e. it inverts the centre.

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In fact for $x, y \in L_1$

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So the L_i are *not* characteristic in $(G, \circ)$, and $\mathrm{Aut}(G, \circ)$ is twice as big as $\mathrm{Aut}(G)$.

That’s All, Thanks!