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Bernhard Hermann Neumann AC. 15 October 1909 –– 21 October 2002

Cheryl E. Praeger

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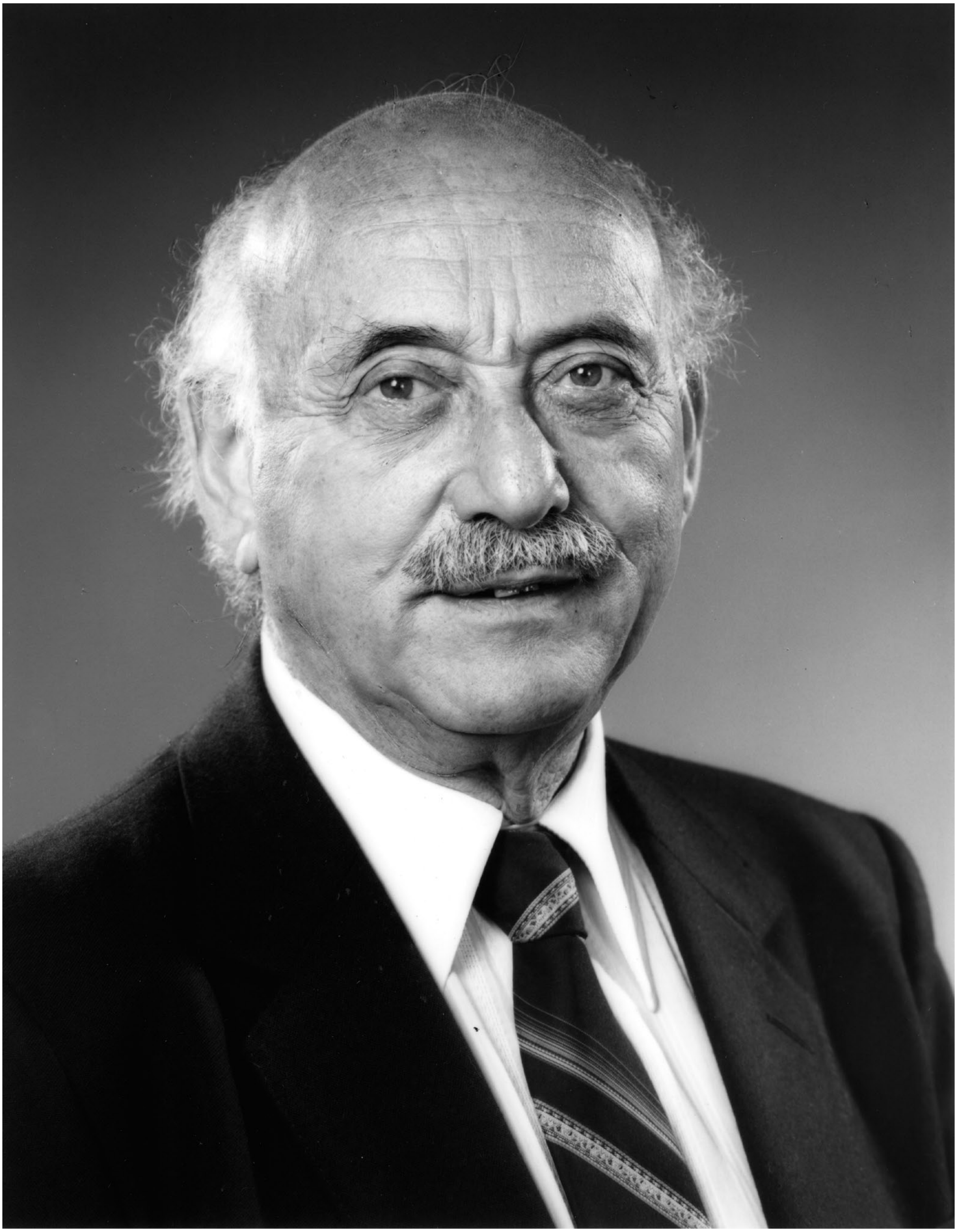

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BERNHARD HERMANN NEUMANN AC

15 October 1909 — 21 October 2002

Elected FRS 1959

BY CHERYL E. PRAEGER

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Bernhard Hermann Neumann was born and educated in Berlin. He held doctorates from Berlin and Cambridge, and mathematical positions at universities in Cardiff, Hull, Manchester and the Australian National University (ANU) in Canberra. Whereas his move to the UK in 1932 was a result of the difficulties he faced as a Jew in finding employment in Germany, his move to Australia in 1962 was to set up a new research Department of Mathematics at the Institute of Advanced Studies at ANU. Bernhard Neumann was famous both for his seminal research work in algebra and also for his strong support of all endeavours in mathematics. His scholarly publications span more than 70 years. His honours include election to the Fellowships of the Royal Society and of the Australian Academy of Science, appointment as Companion of the Order of Australia, and numerous honorary doctorates. To Bernhard it was important to share and spread the joy of doing mathematics.

INTRODUCTION

Bernhard Hermann Neumann was born in Berlin, Germany, on 15 October 1909. After completing his doctorate in Berlin he moved to the UK to seek employment and undertook a second doctorate in Cambridge. His academic work as a mathematician in the UK was interrupted by a period of internment, and then service in the British army, during World War II. Bernhard Neumann was head-hunted from his position in Manchester to become, in 1962, foundation professor and head of a new research mathematics department at ANU. He had an enormous positive influence on the development of mathematics in Australia and the Asia–Pacific region, not only during his time as departmental head at ANU but also in the decades after

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his retirement in 1975. Bernhard died suddenly on 21 October 2002, in Canberra, Australia, shortly after his 93rd birthday. He is survived by his second wife, Dorothea, and the five children from his first marriage (to Hanna, also a mathematician and a Fellow of the Australian Academy of Science at the time of her death in 1971).

Early life and education

Bernhard was named after his two grandfathers. His paternal grandfather, Bernhard Neumann, was a hardware merchant in Karlsruhe in the south of Germany, and Bernhard's father, Richard, was the youngest of four children, born on 15 January 1876. Bernhard's maternal grandfather, Hermann Aronstein, had a large farm, called a *Rittergut*, in Westphalia, and Bernhard's mother, Else, was Hermann's third daughter. Both sides of Bernhard Neumann's family were thus solidly middle-class.

Richard Neumann married Else in 1905 in Berlin. He studied engineering, spent two years in the USA as an engineer and then held a job until the late 1930s in Berlin, in the AEG (Allgemeine Elektrizitätsgesellschaft). Bernhard was the second child and only son. His elder sister Edel-Agathe (known as Eta) was born in 1906 in Berlin, studied physics in Berlin, worked in patent law and patent physics in northeast England until she married Gösta Lindskog, a Swedish pastor and schoolteacher, and then lived in Sweden with Gösta until her death from cancer in 1958. Both of Bernhard's parents were Jewish. Richard was not practising, but Else went to synagogue for the important festivals. Bernhard was also not practising, but despite this he and his first wife Hanna sat on the Council of Jews and Christians in their Manchester days in the late 1950s, wishing to bring reconciliation between the two religions.

Bernhard was four years old when World War I began. The blockade against Germany caused severe shortages and strict rationing, and Bernhard remembered feeling very hungry for much of the war. Thus he knew privation of a type not often met with in Europe today. Bernhard remembered with some affection the arrival of Quakers immediately after the war; they went around schools in Germany feeding the children.

Bernhard attended the Herderschule in Berlin-Charlottenburg from 1916 to 1928. He claims that he was dull as a young child to the extent that an aunt consoled his mother with the words, 'he might yet make an artisan of sorts!!'* A tonsillectomy, which happened at about the time he entered the Herderschule, changed this perception. From then on he took delight in sums and particularly mental arithmetic. Bernhard's father had a student text on calculus that fascinated him. He did all the exercises and thus learned differential calculus (then a university subject) at a very early age, and loved studying the beautiful curves contained in the book. Later, in year 10 of school, he invented three-dimensional analytic geometry for himself.

At school, Felix W. Behrend taught Bernhard mathematics and physics. His teacher's son Felix A. Behrend, a lifelong friend of Bernhard's, became an associate professor of mathematics at the University of Melbourne (see (73)†). The Herderschule also produced G. P. Hochschild, later to become Professor of Mathematics at the University of California,

* Much of the material for this part of the memoir has been drawn from the record of an interview with Bernhard (Crompton 1998) and the essays written by Bernhard in *Selected works of B. H. Neumann and Hanna Neumann* (120).

† Numbers in this form refer to the bibliography at the end of the text.

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Berkeley. Throughout his life, Bernhard made friends quickly and easily, and kept in contact with a large number of people. His remarkable memory was no doubt aided by his famous pocket diary in which he recorded birthdays and other significant details.

