Vrije Universiteit Brussel
Faculty of Science and Bio-engineering Sciences

Identities of Affine Algebras and their asymptotic behaviour

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Pure mathematics is, in its way,
the poetry of logical ideas.

Albert Einstein

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Faculteit Wetenschappen en Bio-ingenieurswetenschappen

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Vakgroep Wiskunde

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As once nicely expressed by Hilbert

"Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country".

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Notational Conventions

$\underline{Alg}_F$	Category of all algebras over F with F-algebra morphisms
$\mathrm{Jac}(A)$	Jacobson radical of A
$\mathrm{Rad}(M)$	Radical of the module M
$\mathrm{Ann}(M)$	Left annihilator of the (left) module M
$\mathrm{supp}(R)$	Support of a given gradation of R
$\mathcal{Z}(R)$	Centre of the ring R
1_R	Unit element of the ring R
$\mathcal{O}(\cdot)$	Big O notation
$\mathcal{M}(G^0; I, J; P)$	Completely 0-simple semigroup
$R(L)$	Solvable radical of the Lie algebra L
$N(L)$	Nilpotent radical of the Lie algebra L
$A^{[-]}$	Associative algebra A endowed with Lie bracket
$\langle u, v\rangle_L$	Lie algebra generated by u and v
$A * B$	The free product of the algebras A and B
$\mathrm{GL}_n(R)$	Group of invertible matrices in $M_n(R)$
$M_n(R)$	Matrix ring over a ring R
$\mathbb{Z}$	Ring of integers
$\mathbb{N}$	Natural numbers, i.e positive integer numbers
$\mathbb{Q}$	Field of rational numbers
$\mathbb{R}$	Field of real numbers
$\mathbb{F}_p$	Field with p elements

$\mathbb{R}^n$	Euclidean space of dimension n
$\mathrm{char}(F)$	The characteristic of the field F
RG	Group ring of G over R
S_n	Symmetric group of degree n
$\lambda \vdash n$	Ordered partition of n
$\mu \models n$	Unordered partition of n
T_λ	Young tableau
D_λ	Young diagram
$h_\lambda(i,j)$	Hook number of the box (i,j) in D_λ
a_{T_λ}	Row symmetrizer corresponding to T_λ
b_{T_λ}	Column symmetrizer corresponding to T_λ
$M^R(\lambda)$	Permutation module over R corresponding to λ
$S^R(\lambda)$	Specht module over R corresponding to λ
$D^F(\lambda)$	Absolutely irreducible module over F corresponding to λ
$Y^F(\lambda)$	Young module over F corresponding to λ
$\leftrightarrow$	Abbreviation for "has the same composition factors as"
$c_n(A,F)$	n-th codimension of A over F
$P_n(F)$	Multilinear polynomials of degree n over F
$\Gamma_n(F)$	Proper multilinear polynomials of degree n over F
$\mathrm{Id}(A,R)$	Polynomial identities of A with coëfficients in R
$\mathrm{Id}(A)$	Polynomial identities of A if scalars are clear from context
$R\langle X\rangle$	Free algebra over R generated by variables in X
Alt_X	Operator of alternation on the set X
$P_n(A)$	The factor space of $P_n(F)$ with $P_n(F) \cap \mathrm{Id}(A)$
Reg_d	Regev polynomial in $2d$ variables
Cap_d	Capelli polynomial in $2d+1$ variables
$[x_1,\ldots,x_n]$	Left-normed commutator

Introduction

Motivation and origins

Given two algebras A and B, the most natural but most difficult question one can ask is whether A and B are isomorphic as algebras. One way of distinguishing A from B is to find a property satisfied by A but not satisfied by B. The idea is to associate to any algebra an invariant, under isomorphism, that takes different values on A and B. Even if one finds an invariant that distinguishes A from B, this does not mean that it would also allow to distinguish A from a third algebra C. Therefore, in order to distinguish a given algebra A one possibly needs more invariants, i.e. "a complete list of invariants". This is of course not feasible in full generality. However, this does not alter the fact that certain invariants may be of interest. In this thesis we will associate with any finite dimensional algebra A over a field of any characteristic a sequence of numbers $(c_n(A))_n$ called codimensions. In turn, this sequence will yield several concrete invariants containing concrete structural information about A.

