The Institute for Advanced Study

LECTURES ON ABELIAN GROUPS

by

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Spring Term 1936



Notes by Dr. Baer

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Abelian group: set A of elements, composition (addition) a + b of the elements of A such that

1) in the equation a + b = c any pair of elements determines uniquely the third one,

2) a + b = b + a,

3) (a+b) + c = a + (b+c) = a + b + c.

There exists 0 satisfying 0 + x = x and -x satisfying x + (-x) = 0.

Isomorphism: f is an isomorphism of the group A upon the group B, if f is defined for x of A and maps x upon a uniquely determined element xf of B, and if furthermore Af = B, the correspondence between A and B defined by f is a one-one-correspondence, and if finally (a+b)f = af + bf.

An isomorphism of A upon A is an automorphism (proper automorphism).

Subgroup: The subset S of A is a subgroup if S contains for any pair x, y also their difference x - y.

Class-group mod a subgroup: if S is a subgroup of A, then $a \equiv b \bmod S$ if, and only if, a - b is in S. The classes of congruent elements form the class-group A/S,

since if $a \equiv b \bmod S$, $c \equiv d \bmod S$, then $a \pm c \equiv b \pm d \bmod S$.

Generators of a subgroup: If M is any subset of the group A, then the intersection of all the subgroups of A which contain M is a subgroup $\{M\}$, the smallest subgroup, containing M, or else, the subgroup generated by M.

Cyclic groups: Groups which are generated by a single element.

Order of an element: Number of elements of the group, generated by the

element. If the order of the element x is infinite, then all the elements mx for integer m are different; if the order of x is a finite number n, then rx = sx if, and only if, r ≡ s mod n.

F(A) = set of all the elements of finite order in A. F(A) is a subgroup of A which contains only elements of finite order and the only element of finite order in the class group A/F(A) is 0.

If A and A' are isomorphic, then F(A) and F(A') are isomorphic and A/F(A) and A'/F(A') are isomorphic. F(A) and A/F(A) are invariants of A.

Group F is a group all of whose elements are of finite order.

Group J is a group such that 0 is the only element of finite order.

Direct sum: A is the direct sum of its subgroups S(v), i.e.

$$A = \sum_{v} S(v)$$

if, and only if, A is generated by the elements of all the S(v) and if the intersection of S(v) and of the group T(v), generated by the S(w) for $w \neq v$, is 0.

To any set of groups A(v) there exists (apart from isomorphic groups) a uniquely determined group which is the direct sum of groups S(v) such that S(v) and A(v) are isomorphic. This group may be defined as the group of all the "vectors" (...a(v)...) with the v-th coordinate a(v) in A(v) such that only a finite number of coordinates of a vector is $\neq 0$ and the addition is defined by (...a(v)...) + (...b(v)...) = (...a(v) + b(v)...).

Direct summand is the subgroup S of A if there exists a subgroup T of A such that A = S + T.

F-J-group is a group A such that F(A) and F, A/F(A) and J are isomorphic. To every group F and every group J there exists at least one F-J-group, namely F + J. There exists essentially exactly one F-J-group if, and only if, F(A) is a direct summand of every F-J-group A. For if F(A) is a direct summand of A, i.e. A = F(A) + B, then A/F(A) and B are isomorphic.

Primary component: If p is a prime number, then the set C(p, F) of all the elements of the group F whose order is a power of p is a subgroup of F and F is the direct sum of its primary components. If x is any element, then $x = \sum_p c(p, x)$ for c(p, x) in C(p, F) and almost all c(p, x) = 0. The c(p, x) are uniquely determined by x.

p^iA, $p^\omega A$: If i is an integer, then p^iA is the subgroup of A consisting of all the elements p^ix for x in A and $p^\omega A$ is the intersection of all the p^iA.

Linear dependence, rank: The (different) elements a(1), ...a(n) of the group J are linearly dependent if there exist integers c(j) not all = 0 such that $\sum_{j=1}^{n} c(j)a(j) = 0$. Any (finite or infinite) subset of J which is not dependent is independent.

There exists always a greatest linearly independent subset of a group J, and any two greatest linearly independent subsets of a group J contain the same number of elements. This number is the rank r(J) of J.

Closed subgroups: C is a closed subgroup of the group J, if 0 is the only element of finite order of J/C (i.e. if J/C is a group J!).

p-complete, almost p-complete: The group J is p-complete if $0 \neq J = pJ$. The group J is almost p-complete if it contains a closed subgroup C of finite rank such that J/C is p-complete.

THEOREM 1: C(p,F(A)) is a direct summand of every F-J-group A (for given groups F and J) if, and only if, at least one of the following two conditions is satisfied:

(a) The orders of the elements of $C(p,F)/p^\omega C(p,F)$ are bounded.

(b) There does not exist any almost p-complete subgroup of J.

Rational coefficients in groups J: If x is any element of the group J, $n \neq 0$ an integer, then there exists at most one solution y of $ny = x$ in J. If a solution of this equation exists it is therefore possible to denote it by $n^{-1}x$. Hence there exists to every element x of J a certain set R(x) of rational numbers such that rx exists in J if, and only if, the rational number r is in R(x). If rx and sx exist, then $(r \pm s)x = rx \pm sx$ and therefore R(x) is an additive group of rational numbers. If $x \neq 0$, then R(x) and the smallest closed closed subgroup of J which contains x are isomorphic, since this smallest closed subgroup consists exactly of the elements rx for r in R(x).

Note that it may happen that rx and ry do not exist in J but r(x + y) exists in J so that the distributive law for these rational operators is not unrestrictedly true.

$n(x,p) = n(x,p; J)$ for elements x of the group J and prime numbers p is the greatest number i such that $p^i y = x$ has a solution y in J. $n(x,p)$ is therefore infinite if every equation $p^i y = x$ has a solution in J. Obviously

$$n(x+y,p) \geq \min (n(x,p), n(y,p))$$

and therefore in particular: if $n(x,p) < n(y,p)$, then $n(x+y,p) = n(x,p)$. Clearly $n(x,p) = n(-x,p)$.

Additive groups of rational numbers: Since direct summands of groups J are closed subgroups, additive groups of rational numbers (= groups of rank 1) are not true direct sums, i.e. direct indecomposable.

Let R be any additive group of rational numbers and $x \neq 0$ an element of R. Suppose that $r \neq 0$ is a rational number such that rx exists in R. Then $n(x,p) \neq n(rx,p)$ if, and only if, $n(x,p)$ is finite and p is a divisor of r, i.e. if $r = st^{-t}$ where s and t are relatively prime integers, then p divides either s or t. If $n(x,p)$ is finite and p divides s, then $n(x,p) < n(rx,p)$, anf if p divides t, then $n(rx,p) < n(x,p)$. Hence: if x and y are two elements $\neq 0$ of R,

then $n(x,p) = n(y,p)$ for almost all prime numbers p and $n(x,p) = n(y,p)$, if at least one of these numbers is infinite. $n(x,p) = \infty$ if, and only if, R is p-complete.