Bernhard left the Herderschule with the Abitur, the qualification that was an entitlement to university entrance. His university education began with two semesters at Freiburg University in 1928/29, studying under Alfred Loewy, Gustav Mie and Lothar Heffter. Heffter, Bernhard's favourite teacher, was mathematically active until his 100th year. Bernhard dedicated his paper (68) for Heffter's 100th birthday. However, on submitting the paper Bernhard learned from the editor 'by return' that Heffter had just died, less than six months before his 100th birthday ((120), p. 587). So the paper is dedicated to his memory. Bernhard was always keenly interested in people who continued to be very active academically after retirement, and later loved the fact that his own life was so long. When a mutual friend died, Bernhard is reported as saying, 'but he was only 85, wasn't he?' The other side of the coin was that he became one of the great supporters of the young, as we shall see below.

After the Freiburg experience, Bernhard returned home to Berlin to study at the Friedrich-Wilhelms-Universität, later to become the Humboldt University. He was delighted to have a degree from both incarnations of the institution, the doctorate from Humboldt being an honorary one. Bernhard studied under such luminaries as Erhard Schmidt, Issai Schur, Robert Remak and Heinz Hopf. He began with pure mathematics, physics and philosophy. Later he concentrated on pure mathematics.

Bernhard was introduced to group theory in the autumn semester of the academic year 1930/31 by Remak, who had given a course on his own (then unpublished) work on direct products of groups. Apparently the attendance started at 30 and ended up with Bernhard as the single attendee. Groups were new and exciting to him, and this excitement never wavered throughout his life. He came across the subject that eventually became part of his Dr.phil. dissertation while attending a seminar on geometry and topology led by Hopf and Georg Feigl.

Bernhard had read a paper of Jakob Nielsen on automorphisms of free groups ((120), p. 1); Nielsen had found a finite presentation for the automorphism group of a free group of finite rank. Bernhard discovered that he could reduce the number of generators and relations. Indeed, he found the minimum generating numbers for free groups of rank at least 4. He told Hopf, who suggested that the proof should be written down; and when Hopf saw the resulting manuscript he suggested that it might be used as Bernhard's doctoral dissertation. Bernhard thought that he was too young and the result too slim, but after comings and goings involving Hopf, Schur and Schur's assistant Alfred Brauer, Schur eventually decided that it was perhaps a little on the slim side. He suggested that Bernhard investigate the minimum generating numbers of what are now called permutational wreath products of finitely presented groups and symmetric groups. In a fortnight, Bernhard had done the required work—'the size doubled' ((120), p. 1)—and now there was enough for the Dr.phil., which Bernhard took at the early age of 22 years. Bernhard's first published paper (1) came out of this.

Life in Britain up to 1946

Bernhard left Germany for England in August 1933, and on his arrival he was given refugee status, somewhat to his surprise. He found that job prospects in England were no better than they had been in Germany, so he 'camouflaged' the fact that he was unemployed. He became a

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research student again, at Fitzwilliam House (now Fitzwilliam College), Cambridge, financed first by his parents in Germany and then by an uncle in South Africa. His supervisor was Philip Hall (FRS 1942), and weekly supervision sessions took the form of dinner in King's College followed by discussion in Hall's rooms in college: 'where we would drink his sherry, smoke his cigarettes and talk about rhizomorphs of plants and political history in Germany and everything'. At about 10 o'clock the discussion would drift towards mathematics. After completing his PhD in 1935, Bernhard stayed on in Cambridge for a further two years until he was appointed in late 1937 as a temporary assistant lecturer in mathematics at University College, Cardiff. He taught in Cardiff until 1940.

Bernhard had met Hanna von Caemmerer in January 1933. Hanna had been born in Berlin on 12 February 1914, the youngest of three children of Hermann and Katharina von Caemmerer. Her father had a doctorate in history, and he was well on the way to establishing himself as an archivist and academic historian when he was killed in the first days of World War I. As a result her mother, brother Ernst (born in 1908) and sister Dorothea (Dora, born in 1910) lived impecuniously on a war pension, and Hanna contributed to the family income by coaching younger school students for up to 15 periods a week.

Bernhard and Hanna got to know each other through the Mathematisch-Physikalische Arbeitsgemeinschaft, a working group within the university in Berlin. Although Hanna was not Jewish, she was very much anti-Nazi. Bernhard and Hanna remained in regular contact by correspondence after Bernhard moved to England. As open correspondence was too dangerous, their letters went secretly via various friendly channels. They became engaged in 1934 and met for a couple of weeks in Denmark in 1936 as Bernhard was returning to England from the International Congress of Mathematicians in Oslo. Then in 1938 Hanna, deciding that war was not unlikely, moved to the UK, where she and Bernhard were married. However, such 'mixed' marriages between Jews and non-Jews were against the law in Germany, and Bernhard and Hanna felt they could not marry openly until his parents were safe from possible reprisals. Thus, after their marriage Hanna lived in Bristol and Bernhard in Cardiff. Bernhard's father could not believe that the level of evil abroad in Germany since 1933 could possibly last, and it was not until February 1939 that Bernhard's parents left Germany. From that time, the four Neumanns lived together in a small house in Cardiff, where Bernhard's and Hanna's first child, Irene, was born in August 1939, just before the beginning of World War II.

During the first part of 1940, Bernhard and Hanna were classified as 'least restricted aliens' and Bernhard continued to work as a lecturer in Cardiff as this was a 'reserved occupation' not liable for army service. However, after the Dunkirk evacuation a large part of the coast was barred to all aliens and they were required to leave Cardiff. They moved to Oxford, and within a week Bernhard was interned. After several months Bernhard was released into the British army. He was trained in Yorkshire and then transferred to the south of England. Most of the time he was within reach of Oxford and could visit his family occasionally. In particular he obtained special leave when his son Peter was born in Oxford at the end of December 1940.

Bernhard served in the Pioneer Corps until 1943, when he was allowed to volunteer for combatant service. He was transferred to the Royal Artillery, then to the Artillery Survey, in which he used theodolites and did some numerical work. Later he entered the Intelligence Corps. During this time Bernhard completed several mathematical papers ((14, 15, 16); see (120), pp. 279 and 740).

Meanwhile Hanna submitted her DPhil thesis in Oxford in mid 1943, and soon afterwards she was allowed to return to Cardiff, where their third child, Barbara, was born in November

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1943. Hanna returned to Oxford for her oral examination in April 1944, and Bernhard, by then in the Intelligence Corps and stationed not very far away, was able to join her for a celebratory lunch after the examination.

During 1945, when the European war was over, Bernhard volunteered to go to Germany with the Intelligence Corps. On a long weekend leave (replacing his leave for VJ Day, which he had missed) he was able to visit Hanna's sister and mother. He was demobilized later that year and came back to England.

Life in England, 1946–62

University College, Cardiff, wanted Bernhard to return to his teaching post there, but Bernhard did not feel obliged to because they had not helped to get him out of internment. Instead he obtained a temporary lectureship at University College in Hull (now the University of Hull) at the beginning of 1946, as a replacement for Jacob Bronowski. At the same time his fourth child, Walter, was born. For the next academic year Bernhard was appointed to a permanent lectureship, and as Bronowski decided not to return to Hull, Hanna applied for and was appointed to the vacant position as a temporary assistant lecturer.

Hanna stayed in Hull for 12 years, but in 1948 Bernhard was enticed to a position at the University of Manchester by M. H. A. Newman. This entailed 10 years of commuting between Manchester and Hull in all the vacations and for many weekends. Bernhard was not only a keen everyday cyclist but also a long-distance one: on several occasions he made the 100-mile trip across the Pennines between Manchester and Hull by bicycle, and once rode from Manchester to give a lecture in London because there was a train strike. In 1951, during this period, his fifth child, Daniel, was born.

This period saw also the publication of several highly influential research papers. Foremost is the famous joint paper (19) of Bernhard and Hanna with Graham Higman, introducing Higman–Neumann–Neumann (HNN) extensions. In the same year Bernhard's paper on ordered groups appeared (17). Published shortly afterwards were (20), in which he answered one of the 1948 Prijsvraagen of the Wiskundig Genootschap te Amsterdam, and Bernhard's celebrated essay (38), for which he won the Adams Prize in 1952. Then in 1954 (36) and (41) on BFC groups appeared (see section F.5 below).

In 1958 Hanna obtained a lectureship in the Faculty of Technology at Manchester College of Science and Technology (now, after several name changes, part of the University of Manchester). At the time the college was part of the University of Manchester, but independently financed and physically separate, with its own department of mathematics. Hanna's role was to set up an honours programme in mathematics.

By then Bernhard had achieved the title of Reader in Mathematics, and in the following year (1959) he was elected to the Fellowship of the Royal Society for his numerous and influential contributions to the theory of infinite groups. Bernhard served as a Vice-President of the London Mathematical Society (1957–59) and as an Editor of *Proceedings of the London Mathematical Society* (1959–61). Bernhard supervised eight PhD students in Manchester, seven of whom became long-term university teachers and researchers in mathematics and the other a senior secondary teacher of mathematics and a mathematics editor. Six of them became professors of mathematics: in Egypt, England, Wales, the USA and two in Australia.