To be more precise, we associate such a sequence with a much larger class of algebras than the class of finite dimensional algebras, namely the class of algebras satisfying a polynomial identity, in short PI-algebras. A polynomial identity of A is a non-zero polynomial $f(x_1, \dots, x_n)$ in non-commutative variables $x_1, \dots, x_n$ such that it vanishes when computed on A, i.e. $f(a_1, \dots, a_n) = 0$ for all $a_i \in A$. As will be explained in Chapter 1, any finite dimensional algebra satisfies such a 'universal relation'. Before discussing the results of this thesis, let us first see where this field of research gets its fuel. The field of PI-theory somehow starts at the end of the 1940s, at the dawn of non-commutative ring theory, with the works of Jacobson [Jac45], Kaplansky [Kap48] and Levitzki [Lev46] in which they solve the bounded Kurosh problem

Theorem (Bounded Kurosh Problem)**.** *Let A be a finitely generated associative algebra over a field F in which every $a \in A$ satisfies some polynomial $x^m + c_{m-1}x^{m-1} + \ldots + c_1x + c_0$, $c_i \in F$, where m and the c_i's may depend on a. If all m are uniformly, i.e. for all a, bounded by a fixed natural number n, then A is finite dimensional over F.*

A counterexample for this problem was given in 1964 by Golod and Shafarevich [GS64, Gol64] if abstraction is made of the uniform bounded assumption, so if A is just a finitely generated algebraic algebra. Interestingly, in the same papers they give a counterexample for the general Burnside problem in group theory whose bounded counterpart, however, is not true as proven by the work of Adian and Novikov [AN68a, AN68b, AN68c].

A first step towards solving the bounded Kurosh problem, as proven by Jacobson, is that an algebraic algebra of bounded degree satisfies a polynomial identity. Next, as proven by Kaplansky and Levitzki, a finitely generated algebraic PI-algebra is finite dimensional. This already gives the impression that the PI-property might be a condition that somehow 'restricts and controls the infinite-dimensionality' of a given PI-algebra. The reader would be right to think so. In fact, a PI-algebra has only finite dimensional simple representations, due to which PI-theory is strictly connected to the study of finite dimensional representations of algebras, a theory which has a strong geometric flavour as shown by Artin and Procesi in the 1960s. This led to a second development of PI-theory. Due to this, commutative geometry can to a certain extent be used in the non-commutative setting. So, even though PI-algebras can be *a priori* very non-commutative, PI-theory mixes methods of commutative algebra with methods of finite dimensional algebras, blended by representation theory.

In a second phase, PI-theory gained interest because of the so-called embedding problem.

Problem (Embedding Problem)**.** *Characterize associative rings which can be embedded in a matrix ring $M_n(C)$ over some commutative ring C.*

A ground-breaking result is the theorem of Amitsur-Levitzki stating that, for every commutative ring C, $M_n(C)$ satisfy a polynomial identity, namely the standard polynomial. So, a ring satisfying the embedding problem must satisfies all polynomial identities of a matrix algebra. Unfortunately, in general this is not a sufficient condition. As we will see in Chapter 1, this is however true for certain 'universal PI algebras', as proven by Kemer in the 1980s. The theory developed hereby will play a central role in Chapter 2.

Hopefully, the reader believes at this point that polynomial identities are at the crossroads of non-commutative algebra, representation theory and algebraic geometry.

Overview of the obtained results

Let us now come back to the main motivation: associating interesting invariants to PI-algebras, especially to finitely generated ones. In this thesis we do so by means of the T-ideal $\mathrm{Id}(A)$, consisting of all polynomial identities of A. Hence we are investigating PI-equivalence classes rather than isomorphism classes. Two algebras A and B are said *PI-equivalent* if $\mathrm{Id}(A) = \mathrm{Id}(B)$.

In case F has characteristic 0 the T-ideal $\mathrm{Id}(A)$ is generated (as a T-ideal) by multilinear polynomials. So, any information on A given by polynomial identities should also be delivered by the multilinear ones. We denote by $P_n(F) = \mathrm{span}_F\{x_{\sigma(1)} \dots x_{\sigma(n)} \mid \sigma \in S_n\}$ the multilinear polynomials of degree n. Then $c_n(A) = \dim_F \frac{P_n(F)}{P_n(F) \cap \mathrm{Id}(A)}$ is called the *n-th codimension of* A and $(c_n(A))_n$ the *codimension sequence of* A. In this thesis we are interested in understanding the asymptotic behaviour of this sequence in terms of algebraic data. In purely analytic terms the behaviour was predicted by Regev. One says that two functions f and g grow asymptotically the same, denoted $f \simeq g$, if $\lim_{n\to\infty} \frac{f}{g} = 1$.