Suppose now that R and R' are two groups of rank 1 containing elements x and x' respectively such that

$n(x,p; R) = n(x',p; R')$ for almost all prime numbers p and in particular for every prime number p such that one of these numbers is infinite.

Then there exist integers v, v' such that $n(vx,p; R) = n(v'x',p; R')$ for every prime number p and in mapping vx upon v'x' an isomorphism of R upon R' can be defined.

Basis of a group J is a greatest linearly independent set G such that $nx = \sum_{j=1}^{m} n_j g_j$ for integer $n \neq 0$, n_j and elements g_j in G, x in J implies that every $n_j n^{-1} g_j$ exists in J.

Then the subset G of J is a basis of J if, and only if, G is the direct sum of the closed subgroups (of rank 1) which are generated by the elements in G, i.e. if $J = \sum_{g \text{ in } G} (g)$.

A basis of the group J exists therefore if, and only if, J is a direct sum of groups of rank 1, i.e. of subgroups of the additive group of rational numbers.

Complete groups are groups J which are p-complete for every prime number p. The only complete group of rank 1 is the additive group of all the rational numbers. If G is a greatest linearly independent subset of the complete group J, then $J = \sum_{g \text{ in } G} (g)$ where g is the smallest closed subgroup of J which contains g, i.e. G is a basis of J. Every (g) is isomorphic to the additive group of all the rational numbers. The number of summands is the rank of J and uniquely determined by J. It is a well known fact of linear algebra , if

the number is finite; and if it is infinite it is exactly the number of elements in J. In particular therefore two complete groups are isomorphic if, and only if, they have the same rank. (An analogous theorem holds if the operators form any field). Furthermore every closed subgroup of a complete group is a direct summand. More generally:

Every complete subgroup of a group J is a direct summand of J.

For if C is a complete subgroup of J, then let S be a greatest linearly independent subset of C and T a greatest linearly independent subset of J which contain S. If T* is the smallest closed subgroup containing all the elements of T but not in S, then $J = C + T^*$.

Every group J is contained in an essentially uniquely determined smallest complete group. For form the set P of all the pairs (x,n) for x in J and integer $n \neq 0$, defining identity by

$$(x,n) = (y,m) \text{ if, and only if, } mx = ny$$

and addition by

$$(x,n) + (y,m) = (mx+ny,nm).$$

Then P is a smallest complete group which contains J (= the elements (x,1)) and if P' is another smallest complete group containing J, then there exists an isomorphism of P upon P' which leaves all the elements of J invariant. From this fact and the invariance of the rank of complete groups follows the invariance of the rank of every group J.

Any two decompositions of a group J into a direct sum of groups of rank are isomorphic, i.e. if $J = \sum_v J(v) = \sum_w J'(w)$, where J(v), J'(w) are of rank 1, then there exists a one-one correspondence between the J(v) and J'(w) such that corresponding summands are isomorphic.

In order to prove this Theorem we have to define the

Invariants of a group J. Let R be any group of rational numbers, $t \neq 0$ an element of R. If x is an element of the group J, then type of $x \geq$ type of R if, and only if,

(a) $n(x,p;J) \geqq n(t,p;R)$ for almost all prime numbers p,

(b) $n(t,p;R) = \infty$ implies $n(x,p;J) = \infty$.

If in (a) for the inequality-sign equality is substituted, then type of x = type of R. Therefore type x = type R, if and only if, type x $\leqq$ type R $\leqq$ type x.

If type x $\geqq$ type R and type x $\neq$ type R, then type x $>$ type R.

Type x $>$ type R if, and only if, (a) and (b) are satisfied and in (a) inequality holds infinitely often.

Equivalent with these definitions are the following statements:

(c) The type of the element $x \neq 0$ in the group J is the type of the smallest closed subgroup of J which contains x.

(d) If R and R' are two groups of rank 1, then

type R = type R' if, and only if, R and R' are isomorphic;

type R $\leqq$ type R' if, and only if, R is isomorphic with a subgroup of R'.

Note that two groups of rank 1 are isomorphic if each of them is isomorphic with a subgroup of the other one.

If R represents any type of group of rank 1, then

$J(R)$ is the set of all the elements x in J satisfying type R $\leqq$ type x,

$J(R)^-$ is the smallest closed subgroup of J containing the set of all the elements x in J satisfying type R $<$ type x. From the formulae derived for the functions $n(x,p;J)$ it follows that $J(R)$ is a closed subgroup of J and that $J(R) \geqq J(R)^-$. They are invariants of the group J and therefore the rank $d(J,R)$ of $J(R)/J(R)^-$ is also a group invariant.

If J has a basis, i.e. if J is a direct sum of groups of rank 1, then there are in every decomposition of J into direct summands of rank 1 exactly $d(J,R)$ summands isomorphic with R.

For let $J = \sum_{v} J(v)$ be a decomposition of J into direct summands J(v) of rank 1. Then $J(R) = \sum_{\text{type } R \leqq \text{type } J(v)} J(v)$ and

$$J(R)^- = \sum_{\text{type } R < \text{type } J(v)} J(v),$$

since $n(\sum_{j=1}^{k} r(j)b(j),p;J) = \min_{1\leq j\leq k} (n(r(j)b(j),p;J))$ holds for elements $b(j)$ of a basis and rational numbers $r(j)$ such that $r(j)b(j) \neq 0$ exists in J; and this proves our statement. From this statement follows the

Theorem 2: Suppose that the group J has a basis. Then the group J and the group J' are isomorphic if, and only if, J' has a basis and $d(J,R) = d(J',R)$ for every type R.

P-adic groups: Suppose that A is a group such that $p^iA \neq 0$, $p^{\omega}A = 0$. All the elements of A/pA are of order (1 or) p and therefore there exists a basis B of A mod pA, i.e. to every element $x \not\equiv 0$ mod pA of A there exists a uniquely determined finite subset $b(1), \ldots, b(t)$ of B and uniquely determined integers $c(j)$ such that $0 < c(j) < p$ and $x \equiv \sum_{j=1}^{t} c(j)b(j)$ mod pA. (Proof exactly like the proof of the existence of a basis of a complete group.) Therefore if x is any element of A and B a fixed basis of A mod pA, then $x = \sum_{b \text{ in } B} e(x,b)b + px(1)$ where $0 \leq e(x,b) < p$ and almost all $e(x,b) = 0$. In applying the same on $x(1)$ and so on it follows: there exist numbers $c(x,b,i)$ with $0 \leq c(x,b,i) < p$ such that for every i almost all $c(x,b,i) = 0$ and $x \equiv \sum_{j=0}^{i-1} p^j \sum_{b \text{ in } B} c(x,b,j)b$ mod p^iA

or $$\equiv \sum_{b \text{ in } B} \left(\sum_{j=0}^{i-1} p^j c(x,b,j)\right)b \text{ mod } p^iA.$$

Since the intersection $p^{\omega}A$ of all the p^iA is 0, not only every element x determines uniquely the coefficients $c(x,b,i)$ but conversely: $x = y$ if, and only if, $c(x,b,i) = c(y,b,i)$ for every b and every i. Furthermore $c(x \pm y,b,i) \equiv$ $\equiv c(x,b,i) \pm c(y,b,i)$ mod p.