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During 1959 Bernhard was granted sabbatical leave from Manchester. He spent three months visiting all the universities in Australia, including a week at ANU in Canberra. After his Australian tour he spent the Monsoon Term of 1959 at the Tata Institute of Fundamental Research in Bombay, and notes of his lectures in Bombay were subsequently published by the Tata Institute (61). These travels are perhaps an early manifestation of his combining travelling activities with the spreading of mathematics.

Bernhard held the position in Manchester until 1962, but the final academic year was spent on study leave working at the Courant Institute of Mathematical Sciences in New York. The year in New York was fruitful mathematically for Bernhard, and also for Hanna and their son Peter, who was by then an undergraduate student in mathematics in Oxford. They wrote the '3N' paper (67) on product varieties of groups and embarked on a joint project with Gilbert Baumslag, one of Bernhard's former PhD students, which culminated a year later in the 'B+3N' paper (75) on varieties generated by a finitely generated group.

The careers of Bernhard's five children

After completing a Master of Arts in English literature at the University of Manchester, Irene (Neumann Brown) was a high-school teacher in Aberdeen, Scotland, and then a lecturer in English at New Mexico State University. Peter became a mathematician at Queen's College, Oxford, with research contributions ranging over several areas of algebra and its history. He was awarded the Lester R. Ford Award by the Mathematical Association of America in 1987 and the Senior Whitehead Prize by the London Mathematical Society in 2003, and was appointed an Officer in the Order of the British Empire in 2008 for services to education. On completing a mathematics degree at the University of Sussex, Barbara (Cullingworth) taught at Saint Bernard's Convent School in Slough, England, specializing in the teaching of statistics, and having an active role in the Mathematical Association. Walter studied mathematics in Adelaide and Bonn. His mathematical interests are in topology, algebraic geometry and geometric group theory, and his academic career has been an international one, with posts in Germany, Maryland, Ohio, Melbourne and New York. He is currently Full Professor at Barnard College, Columbia University. Daniel completed degrees in ancient Greek and pure mathematics at Monash University, and in psychology at the University of New England and Swinburne University. For many years he was a professional musician in Orchestra Victoria, and he now works as a psychologist.

Life in Australia, 1962–74

As a result of Bernhard's Australian visit in 1959, and with some strong encouragement especially from Sir Mark Oliphant FRS, in 1961 Bernhard and Hanna accepted positions at ANU, Bernhard as the Foundation Professor of Mathematics at the Institute of Advanced Studies, a position he held until his retirement at the end of 1974, and Hanna as Reader in the department that Bernhard was appointed to set up. Later Hanna became the first female professor of mathematics in Australia, and Foundation Head of the Department of Pure Mathematics in the School of General Studies at ANU (see figure 1). She was also the first woman mathematician to be elected a Fellow of the Australian Academy of Science (in 1969). Sadly she died in 1971 while on a lecture tour in Canada.

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Figure 1. Bernhard Neumann and his first wife, Hanna, around the time of Hanna's appointment as professor at ANU, Canberra, in 1964.

Bernhard's major task on taking up his position at ANU in 1962 was to build up the new Department of Mathematics, and to establish a healthy PhD programme in mathematics there. He attracted active researchers to his new department. During Bernhard's headship (1962–74) about 50 graduate students completed their degrees, of whom eight were directly supervised by him. He always had an open door and fostered a feeling of family. The students sometimes found his expectation of intellectual rigour daunting but came to appreciate it. They have gone on in diverse ways—quite a few of them made their careers in Australian universities, and Bernhard lived to see many of them make a significant mark.

Bernhard married Dorothea Frieda Auguste Zeim in 1973 (see figure 2). Dorothea was born in Berlin on 20 January 1939. Her father, Albert Zeim, was a classics scholar and secondary school teacher. Her mother, Hildegard Rockstroh, studied modern languages (French and English) but did not finish her doctorate for family reasons. She started working, as a contract teacher at secondary school, in the 1960s when her three children were all at university. Dorothea had known the Neumann family for many years. Her mother had gone to school with Hanna's elder sister, Dora, and Dorothea had spent some time with the Neumann family in Manchester working as an au pair and becoming familiar with English as part of her education. She had remained in contact with the family from that time. After her marriage Dorothea completed a PhD in linguistics at ANU.

Bernhard travelled widely, giving many talks, attending many conferences and spreading the message that good mathematics was being done in Australia and that Australia was a good place in which to do mathematics.

Bernhard expected intellectual rigour not only in mathematics but also more widely. He was known for his rigour in other activities such as editing typescripts and recording of minutes of meetings.

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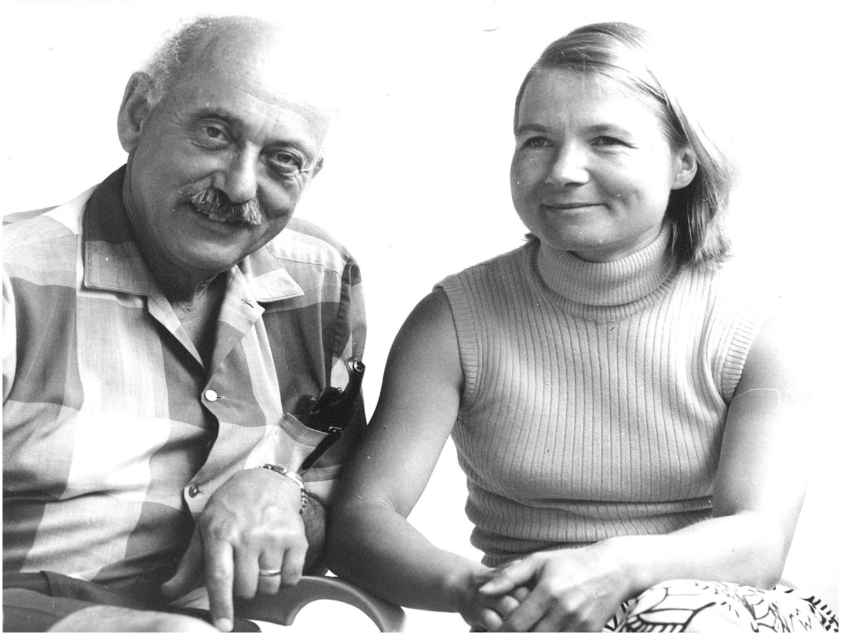

Figure 2. Bernhard Neumann and his second wife, Dorothea, in Canberra in 1974.
(Photograph by Professor Kurt Mahler FRS FAA.)

The first two international conferences in Australia on a mathematical topic were held under his leadership, on the theory of groups, in 1965 and 1973. Both were notable for the quality of the main speakers and the range of countries from whence they came. The first was also notable because Bernhard was able to arrange for young people from overseas to earn their way to Australia by teaching at an Australian university. A third international conference on the theory of groups was held in 1989 to mark his 80th birthday.

Bernhard was active in many aspects of mathematical life in Australia, holding numerous leadership positions, which were later appropriately acknowledged. He became the fifth President of the Australian Mathematical Society (1964–66), was elected an honorary member in 1981, and was further honoured by having a prize named after him for the most outstanding talk by a student at the Annual Meeting of the Society. Two of his mathematical grandchildren (doctoral students of his own PhD students) have won that prize. As founding editor (1969–79) of *Bulletin of the Australian Mathematical Society*, which he developed as a quality international journal, Bernhard succeeded in creating a vehicle for the fast publication of good mathematics, although he did not quite achieve his hoped-for five months, on average, from submission to publication. Bernhard also helped found the Australian Association of Mathematics Teachers and was its first president (1966–68).

Bernhard was elected to the Fellowship of the Australian Academy of Science in 1964. The citation noted his contributions to group theory, his having established himself as one of the leaders of mathematics in the country and his determination to foster mathematics on a national scale. He served on the Academy's Council (1968–71), was a Vice-President (1969–71), and gave the Matthew Flinders Lecture in 1984. He served an extended term on the National Committee for Mathematics (1963–75) and on Australian delegations to many meet-

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ings of the International Mathematical Union (IMU), held in conjunction with International Congresses of Mathematicians (he attended 13 of these congresses, surely close to a record). He gave an invited lecture at the Nice congress in 1970. Also at the Nice congress, he was appointed to improve communication among mathematicians. This led to his founding the *IMU Canberra Circular*, which he edited almost single-handedly from 1972 to 1999; it provided timely information about mathematical meetings as well as announcements of honours and deaths within the mathematical community internationally. Its circulation rose to more than 1100 before becoming largely electronic, and it was especially valued by colleagues in countries with little contact with the wider world. Bernhard also served (1975–79) on the Exchange Commission of the IMU.

Through the Australian Academy of Science he initiated the Australian Subcommission of the International Commission on Mathematical Instruction (ICMI), chaired it (1968–75), and was the Australian representative on ICMI. He was a member-at-large of ICMI (1975–82) and of its Executive Committee (1979–82). This was the basis for Australia's hosting of the Fifth International Congress on Mathematical Education, ICME5, in Adelaide in 1984. In addition he was active in getting the Academy involved in providing materials for schools, and after a long gestation period, six volumes of *Mathematics at work* appeared in 1980–81.

