On groups with automorphisms whose fixed points are Engel

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Higman's Theorem

If G is a nilpotent group admitting an automorphism ϕ of prime order q and such that $C_G(\phi) = 1$, then G has nilpotency class bounded by some function $h(q)$ depending on q alone.

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- $C_{G/N}(A) = C_G(A)N/N$, for any A-invariant normal subgroup N
- $G = \langle C_G(B) \mid A/B \text{ is cyclic} \rangle$ whenever A is a noncyclic abelian group

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By an automorphism of a profinite group we always mean a continuous automorphism.

We say that a group A acts on a profinite group G *coprimely* if A is finite while G is an inverse limit of finite groups whose orders are relatively prime to the order of A.

Engel groups

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Note that the nilpotency class $c(n)$ and the exponent $e(n)$ exclusively depends on the Engel class n.

Coprime automorphisms whose fixed points are Engel

We consider the situation where A is an elementary abelian q-group acting coprimely on a (pro)finite group G and the centralizers $C_G(a)$ consist of Engel elements.

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Let $A^{\#}$ denote the set of non-identity elements of A.

Centralizers consisting of Engel elements in G

Theorem A1

Let q be a prime, n a positive integer and A an elementary abelian group of order q^2. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^{\#}$ every element of $C_G(a)$ is n-Engel in G. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.

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Relaxing the hypothesis on the fixed point subgroups

If, in Theorem A1, we relax the hypothesis that every element of $C_G(a)$ is n-Engel in G and require instead that every element of $C_G(a)$ is n-Engel in $C_G(a)$, then the result is no longer true.

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Example

a finite non-nilpotent group G admitting a four-group of automorphisms A such that $C_G(a)$ is abelian for each $a \in A^{\#}$ can be constructed.

Centralizers consisting of Engel elements

Theorem A2

Let q be a prime, n a positive integer and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a finite group G and assume that for each $a \in A^{\#}$ every element of $C_G(a)$ is n-Engel in $C_G(a)$. Then the group G is k-Engel for some $\{n, q\}$-bounded number k.

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Theorem A1 can be generalized as follows:

Let q be a prime, n a positive integer and A an elementary abelian group of order q^r with $r \geq 2$ acting on a finite q'-group G.

(1) If all elements in $\gamma_{r-1}(C_G(a))$ are n-Engel in G for any $a \in A^\#$, then $\gamma_{r-1}(G)$ is k-Engel for some $\{n, q, r\}$-bounded number k.

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(2) If, for some integer d such that $2^d \leq r - 1$, all elements in the dth derived group of $C_G(a)$ are n-Engel in G for any $a \in A^{\#}$, then the dth derived group $G^{(d)}$ is k-Engel for some $\{n, q, r\}$-bounded number k.

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Analogous generalizations of Theorems A2, B1 and B2 also hold.

Idea of the proof of Theorem B2

Recall that q is a prime and A is an elementary abelian group of order q^3. Assume that A acts coprimely on a profinite group G and assume that $C_G(a)$ is locally nilpotent for each $a \in A^{\#}$.

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- the Lubotzky–Mann theory of powerful p-groups;
- a Lazard's criterion for a pro-p group to be p-adic analytic;
- a theorem of Bahturin and Zaicev giving an important criterion for a Lie algebra to satisfy a polynomial identity.

Idea of the proof

Theorem (Ward, 1971). *Let q be a prime and A an elementary abelian group of order q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent for each $a \in A^\#$. Then G is nilpotent.*

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If G is a profinite group admitting a coprime group of automorphisms A which is elementary abelian of order q^3 and such that $C_G(a)$ is pronilpotent for all $a \in A^\#$, then G is pronilpotent.

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In view of the profinite version of Ward's Theorem the group G is pronilpotent and therefore G is the Cartesian product of its Sylow subgroups.

Idea of the proof

Proposition

For any locally nilpotent profinite group K there exist a positive integer n, elements $g_1, g_2 \in K$ and an open subgroup $H \leq K$ such that the law $[x,_n y] \equiv 1$ is satisfied on the cosets g_1H, g_2H, that is $[g_1h_1,_n g_2h_2] = 1$ for all $h_1, h_2 \in H$.

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Choose $a \in A^{\#}$. Since $C_G(a)$ is locally nilpotent, from the proposition above it follows that $C_G(a)$ contains an open subgroup H and elements u, v such that for some n the law $[x,_n y] \equiv 1$ is satisfied on the cosets uH, vH.

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Let $[C_G(a) : H] = m$ and let $\pi_1(a) = \pi(m)$. Put $T = O_{\pi_1'}(C_G(a))$.

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Since T is isomorphic to the image of H in $C_G(a)/O_{\pi_1}(C_G(a))$, it is easy to see that T satisfies the law $[x,_n y] \equiv 1$, that is, T is n-Engel.

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By the result of Burns and Medvedev the subgroup T has a nilpotent normal subgroup U such that T/U has finite exponent, say e. Set $\pi_2(a) = \pi(e)$.

Idea of the proof

Note that the finite sets $\pi_1(a)$ and $\pi_2(a)$ depend on the choice of $a \in A^{\#}$. Let

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The choice of the finite set π guarantees that $C_K(a)$ is nilpotent for every $a \in A^{\#}$.

Theorem (Shumyatsky, 2001). *Let* q be a prime and A an elementary abelian q-group of order q^3. Suppose that A acts coprimely on a finite group G and assume that $C_G(a)$ is nilpotent of class at most c for each $a \in A^{\#}$. Then G is nilpotent and the class of G is $\{q, c\}$-bounded.

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By the profinite version of the result above, the subgroup K is nilpotent.

Idea of the proof

We have

$$G = P_1 \times P_2 \times \cdots \times P_r \times K,$$

where $p_1, p_2, \ldots, p_r$ are the finitely many primes in π and $P_1, P_2, \ldots, P_r$ are the corresponding Sylow subgroups of G.

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Since every finite subset of G is contained in a finitely generated A-invariant subgroup we can also assume that G is finitely generated.

Idea of the proof

We denote by $D_j = D_j(G)$ the terms of the p-dimension central series of G, i.e.

$$D_j = \langle [g_1, \dots, g_s]^{p^t} \mid sp^t \geq j, g_k \in G \rangle = \prod_{sp^t \geq j} \gamma_s(G)^{p^t}.$$

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Consider $L_p(G)$, the subalgebra generated by the first homogeneous component D_1/D_2 in $L(G)$.

Since any automorphism of G induces an automorphisms of $L(G)$, in particular the group A naturally acts on $L_p(G)$.

Idea of the proof

By applying a combination of results due to Wilson, Zelmanov, Khukhro and Shumyatsky and the Lie-theoretical techniques created by Zelmanov in the solution of the RBP we can show that $L_p(G)$ is nilpotent.

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Proving that $L_p(G)$ is nilpotent relies on one of the most general form of the positive solution of the restricted Burnside-type problems for Lie algebras.

Theorem (Zelmanov, 1992)

Let L be a Lie algebra generated by finitely many elements $a_1, a_2, \ldots, a_m$ such that each commutator in these generators is ad-nilpotent. If L satisfies a polynomial identity, then L is nilpotent.

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Theorem (Lazard, 1965)

A finitely generated pro-p-group G is p-adic analytic if and only if $L_p(G)$ is nilpotent.

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Finally applying the profinite version of Shumyatsky's result we get that G is nilpotent. This concludes our proof.

Thank you!