Therefore it is possible to write

$x = \sum_{b \text{ in } B} d(x,b)b$ where $d(x,b) = \sum_{i} p^i c(x,b,i)$ is an integral p-adic number and for every integer i almost all $d(x,b)$ are $\equiv 0$ mod p^i.

Note firstly that it is quite possible that an infinity of $d(x,b) \neq 0$ and that secondly not all possible developments $\sum_{b \text{ in } B} d(b)b$ where the $d(b)$ are integral

p-adic numbers almost all of which are $\equiv 0 \bmod p^i$ exist in A.

But there exists an essentially uniquely determined group $\overline{A}$ which contains A such that every basis B of A mod pA is a basis of $\overline{A}$ mod $p\overline{A}$ and such that in $\overline{A}$ for every basis of $\overline{A}$ mod $p\overline{A}$ all possible developments exist. An invariant construction of $\overline{A}$ is the following: a defining series of A is a series C(i) such that C(i) is a class of A mod p^iA and $C(i+1) < C(i)$. The intersection of the classes of a determining series is at most one element of A and, if it is an element of A, then this element may be identified with its defining series. The sum of the defining series C(i) and D(i) is the defining series C(i) + D(i). Then the set of all the defining series becomes a suitable group $\overline{A}$, as can be verified easily.

$\overline{A}$ is the p-adic closure of A.

If in particular Z is an infinite cyclic group, then its p-adic closure $\overline{Z}$ exists and is exactly the additive group of all the integral p-adic numbers. $\overline{Z}/p\overline{Z}$ is a cyclic group of order p and therefore holds: Every closed subgroup of $\overline{Z}$ is a direct indecomposable group. There exists an infinity of different closed subgroups of $\overline{Z}$, but a characterization of the possible different types of closed subgroups of $\overline{Z}$ is an unsolved problem.

Note that all the (formerly defined) type-invariants $d(\overline{Z},T) = 0$ with the exception of one $d(\overline{Z},R) = 2^{\aleph_0}$ But $\overline{Z}$ is not a direct sum of a continuum of groups of rank 1 and therefore not characterized by these type-invariants.

Similarly it is possible to derive from the p-adic closures of groups J (if existent) indecomposable groups. Whether there exist other types of undecomposable groups seems unknown though rather probable.

Reducible is the element $x \neq 0$ in the group J if $x = \sum_{i=1}^{k} x_i$ and type x $<$ type x_i for every i. The element $x \neq 0$ is irreducible if $x = \sum_{i=1}^{k} x_1$ and type $x \leq$ type x_i for every i implies that type x = type x_i holds for at least one i.

Theorem 3: Suppose that the group J is a direct sum of groups of rank 1. Then the subgroup S of finite rank is a direct summand of J if, and only if,

(a) S is closed in J, mod $J(R)^-$ for every type R;

(b) the elements irreducible in S are also irreducible in J.

Proof: 1. If J = S + T, then S satisfies (a) and (b). For if s, s' are in S, t, t' in T and n(s'+t') = s+t, then ns' = s. Furthermore, n(s+t, p; J) = n(s, p; J) if n(s, p; J) = n(t, p; J) and therefore type (s+t) $\leq$ type s and type (s+t) $\leq$ type t. Hence $J(R)^- = S(R)^- + T(R)^-$ for every type R and therefore $J/J(R)^-$ and $S/S(R)^- + T/T(R)^-$ are isomorphic and this implies (a). If finally the element s in S is reducible in J, then there exist elements s_i in S, t_i in T such that $s = \sum_{i=1}^{k} (s_i + t_i)$ and type s $<$ type (s_i+t_i) for every i. But since J is the direct sum of S and T, it follows that $s = \sum_{i=1}^{k} s_1$ and, since type s $<$ type $(s_i + t_i) \leq$ type s_i (as proved before), it follows that the element s of S is also reducible in S, i.e. (b).

2. If S is of finite rank and J a direct sum of groups of rank 1, then S is a subgroup of a direct summand of J which is a direct sum of a finite number of groups of rank 1.

For let G = (s(1), ..., s(k)) be a greatest linearly independent subset of S and B a basis of J. Then there exists a finite number of elements b(j) in B and rational numbers r(i, j) such that r(i, j)b(j) exists in J and $s(i) = \sum_{j=1}^{h} r(i,j)b(j)$. Furthermore, if s is any element in S, then there exist integers $w \neq 0$, w(i) such that $ws = \sum_{i=1}^{k} w(i)s(i)$ and therefore s is contained in the direct sum of the closed subgroups of J which are generated by the elements b(j).

3. In this section of the proof it will be shown that it is sufficient to prove the following statement:

(3:1) Suppose that the group J is the direct sum of a finite number of groups of rank 1, that the subgroup S of J satisfies the conditions (a) and (b) and that

(c) all the elements $s \neq 0$ of S are of the same type S* and that type S* $\leq$ type R for every type R of groups of rank 1 such that $d(J, R) \neq 0$, and that $d(J,S^*) \neq 0$.

(As proved before, $d(J,R) \neq 0$ is equivalent with the fact that in every decomposition of J into direct summands of rank 1 there occur summands isomorphic with R.)

Then S is a direct summand of J.

For let us suppose that (3:1) holds true and that S is a subgroup of the direct sum J of groups of rank 1 which satisfies (a) and (b). Then from sec. 2 of the proof we can assume that J is of finite rank and since our Theorem clearly holds true if J is a group of rank 1 we may assume (by induction) that the Theorem holds true for all the groups J of a rank smaller than the rank of the considered group and therefore in particular for every proper direct summand of J.

Since J is of finite rank and a direct sum of a finite number of groups of rank 1, it follows that $J = \sum_{i=1}^{k} R_1$ where the R_1 are groups of rank 1 whose type is uniquely determined and therefore there exists at least one number w such that type $R_1 \leq$ type R_w implies type R_i = type R_w, i.e. such that R_w is of minimal-type amongst the R_i.

Suppose now that R_v is also of minimal-type though type $R_w \neq$ type R_v. Then $J(R_w)$ is a proper subgroup and direct summand of J and therefore the intersection S' of S and $J(R_w)$ is a direct summand of $J(R_w)$, i.e. $J(R_w) = S' + T$. Therefore if $J = J'' + J(R_w) = J'' + T + S'$, and, if S" is the intersection of $J'' + T$ and S, it follows that $S = S' + S''$. But since the rank of $J'' + T$ is also

smaller than the rank of J, provided $S' \neq 0$ it follows from the induction hypothesis that S" is also a direct summand of J" and therefore that S is a direct summand of J. Since by (b), if $S \neq 0$, for at least one minimal-type R_w, the intersection of S and $J(R_w)$ is $\neq 0$, it can be assumed that

$$\text{type } R_w \leq \text{type } R_i \text{ for every } i.$$

Then $J = U + J(R_w)^-$ where U is a direct sum of groups of type R_w. Let S_w be the intersection of S and $J(R_w)^-$. Since the rank of $J(R_w)^-$ is smaller than the rank of J, it follows from the induction-hypothesis that S_w is a direct summand of $J(R_w)^-$ i.e. $J(R_w)^- = S_w + V$. Therefore if S_v is the intersection of S and U + V it follows that $S = S_v + S_w$. Since the rank of U + V is smaller than the rank of J provided that $S_w \neq 0$, S_v is, by the induction-hypothesis, a direct summand of U + V and therefore S a direct summand of J if only $S_w \neq 0$. But if $S_w = 0$, then because of (b) the condition (c) is satisfied and therefore by (3:1) S is a direct summand of J.