Life in Australia, 1974–2002

On retiring as Professor and Head of the Department of Mathematics at the end of 1974, Bernhard was made Professor Emeritus and an Honorary Fellow of ANU. He was also appointed a Senior Research Fellow at the Commonwealth Scientific and Industrial Research Organisation (CSIRO) for three years and then became an Honorary Research Fellow, reappointed annually until his death. In 1975 he also became an Honorary Member of the Canberra Mathematical Association, of the Australian Association of Mathematics Teachers, and of the New Zealand Mathematical Society.

In 'retirement' he continued his own research in mathematics and his support of work in mathematics by others, directly, through teaching and through editorial and committee work. For example, he served as a member of the Academic Advisory Council of the Royal Australian Naval College. He continued to be an ambassador for mathematics and for Australia.

Together Bernhard and Dorothea provided a steady, welcoming and supportive environment especially for visitors and young people, in mathematics, in music and quite generally. The annual Neumann wine and cheese parties were legendary. Initially they were yearly gatherings of Bernhard's and Hanna's students. By the 1970s the guest list included at least 100 students and former students, together with their friends and family.

Dorothea gave Bernhard outstanding non-mathematical support, by joining him in exploring and pursuing activities outside mathematics. In particular she joined him in regular chamber music evenings, and in an orchestra in which Bernhard played the cello, and strongly encouraged him to continue. Bernhard's approach to new challenges and suggestions was always 'why not?' After his son Daniel and grandchildren took him camping for the first time when he was over 70 years old, Dorothea and he planned and executed numerous camping expeditions that sometimes included mathematics conferences. Dorothea developed ingenious plans to enable Bernhard to keep cycling until a few months before his death (see figure 3), and to continue his active enjoyment of the Australian bush. On his eightieth birthday he went

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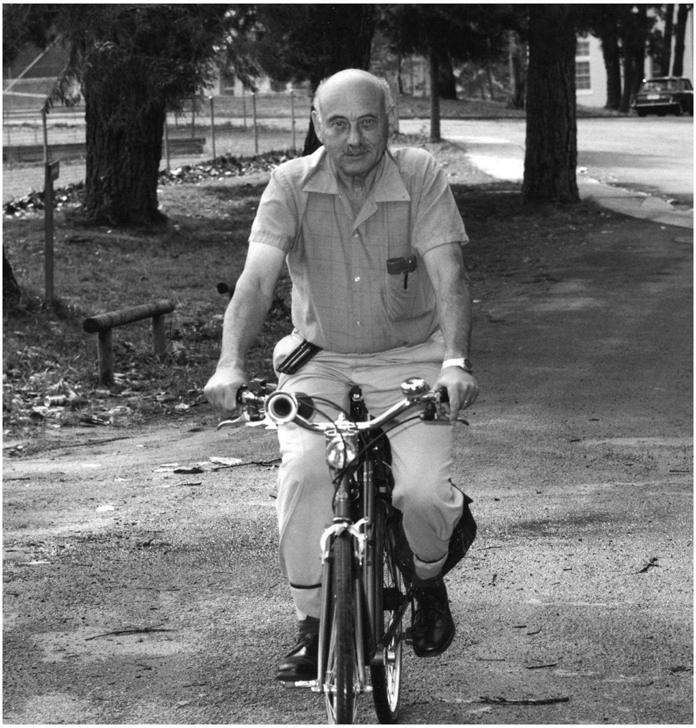

Figure 3. Bernhard Neumann cycling from his CSIRO office located in the Division of Forest Research, Yarralumla, ACT. (Photographed in 1978 by Alan G. Edward AIAP. Image courtesy of CSIRO.)

ballooning; on his ninetieth he went abseiling. One cannot help but wonder what he would have done had he made his hundredth birthday!

Bernhard continued to support activities aimed at stimulating and developing mathematics talent. He gave considerable encouragement to Peter O'Halloran and his colleagues involved in the formative stages of what is now the Australian Mathematics Competition, and maintained an active interest in it. He chaired the Australian Mathematical Olympiad Committee from its inception in 1980 until 1986. In his role as Chair of a Site Committee (1981–83) during his term on the Executive Committee of ICMI (1979–82), he ensured better structure and operation of the International Mathematical Olympiads. The holding of the 1988 International Mathematical Olympiad in Australia during Australia's bicentennial year owed much to him. Through the B. H. Neumann Awards, the Australian Mathematics Trust recognizes significant and sustained contributions to the enrichment of mathematical learning in Australia and its region. When the Australian Mathematics Trust commissioned the Sydney portrait artist Judy Cassab to paint his portrait, Bernhard was delighted not only by the painting but also by Judy Cassab's approach to portrait painting: 'she talked all the time and asked me questions, and I talked all the time—and so the portrait is not a photograph, it's myself.'

Most of his mathematical life was spent in Australia. During that time he contributed much to mathematics in Australia and became a much-loved and respected figure. He was a positive influence on a great number of people. He was made a Companion of the Order of Australia in 1994 for service to the advancement of research and teaching in mathematics.

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He was awarded honorary doctorates from several universities: the University of Newcastle (New South Wales), Monash University (Victoria), the University of Western Australia and ANU within Australia; and the University of Hull (UK), the University of Waterloo (Canada) and the Humboldt University of Berlin (Germany). Bernhard was well known in circles outside mathematics. On his first visit to Australia he played in many chess clubs. At the time of his death he was the oldest rated chess player in the country. He was known to the broader Canberra public as a familiar figure cycling on its roads, in more recent times wearing an electric blue helmet. He was for many years Vice-President of the Friends of the Canberra School of Music; he helped judge an annual chamber music competition on the day before he died. He enjoyed exploring the countryside and showing visitors around it. In spite of all this visible activity he was perhaps at his best giving quiet, often unnoticed and unrecognized help to individuals.

Bernhard Neumann was a strong supporter of all endeavours in mathematics—he supported people who did mathematics for its own sake, people who applied mathematics and people who taught mathematics. To him it was important to share and spread the joy of doing mathematics. Bernhard Neumann was the right person at the time for mathematics in Australia with his energy, enthusiasm and commitment to the subject and to all people involved in mathematics. During his lifetime, Bernhard saw Australia change from a mathematically underdeveloped country to one with a significant mathematical profile, and he was instrumental in that change.

Scientific work

The scientific work of Bernhard Neumann ranges widely in mathematics. The core is his published research. In addition he made many other contributions through his editorial work, his lectures, his reviews (of which he wrote many hundreds), his training of students, his participation in and organization of conferences and his ready availability to all who sought him out.

The published research of Bernhard Neumann is primarily in algebra with occasional excursions into geometry and other parts of mathematics. Within algebra his main work was on the theory of groups, although there was often work in more general contexts, such as division rings, near-rings, semigroups, universal algebra and ordered algebras. Groups are mathematical structures satisfying a small number of axioms that formalize the essential aspects of symmetry. Thus one may view the focus of Bernhard Neumann's research as the 'mathematical science of symmetry'.

Much of Bernhard Neumann's published work still continues to be cited regularly, most notably the HNN embedding paper (19) and the 'Mal'cev–Neumann construction' (18) for ordered division rings, both dating from 1949; his seminal paper on infinite groups (6) and his paper on varieties of groups (9), both published in 1937; his essay on group products with amalgamations (38); and his papers on group coverings (36, 39), on various finiteness conditions for groups (24, 42), and on the 'Erdős Problem' (104). Other 'mathematical inventions' of his, such as the twisted wreath product, are now household names in mathematics and are routinely used without explicit reference.

Bernhard Neumann was punctilious about publication names. For example, in 1970, when I was writing my first research article arising from an undergraduate summer vacation research

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project supervised by Bernhard, he counselled me about keeping one publication name throughout my career. Moreover, that name, he said, should as far as possible uniquely identify me as author. Thus I became Cheryl E. Praeger and, as Bernhard explained, he was from 1935 onwards B. H. Neumann in print. Therefore, in this scientific evaluation, I refer to Bernhard as BHN, a name used by his colleagues when discussing Bernhard as mathematician.

In this memoir the use of references to other sources has been kept to a minimum. Where possible such references have been restricted to material that can be readily traced by using the mathematical reviewing journals *Mathematical Reviews* (published by the American Mathematical Society, Providence, RI; for *Mathematical Reviews on the Web* see http://www.ams.org/mathscinet/) and *Jahrbuch über die Fortschritte der Mathematik* (1868–1942, available electronically through the Electronic Research Archive for Mathematics at http://www.emis.de/projects/JFM/), or the bibliography of the monograph by Chandler & Magnus (1982) (from which one may gain a good view of BHN's contribution to combinatorial group theory). Publications reviewed by *Mathematical Reviews* (MR) or *Jahrbuch* (JFM) will be referenced in the text by their review number and year of publication. As examples of this usage, the history by Chandler & Magnus (1982) might be referenced as Chandler and Magnus (MR0680777, 1982), and the paper by Burnside mentioned in section C.1 as Burnside (JFM33.0149.01, 1902).