Conjecture (Regev)**.** *Let A be an F-algebra with char$(F) = 0$, then*

$$c_n(A) \simeq cn^t d^n$$

for constants $c \in \mathbb{Q}(\sqrt{2\pi}, \sqrt{v}), v \in \mathbb{N}, t \in \frac{\mathbb{Z}}{2}$ and $d \in \mathbb{Z}$.

In section 1.3.1 we give a precise account on the state of art of this conjecture. Among other, this conjecture has been confirmed by Berele and Regev [BR08] for finitely generated unital algebras and for non-unital finitely generated algebras they showed that $c_n(A) \simeq \mathcal{O}(n^t d^n)$, where $\mathcal{O}(\cdot)$ denotes the big O-notation. Thanks to this, one can associate to any PI-algebra two invariants, namely the constants t and d. We refer to these numbers as, respectively, *the polynomial* and *exponential part of A*. Note that the integrality of d is really a striking result. It indicates that this growth function is very different from other ones such as the Gelfand-Kirillov or the word growth function in group theory where almost any real number can appear as an exponential growth rate.

Clearly, the numbers are not intrinsic to A, but to its PI-equivalence class which turns them into potential interesting invariants. So, the next question is which algebraic information, if any, is contained in these numbers. Concerning the exponential part, also called the *PI-exponent*, Giambruno and Zaicev proved in their seminal paper [GZ98] that the PI-exponent of a finite dimensional algebra A is connected to its Wedderburn-Malcev decomposition in the following way:

$$d = \max\{dim_F(A_{i_1} \oplus \cdots \oplus A_{i_r}) \mid A_{i_1} J(A) A_{i_2} \cdots J(A) A_{i_r} \neq 0 \text{ with } i_j \neq i_k \text{ for } j \neq k\},$$

where $A \cong A_{ss} \bigoplus J(A)$, $J(A)$ is the Jacobson radical of A and $A_{ss} \cong A_1 \oplus \ldots \oplus A_q$ is a maximal semisimple subalgebra of A. Note that this result was obtained before the aforementioned result by Berele and Regev and actually Giambruno and Zaicev proved in [GZ98, GZ99] the existence and integrality of $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ for any PI-algebra, even when not necessarily finitely generated. In order to make the transition to the non-finite dimensional setting the authors used Kemer theory, which leads to losing a concrete interpretation. Contrary to the proof of Giambruno and Zaicev's result, the proof of Berele and Regev's theorem is geometric by nature, allowing us to understand the asymptotic growth, but it gives no insight on the algebra side. One of the goals of this thesis is to fill this gap.

The thesis can be subdivided into two different parts. *In the first part*, i.e. Chapter 1 till Chapter 3, we work solely with the algebra A itself and aim at understanding the polynomial part t and at investigating whether such invariants can also be introduced and used over principal ideal domains and especially over $\mathbb{Z}$ (i.e. for rings). *In the second part*, i.e. Chapter 4 and Chapter 5, we take into account that actions on an object often contain interesting information about the object under consideration. We will mainly focus on actions by the dual of the semigroup algebra FS of a finite semigroup S, which alternatively can be rephrased in the language of gradings. This is done in terms of graded S-polynomials and analogues of the codimension sequence and its polynomial and exponential part.

Classical, non-graded, part

Let us now review the main results obtained in the ungraded part of this thesis.

The first goal is to connect the polynomial part t to the algebraic structure of A. To do so, we first reduce the problem to certain building blocks for working up to PI-equivalence, the so-called *'basic algebras'* which have been introduced by Kemer in his solution to the Specht problem. We now recall their definition.

For this purpose we decompose $A \cong A_{ss} \oplus J(A)$ according to the *Wedderburn-Malcev theorem*. The tuple $\mathrm{Par}(A) = (\dim_F A_{ss}, \mathrm{nildeg}(J(A)) - 1)$, where $\mathrm{nildeg}(J(A))$ is the smallest positive integer s such that $J(A)^s = 0$ but $J(A)^{s-1} \neq 0$, is called *the parameter of A*. Now, a finite dimensional algebra is called basic if it is not PI-equivalent with a direct sum of algebras $C_1 \oplus \ldots \oplus C_l$ where $\mathrm{Par}(C_i) < \mathrm{Par}(A)$ for the left lexicographic order and for all i. As shown in work of Kemer each finite dimensional algebra A is PI-equivalent to a finite direct sum $B_1 \oplus \ldots \oplus B_q$ of basic algebras. Since $t(A) = \max_i\{t(B_i) \mid d(B_i) = d(A)\}$, see Corollary 1.3.7, one first has to find an interpretation for the polynomial part of a basic algebra. An algebraic interpretation was conjectured by Giambruno. In Chapter 2, which is the result of joint work with Aljadeff and Karasik, we prove this conjecture.