Thus it has been proved that it is sufficient to prove (3:1).

4. In this section it shall be shown that it is sufficient to prove the following particular case of (3:1):

(4:1) Suppose that J is a direct sum of a finite number of groups of rank 1, that the subgroup S of J satisfies the conditions (a), (b), (c) and (d) If $s \neq 0$ is an element of S, then $n(s, p; J)$ is for almost every prime number p either 0 or infinite.

Then S is a direct summand of J.

For let P be the set of all the prime numbers p such that S is not p-complete, let $n(p) = n(s, p; J)$ for a fixed element $s \neq 0$ in S and every p in P, and for every closed subgroup T of J let T_P be the intersection of all the groups $p^{n(p)}T$ for p in P. If S is a closed subgroup of J, then S_P is the intersection of S and J_P. If J and S satisfy the conditions (a) - (c), then J_P and S_P satisfy

the conditions (a) - (d) and therefore if (4:1) holds true it follows that S_p is a direct summand of J_p, i.e. $J_p = S_p + T'$. If T is the smallest closed subgroup of J which contains T', then $T_p = T'$ and therefore $J = S + T$. Hence it is sufficient to prove (4:1).

5. Proof of (4:1):

From the assumptions it follows that

$$J = \sum_{i=1}^{k} (b(i)) + J(S^*)^-$$

(*) : where type b(i) = type (b(i)) = type s for every $s \neq 0$ in S

and $n(b(i), p; J)$ is either 0 or infinite. In particular therefore J is p-complete for every prime number p such that S is p-complete.

If $b(i)' = b(i)$ for $i \neq h$

$= b(h) + f$ with f in $\sum_{i \neq h} (b(i)) + J(S^*)^-$ for $i = h$;

then the elements b(i)' satisfy also the conditions (*).

We are now going to prove that there exists a set of elements b'(i) which satisfy (*) such that at least one of these elements b'(i) is contained in S. To prove this, note that every element $s \neq 0$ of J has the form

$$s = \sum_{i=1}^{k} h(s,i)m(s,i)^{-1}b(i) + s^*$$

where h(s,i) and m(s,i) are relatively prime integers (m(s,i) = 1 if h(s,i) = 0), m(s,i) is prime to every prime number p in P [such that S and J are not p-complete] and where s^* is in $J(S^*)^-$. From the conditions (b) and (c) no element $s \neq 0$ in S is contained in $J(S^*)^-$ and therefore, if $S \neq 0$, there exist elements s in S such that h(s,i) is positive for at least one i. Therefore there exists a number $g = g(b(1), \ldots, b(k))$ and an element w in S such that $0 < g \leq |h(s,i)|$ for every s in S and $h(s,i) \neq 0$ and $w = gb(i_o) + \sum_{i \neq i_o} h(w,i)m(w,i)^{-1}b(i) + w^*$. If g = 1, then the elements $b'(i_o) = w$, $b'(i) = b(i)$ for $i \neq i_o$ satisfy (*) and one of these elements is in S. If $1 < g$, then

$$h(w,i) = gh'(i) + h''(i) \text{ with } 0 \leq h''(i) < g$$

and eht elements $b'(i) = b(i)$ for $i \neq i_o$

$$= b(i_o) + \sum_{i \neq i_o} h'(i)m(w,i)^{-1}b(i)$$

satisfy (*). Then the element w in S has the form:

$$w = gb'(i_o) + \sum_{i \neq i_o} h''(i)m(w,i)^{-1}b(i) + w^*$$

If $h''(i) = 0$ for every $i \neq i_o$, then*) from (a) the element $g^{-1}w$ exists in S and

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)More accurately: there exists by (a) in S an element u such that gu-w is in J(S). But since the intersection of S and J(S*) is 0 because of (r) it follows that gu = w or that $g^{-1}w$ exists in S.

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this is not possible, since $1 < g$. Therefore at least one of the $h''(i)$ is $\neq 0$ and therefore $g(b'(1), \ldots, b'(k)) < g(b(1), \ldots, b(k))$. Hence it follows by complete induction with regard to g that there exists a set $b''(1), \ldots, b''(k)$ satisfying (*) and containing at least one element of S. Then

$$J = (b''(z)) + \sum_{i \neq z} (b''(i)) + J(S^*)^- = (b''(z)) + Z$$

where $b''(z)$ in S and therefore, if S' is the intersection of Z and S, $S = (b''(z)) + S'$. But since Z is of lower rank than J it follows that S' is a direct summand of Z and therefore that S is a direct summand of J and this completes the proof of the Theorem.

Note that we have actually proved slightly more; namely the

Corollary 1: If J is a direct sum of groups of rank 1 and if the subgroup S of finite rank satisfies the conditions (a) and (b) of the Theorem, then there exists a basis of J which contains a basis of S.

Corollary 2: Suppose that the group J is a direct sum of groups of rank 1. Then every closed subgroup of finite rank of J is a direct summand of J if, and only if, all the non-complete direct summands of rank 1 of J are isomorphic, i.e. if, and only if, all the elements $\neq 0$ of J are of the same type, mod the greatest complete subgroup C of J.

Proof: Obviously every closed subgroup of J satisfies (a) and (b) of the Theorem if the condition of the Corollary is satisfied, since the greatest closed subgroup C of J is a direct summand of J. — If conversely the condition of Corollary 2 is not satisfied, then $J = U + V + W$ where U and V are groups of rank 1 of different type and neither of them is complete.

Case 1: type U < type V.

Since V is not complete, there exists a prime number p such that V (and therefore U) is not p-complete and therefore there exist elements u and v in U and V respectively such that $n(u, p; J) = n(v, p; J) = 0$. The closed subgroup (pu+v) of J and U + V which is generated by pu+v does not satisfy (a) though (b) is satisfied and though, which is noteworthy, the types of U and of V form an ordered set.

Case 2: type U ≠ type V but neither is smaller than the other one.

Then let u and v be elements ≠ 0 in U and V respectively. The closed subgroup (u+v) of J and U+V which is generated by u+v does not satisfy (b) since type (u+v) < type u and < type v as has been proved in sec. 1 of the proof of the Theorem.

(u+v) does not satisfy (a) and therefore it is undecided whether condition (b) depends on (a) or not.