Within group theory several topics are specifically associated with BHN. He is noted for the inventiveness of his constructions and his analysis of constructions and their application to embedding problems. He is a founder of the theory of varieties of groups.

Several of his contributions have seen his name used (in conjunction with others) and these are discussed in the following sections: the Birkhoff–Neumann embedding, section E.1; the Douglas–Neumann Theorem, section D.3; the Grushko–Neumann Theorem, section F.3; the HNN construction/embedding, section E.2; the Macintyre–Neumann Theorem, section F.6; the Mal'cev–Neumann construction, section E.5; the Moser–Neumann question, section D.2; the Neumann–Rado Theorem, section D.1; and the Neumann semidirect product, section E.3.

A. Automorphisms and endomorphisms

A.1. Automorphism groups of free groups

As mentioned already, BHN's interest in groups was aroused by a course in Berlin given by Remak in 1930. While he was participating in a seminar on geometry and topology conducted by Feigl and Hopf he came across a paper by Nielsen (MR1512188, 1924) in which finite presentations for the automorphism groups of finitely generated free groups are determined. BHN was able to reduce these presentations. In particular he showed that the automorphism groups of the free groups of rank at least four can all be generated by two elements. This was the core of his doctoral dissertation (1931) in Berlin and of his first paper (1). The heart of the work was manipulation of group presentations. This is something that continued to be a significant part of his work for his whole career. Presentations for the automorphism class groups of the finitely generated free groups were determined in (23) with Hanna. The main thrust of that paper was characteristic subgroups and systems of generating vectors. Characteristic subgroups were also studied in (82). The theme was extended to general categories in (89).

A.2. Fixed-point-free automorphisms

BHN reported in *Selected works* ((120), introductions to chapters 2 and 9) that questions in the foundations of geometry had a revival of interest in the 1920s and 1930s arising from work

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on the 7th edition of Hilbert's *Grundlagen der Geometrie*. BHN also reported that much of the work he did on these foundational questions was never published. BHN was fascinated by the algebra underlying geometrical systems. In (11) he studied what are now called near-rings, generalizing results of Zassenhaus about finite near-fields to the infinite case. The questions were boiled down to proving that a group satisfying certain conditions is abelian. One of these conditions is the existence of an automorphism with order 2 that leaves fixed only the identity element; this is nowadays usually referred to as a fixed-point-free automorphism. BHN proved that a group that admits a fixed-point-free automorphism with order 2 and in which every element has a unique square root is abelian. In (14) he showed that the existence, but not uniqueness, of square roots is not sufficient. In (44) BHN gave a description of finite groups that admit a fixed-point-free automorphism with order 3. The study of groups and other algebraic systems admitting automorphisms with few fixed points has developed into a substantial topic; see, for example, monographs by Khukhro (MR1224233, 1993; MR1615819, 1998).

A.3. Finite groups

In (43) and (45) Ledermann and BHN studied automorphism groups of finite groups. They were inspired by a problem of Birkhoff and Hall (MR1501860, 1936) that asked for a lower bound for the orders of automorphism groups of finite groups. In (43) Ledermann and BHN showed that large finite groups have large automorphism groups or, more precisely, that there is a bound $f(n)$ such that every finite group with order not less than $f(n)$ has at least n automorphisms. Recent work of Bray and Wilson (MR2243245, 2006) sheds further light on this. In (45) Ledermann and BHN considered what they called a local version and showed that, for each prime p, there is a function $g(h)$ such that every group whose order is divisible by $p^{g(h)}$ has p^h dividing the order of its automorphism group. An important ingredient in the work on the local problem is Schur's multiplicator. Not long afterwards, Green (MR0081901, 1956) made improvements showing that the function $h(h + 3)/2 + 1$ suffices. There have been some further improvements in the bound but not asymptotically. Nevertheless there is a popular conjecture, seemingly dating from about the same time, that the order of a non-abelian group with prime-power order divides the order of its automorphism group (Schenkman MR0067111, 1955; Mann MR1716701, 1999).

A.4. Semigroups

The story is quite different in semigroups, as Dlab and BHN showed in (91). They exhibited arbitrarily large finite semigroups with just eight endomorphisms, and their question of whether a smaller number can be achieved was answered by Kublanovsky (MR734514, 1983), who showed that for semigroups containing at least four elements, the minimum possible number of endomorphisms is four, and there are infinitely many finite semigroups with exactly four endomorphisms. Dlab and BHN also exhibited an infinite rigid semigroup: one whose only endomorphism is the identity.

B. Presentation manipulation

Presentation manipulation is a significant feature of BHN's work. It is the core of (1), and in particular is a critical ingredient of (6). About paper (6), Baumslag and Miller (MR2554768, 2009) say:

> In [this] brief, remarkable paper [BHN] established several facts which are now widely known and used in combinatorial group theory: (i) a proper subgroup of a finitely generated group is

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> contained in a maximal proper subgroup; (ii) if a group can be defined by finitely many relations on one finite set of generators, then it can be defined by finitely many relations on any other finite set of generators; (iii) there exist finitely generated groups which are not finitely presented; and (iv) there exists [by construction] a family of continuously many non-isomorphic two generator groups.

This important construction is discussed further in section E.1.

Part of his joy with presentation manipulation came via coset enumeration. BHN was one of the early practitioners and he continued to enjoy it for a long time. He saw that it would benefit from the use of electronic computers. He argued for computational resources at ANU in the 1960s and established a post to support computation. He set interesting challenges for the development of better implementations (see (107c) for an example, and also MR1775892, 2000). He continued to record such manipulations in his notebooks and publish consequences he thought of interest.

In (47) BHN considered the question of whether a finite group with trivial Schur multiplicator has a presentation with an equal number of generators and relations, in other words has deficiency zero. This was a growing subject, to which BHN contributed further in (106), dedicated to H. S. M. Coxeter FRS—another great presentation manipulator (117, 122, 123).

In the course of proving that it is possible to adjoin nth roots of elements to semigroups in (85) and (86), BHN suggested a computational procedure, analogous to the Todd–Coxeter procedure for groups, to enumerate the elements of a finitely presented monoid. The procedure was proved by Jura (MR0486223, 1978), and various variants have been proposed, culminating in a general framework of Linton (MR1344920, 1995) that encompasses double coset enumeration in groups and vector enumeration, as well as the Todd–Coxeter procedure for groups and for monoids as special cases.

C. Varieties

C.1. Varieties of groups

BHN initiated the study of identical relations in groups in his Cambridge dissertation (1935), ‘Identical relations in groups’. The investigation was set in motion from his study of Boolean matrices (2). The introduction to (7), which arose from the thesis, suggests that the famous paper (JFM33.0149.01, 1902) of Burnside played a part in setting the direction of the work. At that time not much progress had been made on the questions Burnside had posed though they were being thought about (see footnotes on p. 506 of (7)). What mattered for BHN was that the groups satisfied a relation $x^k = 1$ identically and, related to that, the idea of a ‘maximal group’ generated by n elements and having the order of all its elements divide a fixed number k. BHN asserted ((7), p. 507):

> The identical relations which hold in a given group are well worth being closely studied; whereas the relations which connect the generators of a group depend on the choice of the generators, the identical relations clearly depend only on the group itself: they are an invariant property of the group.

Paper (7) is labelled part I; it deals with the general theory. It introduced ‘word-groups’, now called verbal subgroups, and the V-groups $V_n(G)$, now called relatively free groups—that is, the n-generator group defined by the identical relations in n variables that hold in G. BHN proved that for a finite group G the groups $V_n(G)$ are finite; more precisely, the order of $V_n(G)$ divides $|G|^{|G|^n}$. He also observed that $V_n(G)$ can be determined from G and n in a finite

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number of steps. He then said ((7), p. 520): 'But to get *all* the identical relations we have to know something about $V_n(G)$ for *every n*. Whether this question can still be reduced to a finite problem cannot be decided now.'

This comment evolved into the *finite basis problem*: does every group have a finite basis for its identical relations; that is, is there a finite set of identical relations from which all the identical relations follow?

The second part of the thesis, which contains detailed computations of the identical relations or laws of a number of finite metabelian groups, was never edited for publication ((120), p. 144). BHN also pointed out ((7), p. 509) that in Hall's paper (JFM62.0082.02, 1936) on Eulerian functions the V-groups for the icosahedral group are implicitly determined.

Nowadays this field of study is given the name 'varieties of groups'. The term 'variety' in this context was introduced by P. Hall in 1949.

Graham Higman and BHN opened another aspect of the subject by showing in (28) that the variety of groups can be described by a single law in terms of the binary operation of right division.

Encouraged by R. Padmanabhan, BHN returned in (111) and (118) to finding single laws for the variety of groups, this time in terms of the natural operations. His technique was based on the use of left and right mappings. This theme has been taken up for other varieties and in particular using automated theorem provers.