Theorem 2.2.13. *[AJK17] Let A be a basic algebra with Wedderburn-Malcev decomposition $A \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_q}(F) \oplus J(A)$ and Par$(A) = (d, s)$. Then*

$$c_n(A) = \mathcal{O}(n^{\frac{q-d}{2}+s} d^n).$$

In the special case where the algebra A has a unit we have

$$c_n(A) \simeq C n^{\frac{q-d}{2}+s} d^n,$$

for some constant $0 < C \in \mathbb{R}$.

So in order to find an interpretation of t internal to A, one is now left with the problem of finding a constructive algorithm to decompose an algebra into basic algebras. We plan to investigate this in the future. A logical next phase would be using these invariants in order to distinguish PI-equivalence classes or investigating how the ground field can be weakened. The latter is the subject of Chapter 3 and is the result of joint work with Gordienko.

Let R be a not necessarily unital ring. In this case we consider $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R)}$ as a finitely generated abelian group. Its decomposition as an abelian group yields different codimension sequences $(c_n(R, \mathbb{Z}, p^k))_n$, one for each prime power p^k arising in the decomposition. In section 1.1.2 and in section 3.1 we investigate how these codimensions behave under scalar extension and restriction. As a consequence, we obtain Regev's conjecture for unital torsion-free rings.

Theorem 3.1.3. *[GJ13] Let R be a torsion-free ring satisfying a non-trivial polynomial identity. Then,*

1. *if $p^k \neq 0$, then $c_n(R, \mathbb{Z}, p^k) = 0$.*
2. *either $c_n(R, \mathbb{Z}, 0) = 0$ for all $n \geqslant n_0$, $n_0 \in \mathbb{N}$, or there exist $d \in \mathbb{N}, t \in \frac{\mathbb{Z}}{2}$ and $C_1, C_2 > 0$, such that $C_1 n^t d^n \leqslant c_n(R, \mathbb{Z}, 0) \leqslant C_2 n^t d^n$ for all $n \in \mathbb{N}$; in particular $\lim_{n\to\infty} \sqrt[n]{c_n(R, \mathbb{Z}, 0)} \in \mathbb{N}$ exists and is a positive integer.*
3. *if R is unital, then there exist $C > 0$ and $t \in \frac{\mathbb{Z}}{2}$ such that $c_n(R, \mathbb{Z}, 0) \simeq C n^t d^n$ as $n \to \infty$.*

Unfortunately, if R contains additive torsion, this will be lost if one does an extension of scalars $R \otimes_{\mathbb{Z}} \mathbb{Q}$. So, in this case one can not hope, as opposed to the torsion-free case, that the use of the classical theory for fields of characteristic 0 is enough. Also, in general, for rings with additive torsion codimensions for several prime-powers p^k are non-zero. The first main problem when working over $\mathbb{Z}$ is that the modules are no longer semisimple. Therefore in Chapter 3 we investigate the existence of 'nice' $\mathbb{Z}S_n$-filtrations that can take over the role of direct sum decompositions. More precisely, we ask the following question.

Question. *Let R be a ring. Does $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ have a chain of submodules with factors that are isomorphic to $S(\lambda)/mS(\lambda)$, where λ is a partition of n and $m \in \mathbb{Z}$ is related to the torsion of R?*

The $\mathbb{Z}S_n$-modules $S(\lambda)$ are called Specht modules and over a field of characteristic 0 they yield a full set of non-isomorphic absolutely irreducible S_n-representations. In section 1.2 we review all necessary S_n-representation theory. In the following result we reduce the problem to proper polynomials, i.e. products of long commutators. The $\mathbb{Z}$-module generated by the proper polynomials of length n is denoted by $\Gamma_n(\mathbb{Z})$.

Theorem 3.2.1. *[GJ13] Let R be a unital ring and $\operatorname{char} R = \ell$ a positive integer. Consider for every $n \in \mathbb{N}$ the series of $\mathbb{Z}S_n$-submodules*

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R, \mathbb{Z})} \supsetneq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(R, \mathbb{Z})}$$

where each M_k *is the image of* $\bigoplus_{t=k}^{n} \mathbb{Z}S_n(x_{t+1}\dots x_n\Gamma_t(\mathbb{Z}))$ *and* $M_{n+1} := 0$. *Then* $M_0/M_2 \cong \mathbb{Z}/\ell\mathbb{Z}$ *(trivial* S_n*-action) and*

$$\begin{aligned} M_t/M_{t+1} &\cong \left(\tfrac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \uparrow S_n \\ &:= \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t\times S_{n-t})} \left(\tfrac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right), \end{aligned}$$

for all $2 \leqslant t \leqslant n$ *where* S_{n-t} *is permuting* $x_{t+1},\dots,x_n$ *and* $\mathbb{Z}$ *is a trivial* $\mathbb{Z}S_{n-t}-$*module. If* $\ell = 0$ *we set by definition* $\mathbb{Z}/\ell\mathbb{Z} = 0$.