Remark: Note that the finite-rank-condition is essential as the following example shows:

$J = \sum_{i=1}^{\infty} (a_i)$ is the direct sum of the infinite cyclic groups which are generated by the elements a_i and the subgroup S is the direct sum of the cyclic subgroups generated by the elements $a_{i-1} - 2a_i$. Then S is a closed subgroup of J, but J/S is 2-complete and therefore S not a direct summand of J.

Theorem 4: The group J is a direct sum of isomorphic groups of rank 1 if, and only if,

(a) the elements $\neq 0$ of J and of J/S have the same type for every closed subgroup S of finite rank;

(b) J is a direct sum of countable groups.

Proof: If J is a direct sum of isomorphic groups of rank 1, then all the elements $\neq 0$ of J have the same type as these direct summands of rank 1. Since by Corollary 2 of Theorem 3 every closed subgroup S of finite rank is a direct summand, every class-group J/S is isomorphic with a subgroup of J and this proves the necessity of the condition (a). Condition (b) is obvious since groups of rank 1 are countable.

Suppose now conversely that the group J satisfies the conditions (a) and (b). Since every direct summand of J is a closed subgroup of J and since therefore every direct summand of J satisfies also the condition (a), it follows from (b) that it is sufficient to consider countable groups J.

Let $b \neq 0$ be any element of J (since from (a) all the elements $\neq 0$ have the same type, the choice of b is of no importance), let P be the set of all those prime numbers p such that J is not p-complete, and let $n(p) = n(b,p;J)$ for every p in P. If T is any subgroup of J, then T_P is the intersection of all the subgroups $p^{n(p)}T$ for p in P. If S is a closed subgroup of J, then S_P is the intersection of S and J_P. Therefore the condition (a) is satisfied in the group J_P and if $c \neq 0$ is any element of J_P, then $n(c,p;J_P) = 0$ for almost every p in P (of course J and J_P are q-complete for the same prime numbers q). Since, furthermore, every basis of J_P is a basis of J, it follows that it is sufficient to prove our statement for countable groups J, satisfying (a), such that for every element $x \neq 0$ of J holds: n(x,p;J) is either 0 or infinite for almost every prime number p.

Every countable group J contains closed subgroups J(i) such that the rank of J(i) is i,

$$J(i) < J(i+1) \text{ and } J(i+1)/J(i) \text{ is a group of rank 1,}$$

either there exists an integer m such that J = J(m) or J is the join of all the groups J(i).

(*) for every i there exists a basis of J(i) which contains a given basis of J(i-1).

To prove (*) remember that J(i)/J(i-1) is from (a) a group of the same type as the closed subgroups of rank 1 of J. Therefore J(i)/J(i-1) contains an element $x^* \neq 0$ such that $n(x^*,p;J(i)/J(i-1))$ is either 0 or infinite. Let B(i-1) be any basis of J(i-1) and x any element in the class x^* of J(i)/J(i-1). If r runs through all the rational numbers with a denominator prime to every p in P (where P contains all those prime numbers p such that J is not p-complete), then every rx exists in J and these elements represent exactly the classes of J(i)/J(i-1). Therefore J(i) = J(i-1) + (x) where (x) is the closed subgroup generated by x, and the elements of B(i-1) and the element x form therefore a basis of J(i), and this proves (*).

Since J(1) is a group of rank 1, there exists a basis of J(1) and therefore it follows from (*) by complete induction that there exist elements b(1), b(2), ..., b(i), ... such that the elements b(j) with $1 \leqq j \leqq i$ form a basis of J(i). Then the set B of all these elements b(i) forms a basis of J and this completes the proof of the Theorem.

Remark 1: The following statement contains a more constructive proof of the existence of a basis in some special cases:

Suppose that the group J is of finite rank, is q-complete for every prime number $q \neq p$ and does not contain any almost p-complete subgroup. Then every basis for J mod pJ is a basis of J.

Proof: Suppose that B is a basis of J mod pJ. Then B is linearly independent, since every linear relation between elements of B leads to a dependence mod pJ. Therefore B is finite since J is of finite rank, i.e. B = (b(1),...,b(k)).

Let B' be the smallest closed subgroup of J which contains B. Then B' is of finite rank. Furthermore if x is any element in J, then there exists an element y in B' such that $x \equiv y \bmod pJ$ i.e. $x \equiv pz \bmod B'$ and therefore $J/B' = p(J/B')$. Since J is not almost p-complete it follows that $J = B'$.

Hence if x is any element in J, $x \neq 0$, then there exist integers $n \neq 0$, $n(i)$ such that $nx = \sum_{i=1}^{k} n(i)b(i)$. Since B is a basis of $J \bmod pJ$, it follows that $n \equiv 0 \bmod p$ implies $n(i) \equiv 0 \bmod p$ for every i. Since J is q-complete for every $q \neq p$, this implies that the elements $n^{-1}n(i)b(i)$ exist in J and therefore B is a basis of J.

Note that the statement and its proof become false if the rank of j is infinite.

2. That the condition (a) is indispensable show e.g. suitable subgroups of the group of all the integer p-adic numbers.

3. If the type of all the elements $\neq 0$ of J is the type of the group of all the rational numbers, then J is complete and therefore a direct sum of groups of rank 1 so that in this case the condition (b) is superfluous. But apart from this case things behave quite differently. For let R be any group of rational numbers which is not complete, p a prime number such that R is not p-complete. Denote by $W(R)$ the group of all the sequences of elements in R and by $W(R,p)$ the group of all those sequences r_i of numbers of R which satisfy: For every integer k there exists to almost every r_i a solution of $p^k x = r_i$ in R. $W(R,p)$ is a closed subgroup of $W(R)$,

$W(R)$ and $W(R,p)$ both contain a continuum of elements. $W(R,p)$ contains the countable subgroup $V(R)$ of all those sequences r_i such that almost every $r_i = 0$. If x is any element of $W(R,p)$, then there exists an element y in $V(R)$ such that

$x \equiv y \bmod pW(R,p)$ and therefore $W(R,p)/pW(R,p)$ .countable. Therefore $W(R,p)$ cannot have a basis, since a basis of $W(R,p)$ would contain a continuum of elements and would also form a basis of $W(R,p)$ mod $pW(R,p)$ and this is impossible.

Since $W(R,p)$ is not a direct sum of groups of rank 1, $W(R)$ is also not a direct sum of groups of rank 1, as follows from

Theorem 5: Suppose that J is a direct sum of isomorphic groups of rank 1, that $P = P(J)$ is the set of all the prime numbers p such that J is not p-complete, that the numbers $n(p)$ $(=n(p,J))$ for p in P are determined in such a way that for at least one element $x \neq 0$ holds $n(p) = n(x,p;J)$ (or nearly equivalent: that for every element $x \neq 0$ in J holds $n(p) = n(x,p;J)$ for almost every p in P).

A. If $n(p) = 0$ for almost every p in P, then a subgroup S of J which is q-complete for every prime number q such that J is q-complete is a direct sum of groups of rank 1 (which are isomorphic with the direct summands of rank 1 of J).