All BHN's other publications on varieties dealt with problems that arose in Hanna's major paper (MR0078374, 1956) on varieties.

BHN showed in (49) that there is a 3-generator non-metabelian group all of whose 2-generator subgroups are metabelian and a 4-generator non-metabelian group all of whose 3-generator subgroups are metabelian. These groups have exponent 8 and were the first of several examples demonstrating the complexity of groups with exponent 8. The most recent is by Krasilnikov (MR1994688, 2003).

The visit of BHN, Hanna and their son Peter to New York for the academic year 1961/62 provided a flourish of activity on varieties that was reported in two papers. In (67) the three Neumanns settled questions about the structure of the lattice and algebra of varieties left open in MR0078374. In particular the semigroup of varieties of groups (omitting the variety of all groups and that of trivial groups) is free and freely generated by indecomposable varieties, a result obtained independently by Shmel'kin (MR0151539, 1963). The important new idea used was to relate wreath products of groups to products of varieties of groups.

In (75) Baumslag and the three Neumanns considered conditions under which a variety is generated by a relatively free group of finite rank. They introduced, and made effective use of, the notion of a 'discriminating' set of groups. For example, they showed that the variety of all groups with given soluble length is generated by the relatively free group of rank 2.

In 1967 BHN gave an invited lecture to the American Mathematical Society in which he surveyed work on varieties, referring his listeners to Hanna's monograph, which was about to appear; paper (83) is the text.

The finite basis problem was the impetus for a lot of work that established 'varieties of groups' as a separate discipline in the 1950s (see the index of *Mathematical Reviews*), with its status reported in a monograph by Hanna (MR0215899, 1967). Oates and Powell (MR0161904, 1964), students of Higman's, proved that the identical relations of a finite group always have a finite basis. Then in 1970, in quick succession, A. Yu. Ol'šanskiĭ, S. I. Adian and M. Vaughan-Lee (MR0286872, MR0257189, MR0276307) published quite different

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proofs that there are groups that do not have a finite basis for their identical relations and indeed that there are continuously many varieties of groups.

Many of the results about varieties of groups have been transferred to varieties of Lie algebras, especially over infinite fields.

BHN and Macdonald took up a theme from BHN's PhD—to study the interdependence and independence of commutator laws in groups in a series of papers (121, 124, 127).

C.2. Varieties of other algebras

BHN and Evans (30), answering a question in (25), showed that there are continuously many varieties of groupoids and loops. BHN and Christine Wiegold (81) reported on semigroup representations of varieties of algebras. In (87) BHN applied the techniques of (28) to improve a result of Grätzer concerning a variety $\mathfrak{R}$ of algebras in a certain species such that $\mathfrak{R}$ is defined by a finite set of laws. Grätzer had constructed a new species by adding two binary operations, and a variety $\mathfrak{R}_1$ in that species, such that both $\mathfrak{R}_1$ restricted to $\mathfrak{R}$ in the original species, and also $\mathfrak{R}_1$ could be defined by two laws. In (87) BHN showed that $\mathfrak{R}_1$ can be defined by a single law. BHN was an early contributor in (116) to the (now extensive) theory of knot quandles. BHN also studied other varieties of algebras (109, 125, 133, 134).

D. A little geometry

The foundations of geometry were a significant source of inspiration early in BHN's career. His interest in geometry went back to his early days as an undergraduate at Freiburg in 1928. Most of his publications that related to geometry deal with algebraic questions that arise—primarily in foundations (11, 17, 18, 44). There are two other themes—convex plane regions and Napoleon's Theorem. The former was the basis for various lectures including the Matthew Flinders Lecture to the Australian Academy of Science. It has also led, via a question popularized by J. Moser, to developments in dynamical systems—the theory of outer billiards. The approach of BHN to Napoleon's Theorem is quite algebraic.

D.1. Convex regions

BHN showed in (10) that a certain parameter associated with the lengths of chords of a plane closed convex curve must lie between $\frac{1}{3}$ and $\frac{1}{2}$. Then in (16) BHN found that the same bounds hold for an analogous parameter related to the area of intersection with a half-plane of a closed finite planar region. His proof applies without alteration if the planar region is replaced by an appropriate planar mass distribution. Moreover, in the case of convex planar regions BHN gave a simplified proof, improved the lower bound to $\frac{4}{9}$, and conjectured that a lower bound of $(3-\sqrt{5})/2$ may hold. Recent computational work by Kaiser (MR1424359, 1996) suggests that the conjecture is true and may be exact in some cases. The main result from (16) and its generalization by Rado (MR0021962, 1946) for higher dimensions is now viewed as a result on measures in Euclidean spaces, called the Neumann–Rado Theorem (see Dolnikov MR1187871, 1992), and has been generalized by many others, most notably by Tverberg (MR0187147, 1966).

D.2. Outer billiards

In about 1960 BHN gave some popular lectures on Dupin curves under the title 'Sharing ham and eggs'. There is a written account (52b) of one such lecture in the magazine *Iota*, published in Manchester. In this account BHN asked a question about what is now known as outer or

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dual billiards. This question was popularized by Moser as a crude model for planetary motion (MR0442980, 1978). Recently Schwartz (MR2299242, 2007) answered what he now calls the Moser–Neumann question and published a monograph on outer billiards (MR2562898, 2009).

D.3. The Douglas–Neumann Theorem

The (Petr–)Douglas–Neumann Theorem is an 'astounding' generalization (see MR1959194, 2003) of what is usually referred to as Napoleon's Theorem: if on the sides of a triangle are erected (outside or inside) equilateral triangles, then the centroids of these triangles form an equilateral triangle with the same centroid as the original triangle. The generalization can be stated (see Theorem 3.2 of (12)): if isosceles triangles with base angles $\pi/2 - 2\pi v/2n$ are erected on the sides of an arbitrary polygon Π, and if this process is repeated with the polygon formed from the free vertices of the triangles, but with a different value of v, and so on until all values of v except 0 and one other (arbitrary) value, say μ, have been used in arbitrary order, then a regular figure is obtained. Its centroid coincides with that of the vertices of Π, and its sides are equal in length and (parallel but) opposite in direction to the sides of the polygon that would have been obtained from Π by taking $v = \mu$. Jesse Douglas was of one the (two) first Fields Medallists in 1936 for his work on the plateau problem. BHN's interest stemmed from a problem about electrical transformers that he came across in proofreading his father's book, *Symmetrical component analysis of unsymmetrical polyphase systems* (1939). BHN's paper attracted the interest of the well-known Cambridge geometer H. F. Baker (MR0008453, 1942), who gave a direct proof. BHN in turn polished it further (13). BHN gave several lectures on this topic. There is a nice, more recent, account in (113). A Java program that illustrates the theorem can be found on the Web (http://www.maa.org/editorial/knot/Napolegon.html).

E. Constructions and embeddings

A major topic of interest to BHN was the embedding of groups or algebras in other groups or algebras, frequently with the preservation of given properties. Many of his embedding results are achieved by ingenious constructions, and may be viewed either as embedding or construction theorems. Most notable are the famous HNN construction for groups and the Mal'cev–Neumann construction for division rings. Other important examples include the twisted wreath product, the non-Hopfian group constructions, and the 'Neumann semidirect product' for semigroups.

E.1. Subgroups of direct products

Several authors remark on a construction in (6). It is described in detail in a recent book by de la Harpe (MR1786869, 2000). BHN used it to construct continuously many pairwise non-isomorphic finitely generated groups. They are all subgroups of unrestricted direct products of alternating groups that contain the restricted direct product of these groups. Because there are only countably many finitely presented groups, most of BHN's examples must be not finitely presented. Indeed, recently Baumslag and Miller showed that all of them are not finitely presented. Baumslag (and others) have used these groups to provide examples of non-trivial discriminating groups (MR2288461, 2007). They are significant in the theory of amenable groups because they show there are continuously many finitely generated elementary amenable groups.

BHN made other use of his insight into unrestricted direct products. The embedding of a relatively free group $V_n(G)$ into the direct product of $|G|^n$ copies of G introduced by Birkhoff

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(1935) and by BHN (7) has been called the Birkhoff–Neumann embedding (see MR2245589, 2006). The restricted product is usually not complemented in the unrestricted one (76). BHN and Yamamuro used it (80) to show that for one family of subgroups, the Boolean powers, all countable quotients are finite. In an addendum they remarked that this fits into a broader context.

E.2. The Higman–Neumann–Neumann (HNN) construction

Given a group G with subgroups A and B, when does there exist a group H containing G such that A and B are conjugate in H? Necessarily A and B must be isomorphic. The HNN construction (19) demonstrates that this is sufficient, constructing groups that are now known as HNN extensions. They are closely related to amalgamated free products and have many analogous properties. HNN extensions are used in (19) to embed any countable group in a 2-generator group, and to prove that any group G can be embedded in a group in which all elements of G of the same order are conjugate. They are at the heart of G. Higman's famous embedding theorem (MR0130286, 1961) showing that a finitely generated group can be embedded in a finitely presented group if and only if it is recursively presented. Most important in their application these days is Britton's Lemma (MR0168633, 1963) giving a normal form for an HNN extension. The HNN construction is of considerable importance in combinatorial and geometric group theory. It is used in essentially all work on unsolvable decision problems. The use of HHN embedding has a significant role in the theory of amenable and non-amenable groups, and there are many other recent applications (for example MR2294245 and MR2270456, 2007).

