Double chain conditions in infinite groups

Mattia Brescia

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Finiteness Conditions

Let $\mathfrak{U}$ be the universe of all groups and let $\mathfrak{F}$ be the class of finite groups. Any intermediate class between them, i.e. any $\mathfrak{X}$ such that

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Classical non-trivial examples of finiteness conditions are

- Locally finiteness;
- Periodicity;
- Being finitely generated;
- ...

Chain Conditions

Let T be a totally ordered set and χ be a subgroup theoretical property.
We will say that a group G satisfies the T-chain condition on χ-subgroups if there is no increasing function between T and the set of the χ-subgroups of G ordered by inclusion. Such class will be called a chain class and will here be denoted by $C(T, \chi)$.

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Very known examples of chain classes are

- For $T = (\mathbb{N}, <)$ and $\chi =$"being a group", $C(T; \chi) = Max$;

$$H_0 < H_1 < \cdots < H_n < \cdots$$

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- $\mathfrak{Max}$-$\mathfrak{ab}$ (A. I. Malčev, B. I. Plotkin, O. J. Schmidt, ...).

Aiming for (too) general results

Some of the many early relevant questions were

- Is $\mathfrak{M}ax = (\mathrm{P}\mathfrak{C})\mathfrak{F}$?
- Is $\mathfrak{M}in = \check{\mathrm{C}}$?
- Is $\mathfrak{M}ax\text{-}sn = \mathfrak{M}ax$ in the universe of locally soluble groups?

Ending up with monsters

The answers to which are

- Is $\mathfrak{Max} = (\mathrm{P}\mathfrak{C})\mathfrak{F}$? NO. (Thank you, Ol′šanskiĭ)
- Is $\mathfrak{Min} = \check{\mathfrak{C}}$?
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- Is $\mathfrak{Min} = \check{C}$? NO. (Thank you again, Ol'šanskiĭ)
- Is $\mathfrak{Max}$-$\mathfrak{sn} = \mathfrak{Max}$ in the universe of locally soluble groups? Who knows!

Changing the point of view

For many years the quasi-totality of investigations about chain conditions were about changing the group theoretical property χ and showing results about Max-χ or Min-χ, i.e. about the classes of the groups satisfying the maximal or the minimal condition on χ-subgroups.

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D. I. Zaicev (1971), T. S. Shores (1973) - Let G be a locally radical group. Then G satisfies the double chain condition on subgroups iff it satisfies either the maximal or the minimal condition on subgroups.

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So in 2005 there came the first new work on the so-called "Double chain condition".

F. De Mari, F. de Giovanni (2005) - Let $\mathfrak{n}$ be the subgroup property of "being normal". Then, in the universe of residually finite groups, $DC_{\mathfrak{n}} = \text{Max-}\mathfrak{n}$.

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F. De Mari, F. de Giovanni (2005) - Let $\mathfrak{n}$ be the subgroup property of ″being normal″. Then, in the universe of periodic soluble groups, $DC_{\mathfrak{n}} = \text{Min-}\mathfrak{n}$.

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Moreover in the same paper

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So is everything this predictable?

Going on with the ordering type of $\mathbb{Z}$

Theorem

F. de Giovanni, M. B. – 2015

Let G be a radical group. G satisfies $DC_{\mathfrak{sn}}$ if and only if G satisfies one of the following:

- G satisfies Max-$\mathfrak{sn}$;
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Let G be a radical group. G satisfies DC_{sn} if and only if G satisfies one of the following:

- G satisfies Max-sn;
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- $G = HJ$ where J is the finite residual of G, H is polycyclic, $C_H(J)$ is finite and every subnormal subgroup of G is either Min or properly contains J.

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Not everything is this predictable, indeed!

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Lemma

Let χ be an intersection-closed, quotient-transitive subgroup theoretical property and let G be a group satisfying $DC_{\overline{\chi}}$. If H/K is a section of G such that

$$H/K = \mathop{\mathrm{Dr}}_{n\in\mathbb{N}} H_n/K$$

and each H_n is different from K, then K is a χ-subgroup of G and such is every direct term of H/K.

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Proof – Let us split $H/K = X/K \times Y/K$ where

$$X/K = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, H_{2n}/K \quad \text{and} \quad \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, H_{2n-1}/K$$

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(Proof, continued) Notice that, in general, if in a group G we have an infinite direct product

$$\operatorname*{Dr}_{n\in\mathbb{N}} G_n$$

of non-trivial groups, we can define the following

$$U_{-k} = \operatorname*{Dr}_{n>k} G_{2n-1} \quad \text{and} \quad U_k = \big(\operatorname*{Dr}_{n\in\mathbb{N}} G_{2n-1}\big) \times \big(\operatorname*{Dr}_{1\leqslant n\leqslant k} G_{2n}\big)$$

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and hence have

$$\cdots < U_{-k} < \cdots < U_{-1} < U_0 < U_1 < \cdots < U_k < \cdots .$$

Let's get familiar with double chains!

(Proof, continued)

So there are two χ-subgroups P/K and Q/K of X/K and Y/K, respectively. Hence $P\chi G$, $Q\chi G$ and $K = P \cap Q$ is a χ-subgroup of G.

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So there are two χ-subgroups P/K and Q/K of X/K and Y/K, respectively. Hence $P\chi G$, $Q\chi G$ and $K = P \cap Q$ is a χ-subgroup of G.

Finally, the same argument applies to each H/H_n.

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- A section H/K of a group G is said to be *ascendant* if H is an ascendant subgroup of G and K is a normal subgroup of H.
- A subgroup H of a group G is normal in G if and only if H is both pronormal and ascendant in G.

Lemma

F. de Giovanni, M. B. – 2017

Let G be a group satisfying DC_{np} and let H/K be an ascendant section of G which is a direct product of infinitely many cyclic non-trivial subgroups. Then:

1) H and K are normal in G;

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2) All cyclic subgroups of G/H are pronormal;
3) G/K is a $\overline{T}$-group.

Double chains on non-pronormal subgroups

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F.dG, M.B. (2017) - Let G be a radical torsion-free DC_{np}-group. Then G is either polycyclic or abelian.

Double chains on non-pronormal subgroups

F.dG, M.B. (2017) - Let G be a periodic locally radical DC_{np}-group. Then either G is a Černikov group or every subgroup of G is pronormal.

Double chains on non-pronormal subgroups

F.dG, M.B. (2017) - Let G be a locally nilpotent DC_{np}-group. Then G either nilpotent or a Černikov group. In particular, G satisfies either Max-np or Min-np.

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F.dG, M.B. (2017) - Let G be a radical DC_{np}-group. Then G is either minimax or a $\overline{T}$-group.

Chain conditions - (not?) a conclusion

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