B. If the subgroup S of J satisfies the two conditions:

(a) S is q-complete for every prime number q such that J is q-complete;

(b) for almost every p in P and every s in S the element $p^{-1}s$ exists in S if, and only if, it exists in J;

then S is a direct sum of groups of rank 1 (which are isomorphic with the direct summands of rank 1 of J).

Proof: Ad A.: There exists a basis B of J such that $n(b,p;J) = 0$ for every b in B and every p in P. Let B be well-ordered in a certain way: $B = (b(1), \ldots, b(v), \ldots)$.

If x is any element in J, then $x = \sum_v h(x,v)m(x,v)^{-1}b(v)$ where $h(x,v)$ and $m(x,v)$ are relatively prime integers, $m(x,v) = 1$ for $h(x,v) = 0$, $m(x,v)$ prime to every p in P and almost every $h(x,v) = 0$. Therefore to every $x \neq 0$ there exists a uniquely determined number $v = v(x)$ such that $h(x,v(x)) \neq 0$, $h(x,w) = 0$ for $v(x) < w$.

Either there does not exist any element $s \neq 0$ in S such that $v(s) = z$ for a given ordinal z; then $s(z) = 0$. Or there exist elements $s \neq 0$ in S such that $v(s) = z$, and then amongst these elements there exists one such that $h(s,z)$ has a smallest positive value $h(z)$. Then there exists an element $s(z)$ in S such that $s(z) = h(z)b(z) + \sum_{v<z} h(s(z),v)m(s(z),v)^{-1}b(v)$. The set C of all the elements $s(z) \neq 0$ is an independent set.*) Furthermore if $s \neq 0$ is any element in

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*) The idea of determining C in this way is due to Zippin.

S, then $m(s,v(s))^{-1}s(v(s))$ exists in S, since S is p-complete for every prime number p dividing $m(s,v)$. Furthermore $h(v(s))$ is a divisor of $h(s,v(s))$ and therefore there exists a rational number r such that $rs(v(s))$ exists in S and such that $s - rs(v(s))$ does not depend on elements $b(w)$ with $v(s) \leqq w$.

Since by complete (transfinite) induction we may assume that all the elements t in S with $v(t) < v(s)$ are expressible by the elements in C, it follows now that every element of S is expressible by elements in C, and therefore that C is a basis of S.

Ad B.: Suppose that S satisfies the conditions (a) and (b) and that P' is the subset of P such that for every p in P' and every element s in S the element $p^{-1}s$ exists in S if, and only if, it exists in J. Then there exists only a finite number of prime numbers p in P which are not in P'.

For every subgroup T of J let $T_{P'}$ be the intersection of all the groups $p^{n(p)}T$ for p in P'. Then $S_{P'}$ is the intersection of S and $J_{P'}$ and $S_{P'}$ is a subgroup of $J_{P'}$ satisfying the conditions of A. At the same time $J_{P'}$ satisfies the conditions of A, and therefore there exists a basis of $S_{P'}$. But from the construction of $S_{P'}$ it follows that every basis of $S_{P'}$ is also a basis of S, and this completes the proof of the Theorem.

Remark 1: If J is a direct sum of infinite cyclic groups, then every subgroup of J is a direct sum of cyclic groups (by A).

2. Since every group J is a subgroup of a complete group and since complete groups are direct sums of groups of rank 1, in general not every subgroup of a direct sum of isomorphic groups of rank 1 is a direct sum of groups of rank 1.

3. The subgroup $W(R,p)$ of $W(R)$ satisfies the conditions (a) and (b) of B, and therefore $W(R)$ is not a direct sum of groups of rank 1.

Theorem 6: $J(R)$ is the direct sum of $J(R)^-$ and a direct sum of groups of rank 1 if, and only if, $J(R)/J(R)^-$ is a direct sum of groups isomorphic with R.

Proof: If $J(R) = J(R)^- + \sum_v H(v)$ where the groups $H(v)$ are groups of rank 1, then each $H(v)$ is isomorphic with R, since the elements $\neq 0$ in $H(v)$ are contained in $J(R)$ but not in $J(R)^-$. Furthermore $J(R)/J(R)^-$ and $\sum_v H(v)$ are isomorphic, and this proves the necessity of the condition.

Suppose now that the condition is satisfied, that $r \neq 0$ is an element in R, that P is the set of all the prime numbers p such that R is not p-complete, and that $n(p) = n(r,p;R)$ for every p in P. If T is any group, then let T_P be the intersection of all the groups $p^{n(p)}T$ for p in P. This definition implies the following equalities:

$$J(R)_P = J_P(R_P) \text{ and } J(R)^-_P = J_P(R_P)^-.$$

Hence from the condition satisfied by J and R it follows that $J_P(R_P)/J_P(R_P)^-$ is a direct sum of groups isomorphic with R_P. Furthermore $n(r,p;R_P) = 0$ for every p in P.

If now there exists a subset B of $J_P(R_P)$ such that $J_P(R_P) = \sum_{b \text{ in } B} (b)_P + J_P(R_P)^-$, where $(b)_P$ is the smallest closed subgroup of $J_P(R_P)$ containing b, then the above identities imply that

$$J(R) = \overline{\sum_{b \text{ in } B}} (b) + J(R)^-$$

where (b) is the smallest closed subgroup of J(R) containing b.

Therefore it is sufficient to prove our statement in the particular case where n(p) = 0 for every p in P. Then let B* be a basis of $J(R)/J(R)^-$ and B any set of elements representing the classes in B* mod $J(R)^-$. The basis B* can and shall be chosen in such a way that $n(b^*,p;J(R)/J(R)^-) = 0$ for every p in P. Then sb* exists in $J(R)/J(R)^-$ if, and only if, the denominator of the rational number s is prime to every p in P and the existence of sb* implies therefore also the existence of sx for every x in J(R). Hence the closed subgroup of J(R) generated by B is the direct sum of the closed subgroups (b) generated by the elements b in B and since this subgroup represents all the classes of $J(R)/J(R)^-$ it follows that $J(R) = \overline{\sum_{b \text{ in } B}} (b) + J(R)^-$.

Remark: It is not sufficient to assume that $J(R)/J(R)^-$ is a direct sum of groups of rank 1 as the following example shows:

Let (a) be an infinite cyclic group, (b) a 2-complete group such that $n^{-1}b$ does not exist in (b) if n is an odd number ($\neq 1$). Let J be the group which contains the direct sum (a) + (b) and elements c(p) for every odd prime number p such that pc(p) = a + (p+2)b. If Z is the type of the infinite cyclic groups, then J(Z) = J, $J(Z)^-$ = (b), $J(Z)/J(Z)^-$ is of rank 1, but not isomorphic with Z, since $n(J(Z)^- + a,p;J(Z)/J(Z)^-) = 1$ for every odd prime number p.