E.3. Wreath products and generalizations

The wreath product construction has a long history and remains a fundamental tool in both abstract group theory and applications of group actions, in algebra, combinatorics and statistics. BHN and Hanna pioneered the use of wreath products in their work on embeddings in (54) and (59). In (54) BHN and Hanna showed that every countable soluble group G is embeddable into a soluble 2-generated group H. This embedding theorem was extended to generalized soluble groups in (77) (with Kovács). In (59) BHN showed that if the (soluble) group G is fully ordered, then the 2-generated (soluble) group H can be made fully ordered with G order isomorphic to its image in H. This result was extended to generalized soluble groups with a full order by Mikaelian (MR1898374, 2002; MR1970055, 2003).

In (68) BHN introduced a generalization of this construction that he called the *twisted wreath product*, of which the wreath product is a special case. The general notion leads in (65) to a sharpened form of a theorem of Auslander and Lyndon on the lower central series of normal subgroups of free groups. The twisted wreath product gives its name to one of the small number of types of finite primitive permutation groups in the O'Nan–Scott scheme. Several other generalizations of the wreath product have appeared in the literature. Of these the *crown product* introduced in (48) is used to study ascending derived series of groups.

BHN was the first to introduce a semidirect product construction for semigroups. In (57) he used it to define a wreath product of semigroups, to prove that every finite semigroup can be embedded in a finite 2-generated semigroup, and also to give a new proof of Evans's theorem (MR0050566, 1952) that every countable semigroup can be embedded in a 2-generator semigroup. The 'Neumann semidirect product' turned out to be a very powerful tool for semigroup theory. For example, Preston (MR0837162, 1986) investigated a more general product defini-

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tion and showed that only the direct product and the Neumann semidirect product have the crucial property that the product of two semigroups is always a semigroup.

E.4. Group amalgams

Group amalgams arose in Hanna's work on free products with amalgamated subgroups (MR0026997, 1948) and the amalgam concept was introduced shortly after by Baer (MR0030953, 1949). Amalgams have an important role in the study of finite simple groups and associated geometries (for an example see MR1705272, 1999). A group amalgam is an 'incomplete group' in that its set of elements is the union of groups G_i, for $i \in I$, such that for distinct i,j the intersection $H_{ij} = G_i \cap G_j$ is a subgroup of both G_i and G_j. In particular, the product of two elements is only assumed to exist if some G_i contains both elements. An amalgam is embeddable if there is a one-to-one homomorphism from it to some group. BHN in collaboration with Hanna undertook an intense study of group amalgams. In (21) they studied a family of amalgams in which the subgroups H_{ij} are central in both G_i and G_j, and constructed what are now called *central products* of the groups G_i to prove that such amalgams are embeddable. In (32) they introduced uniform notation to state all the then-known embedding results and showed that each was best possible. In particular they asked whether an embeddable amalgam consisting of finite groups is embeddable in a finite group. This question was answered in the negative by Brown (MR1230631, 1992). In (58) BHN introduced the permutational product of an amalgam of two [sub]groups as an embedding group for the amalgam. He studied the extent to which various finiteness properties of the amalgam carry over to the permutational product. Using this construction he showed (56) by example that the properties of local finiteness, or being periodic, need not carry over to the embedding group. BHN (60) explored existence questions for linked products of given groups (introduced by Hanna and Wiegold, MR0124386, 1960), and in (66) he demonstrated that there are no necessary and sufficient criteria involving only commutators, intersections, and multiplications of subgroups for the existence of a generalized free nilpotent product, or a generalized free soluble product, of the amalgam of two groups.

E.5. The Mal'cev–Neumann Theorem

It was the role of ordered groups and ordered division rings in the foundations of geometry that drew BHN's interest ((120), p. 800). In (17) BHN generalized some necessary conditions and some sufficient conditions proposed by Levi for a group to be orderable, producing some general constructions for ordered groups, and the first example of a perfect ordered group. Ordered division rings are even more important in geometry than ordered groups, and in (18) BHN constructed a formal power series division ring and used this to prove that every ordered group can be embedded in an ordered division ring. The construction made independently by Mal'cev (MR25457, 1948) is now known as the Mal'cev–Neumann construction and is of central importance in the theory of division rings (see chapter 2 of Cohn's book MR1349108, 1995). It has had a far-reaching impact in areas as diverse as classical analysis, diophantine equations, combinatorics and commutative algebra.

E.6. The non-Hopfian constructions

A *Hopf group* is one that is not isomorphic to any of its proper quotient groups. Hopf had conjectured in the 1930s that all finitely generated groups should be what we now call Hopf groups. The first suggestion that this was not so came in a paper by Baer that 'shook the

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group-theoretical world' ((120), p. 905). Combining Baer's basic ideas with use of the newly developed HNN-extension of (19), BHN constructed (22) the first finitely generated non-Hopf group (a 2-generator group). In (31) BHN and Hanna answered another question of Hopf negatively by constructing two non-isomorphic (finitely presented) groups, each a homomorphic image of the other. Later (54) BHN and Hanna constructed the first example of a soluble non-Hopf group.

E.7. Essay for the Adams Prize

BHN's largely expository prizewinning essay (38) has been widely read and has had substantial influence. Its origins lie in an appendix that BHN was invited to write as an update for the 1953 German translation of Kurosh's book *Teoriya Grupp*, to 'describe some of the then new results that had been obtained and that answered many of the problems in Kurosh's book' ((120), p. 281). The appendix was updated, translated into English, and submitted to the University of Cambridge for the Adams Prize, which it won. The published essay (38) contains basic facts about amalgamated free products with all sorts of applications and examples, including the HNN construction, 2-generator embedding and 'three remarkable groups of G. Higman', in a very readable exposition.

E.8. More on embeddings of groups and algebras

In (14), written while on leave from the British army, BHN studied the problem of solving equations in groups and adjoining the necessary solutions. He developed an elegant procedure, proving that every group can be embedded in a divisible group (one in which, for every group element g and every positive integer n, the equation $x^n = g$ has a solution x in the group).

In (17) BHN noted that every ordered group can be embedded in the multiplicative group of an ordered division ring, and in the early 1950s BHN asked whether every totally ordered group can be embedded in a totally ordered divisible group. Despite forming the topic for PhD theses in 1974 and 2002, and being listed as the first and most notorious problem in the Black Swamp Problem Book (a problem book on ordered groups carried around to conferences in this area of mathematics and containing contributions from participants), this problem was not solved until Bludov (MR2213301, 2005) produced a totally ordered group for which no such embedding exists.

In (25), as a result of discussions with Evans, BHN showed that a non-associative ring R is embeddable in a division ring D with unique left and right division if and only if R has no zero divisors. Moreover, D may be chosen to retain certain properties of R such as commutativity or being idempotent.

In a subsequent paper (37) BHN derived from a theorem of Steenrod a general embedding result for a large class of universal algebras that contains all (partial) semigroups. He deduced several striking consequences: a group can be fully ordered if and only if all its finitely generated subgroups can be fully ordered; a ring can be embedded in a division ring if and only if every finitely generated subring can be so embedded; the edges of a graph can be coloured with k colours so that no incident edges have the same colour provided the same is true for every finite subgraph. Many of these consequences would these days be derived from the Compactness Theorem of first-order logic. However, several now-senior mathematicians recall the great impact that (37) had on them as students, one finding the paper 'very beautiful and clever', another finding it had 'changed forever [his] view of the mathematical world to see that one could take such a global view of the world and still prove significant theorems'.

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In (70), with Tekla Taylor, BHN characterized those semigroups that can be embedded in nilpotent groups, a characterization similar to an earlier one by Mal'cev (MR75959, 1953).

Other papers of BHN may be regarded as a prelude to (19) ((20), written in June 1948—see the endnote to (20) on (120), p. 118) or a development of (19) (see (27, 40)). In this context, Problem 11.67 in the Kourovka notebook by BHN asks: does there exist a torsion-free group with exactly three classes of conjugate elements such that no non-trivial conjugate class contains a pair of inverse elements?

F. Other contributions

F.1. A group-theoretic arithmetic problem

Paper (3) shows the existence of continuously many maximal non-parabolic subgroups of the modular group. The existence of such subgroups was needed by Arnold Schmidt for his doctoral thesis on the axiomatics of geometry ((120), p. 65). There is an extensive account in a monograph by Wilhelm Magnus on *Noneuclidean tesselations* [*sic*] *and their groups* (MR0352287, 1974).

F.2. A problem like Burnside's

BHN considered, although not as part of his thesis work, a problem similar to Burnside's that he called the *bounded orders problem*. In (8) he considered the simplest non-trivial case: groups all of whose elements have order at most 3.