Corollary 1: Suppose that the types R with $J(R) \neq J(R)^-$ form an ordered set such that every subset contains a greatest element, i.e. an inversely well-ordered set. Then J is a direct sum of groups of rank 1 if, and only if,
(a) for every type R, $J(R)/J(R)^-$ is a direct sum of groups isomorphic with R, and
(b) no element $x \neq 0$ of type X is contained in $J(X)^-$.

Proof: If J is the direct sum of the groups Z(v) of rank 1, then J(R) is the direct sum of those groups Z(v) which satisfy type R $\leq$ type Z(v), $J(R)^-$ is the

direct sum of those groups Z(v) which satisfy type R < type Z(v), as has been proved before, and therefore J(R) is the direct sum of $J(R)^-$ and those groups Z(v) whose type is exactly R. This proves the necessity of the conditions.

If on the other hand the condition is satisfied, then from Theorem 6 there exists a direct sum Z(R) of groups of type R such that

$$J(R) = Z(R) + J(R)^-.$$

The subgroup Z of J which is generated by these groups Z(R) is the direct sum of all these groups Z(R). For if z(1), ..., z(w) are elements belonging to different groups Z(R), then the type of $z(i) \neq 0$ is exactly the type R_i belonging to the group $Z(R_i)$ which contains z(i). Therefore if all these elements z(i) are $\neq 0$, then the type of their sum is exactly the type of the element z(i) with the lowest type (of all the types R(i)) since these types form a finite ordered set. Hence their sum cannot be 0 and therefore $Z = \sum_R Z(R)$.

Since x = u + v with u in Z(R(x)), R(x) the type of x, and v in $J(R(x))^-$ and therefore R(x) < R(v) by (5) and therefore after a finite number of steps it follows that x is contained in Z, i.e. that J is a direct sum of groups of rank 1.

Remark 1: The condition that the types R with $J(R) \neq J(R)^-$ form not only an ordered set, but also do not contain infinitely increasing subsets is essential as the following example shows:

Suppose that the groups R(i) of rational numbers satisfy

type R(i) < type R(i+1) for every integer i

and let W be the group of all the sequences r(i) of rational numbers such that the i-th coordinate r(i) is in R(i). Then the type of the sequence r(i) is exactly the type of the first coordinate $r(j) \neq 0$ in this sequence and therefore $W(R) = W(R)^-$ if $R \neq R(i)$, $W(R(i))^- = W(R(i+1))$ and $W(R(i))/W(R(i))^-$ is isomorphic R(i). Therefore W cannot be a direct sum of groups of rank 1 since W contains a continuum of elements and by Theorem 2 would be countable if it should be a

direct sum of groups of rank 1.

2. That the condition ($\mathfrak{b}$) is indispensable, the following example shows:

Let $p(1), p(2), \ldots, p(i), \ldots$ be any enumeration of the prime numbers and J a group generated by elements $(i_1, \ldots, i_k)$ where k runs through all integers, $i_j = 1, 2$ which satisfy the following relations:

$$(i_1, \ldots, i_k) = (i_1, \ldots, i_k, 1) + (i_1, \ldots, i_k, 2),$$

$$n(i_1, \ldots, i_k), p; J) = 0 \text{ if } p \neq p(2j + i_j) \text{ for every } j = 1, \ldots, k$$

$$= \infty \text{ if } p = p(2j + i_j) \text{ for a certain } j.$$

Then every element $(i_1, \ldots, i_k)$ is the sum of two elements of higher type and therefore $J(R) = J(R)^-$ for every type R. On the other hand J is countable but of infinite rank.

Note for example that ($\mathfrak{b}$) is satisfied if the set of all the types of elements $\neq 0$ in J is ordered.

Corollary 2: Suppose that J is of finite rank and that the types R with $J(R) \neq J(R)^-$ form an ordered set. Then J is a direct sum of groups of rank 1 if, and only if,

for every type R, $J(R)/J(R)^-$ is a direct sum of groups isomorphic with R.

Proof: The necessity of the condition is a consequence of Theorem 6. If on the other hand the condition is satisfied, then we prove:

If X is the type of the element $x \neq 0$ of J, then x is contained in $J(X)$ but not in $J(X)^-$, i.e. the condition ($\mathfrak{b}$) of Corollary 1.

This statement is true if J is of rank 1 and therefore we may assume that it is true for every group of lower rank which satisfies the condition. In particular therefore our statement is true for every subgroup J(Y) where Y is any type such that $X < Y$, since x is contained in J(X) but not in J(Y). If x should be contained in $J(X)^-$, then there exist elements x(i) in J such that $x = \sum_{i=1}^{k} x(i)$

and type x < type x(i). Therefore, as noted before, every x(i) is contained in $J((x(i))$ but not in $J((x(i)))^-$, and hence the types of the elements x(i) form by the general assumption an ordered set, i.e. type x(1) $\leqq$ ... $\leqq$ type x(k). But this implies type x(1) $\leqq$ type x in contradiction to our assumption i.e. x is not contained in $J(X)^-$.

The statement we have just proved implies that the types of elements of J form an ordered set.

There exists an element of lowest type in J, since otherwise there would exist elements x(i) such that type x(i+1) < type x(i) and since that would imply $J((x(i)) < J((x(i+1))$ for every i. But this is impossible, since the groups J(R) are closed and therefore the ranks of the groups $J((x(i))$ would grow infinitely in contradiction with the finiteness of the rank of J.

Let now L be the type of an element of lowest rank in J. Then $J = J(L) \neq J(L)^-$ and therefore from our assumption and Theorem 6: $J = S + J(L)^-$ where S is a direct sum of groups isomorphic with L. S is not vacuous, since there are elements of type L in J and therefore $J(L)^-$ is a group of lower rank than J though satisfying our condition. Hence we may suppose that the Corollary holds for $J(L)^-$, i.e. $J(L)^-$ and therefore J are direct sums of groups of rank 1 and this completes the proof.

Remark: The condition that the types R with $J(R) \neq J(R)^-$ form an ordered set is indispensable as the following example shows:

Let J be a group containing a pair of elements a, b such that a and b are independent,

$n(a,p;J) = 0$ for $p \neq 2$, $n(b,p;J) = 0$ for $p \neq 3$, $n(a+b,p;J) = 0$ for $p \neq 5$,

$= \infty$ for $p = 2$, $= \infty$ for $p = 3$, $= \infty$ for $p = 5$.

If J is characterized by these relations, then

$J(R(2)) = (a)$, $J(R(3)) = (b)$, $J(R(5)) = (a+b)$, $J(R(p))^- = 0$ for $p = 2, 3, 5$

where R(p) contains all those rational numbers whose denominator is a power of p, and $J(R) = J(R)^-$ for all the other types R.

This group J satisfies all the conditions of Corollary 2, and the types R with $J(R) \neq J(R)^-$ form a finite set. But the group J is not a direct sum of groups of rank 1, since the rank of J is 2 but $\sum_R d(J,R) = 3$.

This example shows that we need, instead of the subgroups J(R), another tool for attacking our problem.

J(R+) is the smallext closed subgroup of J which contains all the elements $x \neq 0$ of J satisfying type $x \not< $ type R.

$J(R+)^-$ is the smallest closed subgroup of J which contains all the elements $x \neq 0$ of J satisfying type $x \not\leq$ type R.

Note that J(R+) can contain elements $x \neq 0$ such that type $x <$ type R, and that $J(R+)^-$ can contain elements $x \neq 0$ such that type $x \leq$ type R.

These new characteristic subgroups have the following properties:

(1) $J(R) \leq J(R+)$, $J(R)^- \leq J(R+)^-$, $J(R+)^- \leq J(R+)$.

(2) If J is the direct sum of the groups J(v) of rank 1, then J(R+) is the direct sum of the groups J(v) satisfying type $J(v) \not< $ type R and $J(R+)^-$ is the direct sum of those groups J(v) which satisfy type $J(v) \not\leq$ type R.

(3) If d(J,R+) is the rank of $J(R+)/J(R+)^-$, and if J is a direct sum of groups of rank 1, then all the classes of $J(R+)/J(R+)^-$ are represented by classes of $J(R)/J(R)^-$ and conversely every class of $J(R)/J(R)^-$ represents exactly one class of $J(R+)/J(R+)^-$; and finally d(J,R) = d(J,R+).

Theorem 7: J(R+) is the direct sum of $J(R+)^-$ and of groups of rank 1 if, and only if, $J(R+)/J(R+)^-$ is a direct sum of groups isomorphic with R.

Proof: If $J(R+) = V + J(R+)^-$, then every element $\neq 0$ of V is exactly of the type R and therefore the condition is necessary. Suppose now that the condition is satisfied. Let P be the set of all the prime numbers p such that R is

not p-complete, $r \neq 0$ any element of R and $n(p) = n(r,p;R)$ for every p in P. If A is any group, then let A_P be the intersection of all the groups $p^{n(p)}A$ for p in P. If S is a closed subgroup of J, then S_P is the intersection of S and J_P. If U is a closed subgroup of rank 1 of J, $u \neq 0$ an element of U and $n(u,p;J) < n(p)$ for an infinity of prime numbers p in P, then $U_P = 0$. Hence if $U_P \neq 0$, then there exists an element $u \neq 0$ in U such that $n(p) \leq n(u,p;J)$ for every p in P and $n(u,p;J_P) = n(u,p;J) - n(p)$ for all the other prime numbers p. Hence if U and V are closed subgroups of J of rank 1 such that $U_P \neq 0$ and $V_P \neq 0$, then type U = type V if, and only if, type U_P = type V_P, and type U < type V if, and only if, type U_P < type V_P. Therefore it follows that $J_P(R_P+)$ is the intersection of J_P and $J(R+)$, and that $J_P(R_P+)^-$ is the intersection of J_P and $J(R+)^-$. Since $J(R+)/J(R+)^-$ is a direct sum of groups isomorphic with R, every class of $J_P(R_P+)/J_P(R_P+)^-$ is contained in exactly one class of $(J(R+)/J(R+)^-)_P$ and conversely, i.e. $J_P(R_P+)/J_P(R_P+)^-$ is a direct sum of groups isomorphic with R_P. If therefore our theorem would hold in the special case where all the numbers $n(p) = 0$, then we should have: $J_P(R_P+) = J_P(R_P+)^- + \sum_v J'(v)$ where $J'(v)$ is a group isomorphic with R_P. If $J(v)$ is the smallest closed subgroup of J (or J(R+)) which contains $J'(v)$, then $J'(v) = J(v)_P$. Since, as noted before, essentially the identity

$$(J(R+)/J(R+)^-)_P = J_P(R_P+)/J_P(R_P+)^-$$

holds true, and since the subgroup generated by the $J(v)$ is the direct sum of the $J(v)$, the direct sum of the groups $J(v)$ represents exactly the classes of $J(R+)/J(R+)^-$ and therefore $J(R+) = J(R+)^- + \sum_v J(v)$. So it has been proved that it is sufficient to prove our statement in the special case that $n(p) = 0$ for every p in P.

Then $J(R+)/J(R+)^- = R^*$ has a basis B^* whose elements b^* satisfy $n(b^*,p;R^*) = 0$ for p in P and

$= \infty$ for p not in P.

Then let B be any set of elements in J representing the classes b* in B*. Since B* is independent, B also is independent. Furthermore, every class of $J(R+)/J(R+)^-$ contains exactly one element of the direct sum of (all) groups (b) for b in B and conversely, and therefore $J(R+) = J(R+)^- + \sum_{b \text{ in } B} (b)$. This completes the proof

Corollary: Suppose that the group J is of finite rank. Then J is a direct sum of groups of rank 1 if, and only if, for every intersection S of a finite number of groups $J(R+)^-$ holds

(a) to every element x of S there exists a type R(x) = R such that x is contained in S(R+) but not in $S(R+)^-$;

(b) $S(T+)/S(T+)^-$ is for every type T a direct sum of groups isomorphic with T; and

(c) S(T+) is the intersection of S, and J(T+), $S(T+)^-$ the intersection of S and J(T+).

Proof: If J is the direct sum of the groups Z(v) of rank 1, then $J(R+)^-$ is the direct sum of all those Z(v) whose type is not $\leq$ type R. If S is the intersection of the groups $J(R_i+)^-$ for i = 1, ..., k, then S is the direct sum of the groups Z(v) whose type is neither $\leq$ type R_1, nor $\leq$ type R_2, nor ... $\leq$ type R_k. In particular S is a direct sum of groups of rank 1. To verify (a) consider a basis S* of S and an element $x \neq 0$ of S. Since S* is finite, there occurs in the representation of x by elements of S* an element of lowest type and this type is a suitable type R(x). The condition (b) is essentially a consequence of Theorem 7.

Suppose now that the conditions are satisfied. Then there exists a type R such that $J(R+) \neq J(R+)^-$, provided J contains elements $\neq 0$. Therefore there exists a lowest type L such that $J(L+) \neq J(L+)^-$, since otherwise there would exist a series of types R(i) such that $J(R(i+1)+) < J(R(i)+)$ and this is im-

possible since the rank of J is finite. If L is a lowest type such that $J(L+) \neq J(L+)^-$, then $J = J(L+)$. For if x is in $J(R+)$ but not in $J(R+)^-$, then $J(R+) \neq J(R+)^-$ and therefore type R $\not\leq$ type L, i.e. $J(R+) \leq J(L+)$, i.e. x is contained in $J(L+)$ and therefore from (a), $J = J(L+)$.

From (b) and Theorem 7 there exists a direct sum K of groups isomorphic with L such that $J = J(L+) = K + J(L+)^-$. Since the conditions (a) and (b) are satisfied in $J(L+)^-$ and since $J(L+)^-$ is of lower rank than J, $J(L+)^-$ and therefore J are direct sums of groups of rank 1.

Remark: Condition (a) is needed, since there exist groups J of rank 2 [the example of the remark on p. 26] such that $J(R+) = J(R+)^-$ for every type R.