The last 30 years have seen an extension of the bounded orders problem. The set of orders of elements of a periodic group has been called the *order spectrum* of the group. Given a divisibility-closed subset Ω of the positive integers, the extended problem asks for a description of the periodic groups whose order spectrum is Ω, preferably in the strong form of determining the groups up to isomorphism. There is a recent survey on this topic by Mazurov and Shi (MR2491731, 2009; see also MR1676650, 1999).

There are three types of order spectrum for groups all of whose elements have order at most 3. The solution of the bounded orders problem for groups with order spectrum {1,2} was known when Burnside posed his questions. BHN solved the bounded orders problem for finitely generated groups with order spectrum {1,2,3}. The strong form of the problem has still not been solved for groups with order spectrum {1,3}; that is, groups with exponent 3.

F.3. The Grushko–Neumann Theorem

The Grushko–Neumann Theorem asserts that the number of generators needed to generate a free product of groups is the sum of the number of generators for each of the free factors. The BHN proof (15) of this result was written while he was on leave from the Pioneer Corps and published in 1943, responding to a question of Levi. During the war both communication and publication were difficult, and we note that the proofs by Grushko and BHN were obtained independently even though the former (MR0003412) dates from 1940. In Chandler & Magnus (1982), p. 106, the theorem is described as the next development in the theory of free products after the work of Kurosh. This important result has attracted intense interest since then, with alternative proofs being offered by Wagner (the infinitely generated case, MR0008809, 1957), Lyndon (MR0178073, 1965), Stallings (MR0188284, 1965) and Imrich (MR0773183, 1984).

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F.4. The Frattini subgroup

In (6) BHN proved fundamental properties of what is now called the Frattini subgroup for not necessarily finite groups.

In (35) BHN and Higman solved two questions of Itô about the Frattini subgroup. They showed that the Frattini subgroup of a group need not be nilpotent and that the Frattini subgroup of a non-trivial free product is trivial. They pointed out that, for free products with an amalgamated subgroup, the Frattini subgroup can coincide with the amalgamated subgroup, and they asked whether it can be larger. It has been shown independently, and by different methods, that the Frattini subgroup is contained in the amalgamated subgroup: by Gelander & Glasner (MR2377495, 2008) if the free product is countable, and by Allenby & Tang (MR2419010, 2008) if the amalgamated subgroup is countable.

In (52) BHN studied ascending Frattini series. He concluded by asking whether a finite group that is the Frattini subgroup of some group is also the Frattini subgroup of a finite group. This question was answered in the affirmative by Eick (MR1427706, 1997).

F.5. Almost finite and almost abelian groups

BHN's work over many years on groups that are in some sense close to being finite, or close to abelian, has been very influential. In (24) and (53) he studied groups in which the conjugacy classes of elements are all finite, in particular proving an extension of the Sylow theorems for these infinite groups. In (36) and (41) BHN studied BFC groups, groups for which there is a finite upper bound on the sizes of the conjugacy classes of elements, and proved that these are precisely the groups with finite derived subgroup (a measure of commutativity). The question in (36), p. 239, asking for an explicit upper bound for the order of the derived group in terms of the upper bound on the conjugacy class sizes of such a group has been investigated over many years. BHN's review of a paper by Segal and Shalev (MR1726791, 1999) gives an account of recent results.

In similar vein BHN proved in (42) equality of the class of groups whose centre has finite index (another measure of commutativity) with the class of groups all of whose conjugacy classes of subgroups are finite. Baer's observation that this class is also equal to the class of groups that can be covered by finitely many abelian subgroups prompted BHN's investigation (39, 41) of groups coverable by finitely many cosets of proper subgroups. If such a covering is minimal in the sense that no proper subset of the cosets covers the group, then BHN's results in (36) imply that all the cosets correspond to subgroups of finite index, and bounds for their indices are given in (41). BHN's work on coverings, in the contexts of both abstract groups and group actions, has been applied and developed in many ways by many mathematicians up to the present (see MR0401882, 1976; MR1328918; 1994).

A more combinatorial measure of commutativity is provided by the non-commuting graph $\Gamma(G)$ of a group G in which elements of G are joined by an edge if and only if they do not commute. In 1975, Erdős asked of groups G in which $\Gamma(G)$ contains no infinite complete subgraph or, equivalently, groups in which every infinite subset of elements contains two that commute: does there exist an integer $n(G)$ such that every complete subgraph of $\Gamma(G)$ has at most $n(G)$ vertices? In (104) BHN characterized such groups G as those for which the centre Z has finite index, and proved that $n(G) \leq |G|/|Z| - 1$ (see also MR0506252, 1978). This result inspired many further investigations and is much cited (see the review MR0419283, 1996; the sequel (135); and MR2260490, 2006).

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F.6. Algebraic closure: the Macintyre–Neumann Theorem

BHN's work on algebraic closure, also called existential closure, in groups and general algebras began with a question of Scott (MR0040299, 1951). Scott had introduced notions of algebraic closure and weak algebraic closure for groups. Using the generalized free product construction, BHN proved (26) that these notions are equivalent, and moreover that each algebraically closed group is simple. Recognizing that these notions belong to universal algebra rather than group theory, BHN proved analogous results for algebraically closed semigroups (93) and related results for cancellative semigroups ((92); see (120), p. 1116).

BHN's second major paper (97) on this theme was widely circulated as a preprint several years before it was published. In it he proved (among other things) that a finitely generated group with soluble word problem can be embedded in every algebraically closed group—one half of the Macintyre–Neumann Theorem. Macintyre (1972) proved the converse for finitely generated groups (and more—see Schupp's review MR0414671 of (97)). At that time logicians from Abraham Robinson's school of model theory were developing generic structures using forcing and were interested in existentially closed structures because of work of Macintyre. This result led to work on existentially closed groups by Simmons, Macintyre, Ziegler and others.

F.7. Properties of countable character

In (99), dedicated to the memory of Mal'cev and written a little before paper (96) (see (120), p. 1259), BHN studied group properties of countable character. These are properties possessed by a group if and only if all the countable subgroups possess them. BHN showed in (99) that residual finiteness is a property of countable character. 'Properties of countable character' is the title of BHN's invited lecture at the International Congress of Mathematicians in Nice in 1970. In the account (96) of that lecture BHN considered this theme for general algebras.

F.8. Means in groups

In (33), BHN simplified necessary and sufficient conditions, due to Scott, for the existence on a group of a set of 'Schimmack means'. In doing so he constructed the first example of a non-abelian group admitting such means. Subsequently Wiegold (MR120277, 1960) showed that both Scott's and BHN's conditions are equivalent to requiring that the centre of the group and its factor group are both direct products of groups isomorphic to the additive group of rational numbers.

F.9. Unsolvability and the Axiom of Choice

BHN learnt from Remak in Berlin to be alert to the use of the Axiom of Choice when working with infinite sets, and this was reinforced by lectures there by von Neumann. There are signs of this influence in quite a lot of his papers, for example in (6). It surfaces explicitly in (102), in which Hickman and BHN showed that the answer to a question of Babai depends on the underlying set theory. Another publication at the confluence of algebra and logic arose out of a visit by Boone to Manchester in 1958. Their discussions with Baumslag led to results on the algorithmic unsolvability of questions about groups (55).

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Students

BHN was the supervisor of the work of 17 students for doctorates and was the formal supervisor of several other students. He was interested in the work of a wider group of students. He also gave help to some students, of which the most significant was that given to Kamal Riad Yacoub (see (120), pp. 65–66).

1953 Kamal Riad Yacoub

Manchester		*ANU*	
1953	Chehata Chehata	1965	Narain Gupta
1954	John Britton	1967	Terence Gagen
1957	Christine Dale	1968	Ahmad Shafaat
1958	Gilbert Baumslag	1969	Neil Williams
1959	James Wiegold	1973	Andrew Brunner
1960	Ian Macdonald		Kok Yeo
	Michael Newman	1974	Lambertus Hesterman
1961	László Kovács		Colin Fox
			Tim Brook

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In addition to generous personal contributions, I am indebted to sources of biographical information such as the record of an interview of BHN (Crompton 1998), and the obituaries (MR0342340, 1974; MR1983807, 2003) of Hanna Neumann and BHN.

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A full bibliography is available as electronic supplementary material at http://dx.doi.org/10.1098/rsbm.2010.0002 or via http://rsbm.royalsocietypublishing.org. As a tribute to BHN, the numbering system used for listing his publications here and in the text is based on that used by BHN himself in *Selected works* (120). Several entries in this list have a lower-case roman letter as an additional label following the numeral, and represent papers published or submitted at roughly the same time as the 'unadorned entry'; for example, (4a) and (4) were both published in 1935. The entries (1a) and (5a) are the two doctoral theses. All other such entries up to (115a) are numbered as in *Selected works* (120). BHN explained there that he added letters for items that he considered of lesser importance, and this was the guiding principle for later entries.

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