In case $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is reasonable, i.e. of the form $S(\lambda)/mS(\lambda)$, a generalisation of Young's rule, see Theorem 3.3.1, yields a positive answer to the previously mentioned question. Consequently, we have to investigate when $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is actually reasonable. We prove that this is the case for two important examples. The first one is the generalized upper-triangular matrix ring $R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$, with M an (R_1, R_2)-bimodule for commutative unital rings R_1 and R_2. The second one is the Grassmann algebra G_S over a commutative unital ring S with odd characteristic l. Recall that G_S is generated by the countable set $\{e_n \mid n \in \mathbb{N}\}$ satisfying $e_ie_j = -e_je_i$ for $i \neq j$. In section 3.4 and in section 3.5 we show that the proper polynomial non-identities are indeed reasonable. More precisely, we have the following theorem.

Theorem 3.4.8. *[GJ13] Let* R *be the generalized upper-triangular matrix ring. Let* ℓ *and* m *be the numbers from Subsection 3.4.1. Then there exists a chain of* $\mathbb{Z}S_n$*-submodules in* $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ *with the set of factors that consists of one copy of* $\mathbb{Z}_\ell$ *and* $(\lambda_1 - \lambda_2 + 1)$ *copies of* $S(\lambda_1,\lambda_2,\lambda_3)/mS(\lambda_1,\lambda_2,\lambda_3)$ *where* $(\lambda_1,\lambda_2,\lambda_3) \vdash n$, $\lambda_2 \geqslant 1$, $\lambda_3 \in \{0,1\}$.

In the case of the Grassmann algebra, we get the following theorem.

Theorem 3.5.4. *[GJ13] Let* G_S *be the Grassmann algebra over a commutative unital ring* S *with* $\ell = \operatorname{char} S$. *Then there exists a chain of* $\mathbb{Z}S_n$*-submodules in* $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$ *with factors* $S(n-k,1^k)/\ell S(n-k,1^k)$ *for each* $0 \leqslant k \leqslant n-1$ *(each factor occurs exactly once).*

Graded part

Let us now review the main results obtained in the graded part of this thesis.

In Chapter 4, which is joint work with Jespers and Gordienko, we classify all finite dimensional S-graded-simple algebras graded by a completely 0-semigroup with trivial maximal subgroup, i.e. for $S = \mathcal{M}(\{e\}^0, n, m, P)$. This may sound as a very restrictive class of semigroups, but we explain in section 4.1 that if A is an S-graded-simple algebra S can be reduced to semigroups of the form $\mathcal{M}(G^0, n, m, P)$. This semigroup may be thought of as certain $n \times m$-matrices with entries in $G \cup \{0\}$. The case we deal with, i.e. $G = \{e\}$, is somehow the opposite case of a group grading (in our notation this corresponds to $n = m = 1$). For group gradings a classification was obtained by Bahturin, Zaicev and Sehgal [BZ02, BZS08]. Hopefully both cases can be merged in the future and generate the general answer.

The classification consists of two parts. First we describe finite dimensional S-graded-simple algebras with $S = \mathcal{M}(\{e\}^0, n, m, P)$ and then we prove that any algebra satisfying the description yields a $\mathcal{M}(\{e\}^0, n, m, P)$-graded-simple structure. The former is done by first decomposing A as $B \oplus J(A)$ with B a maximal semisimple graded subalgebra and then by decomposing $J(A)$ into left B-modules which are isomorphic to concrete pieces of B. Before stating the classification, we want to point out that due to the definition of $\mathcal{M}(\{e\}^0, n, m, P)$ the algebra A can be decomposed into subspaces as

$$A = \bigoplus_{\substack{1 \le i \le n, \\ 1 \le j \le m}} A_{ij}$$

with $A_{ij} A_{k\ell} \subseteq A_{i\ell}$. We denote the 'columns' and 'rows' by

$$L_i := \bigoplus_{k=1}^{n} A_{ki} \quad \text{and} \quad R_i := \bigoplus_{k=1}^{m} A_{ik}. \tag{1}$$

Now we advance a graded Wedderburn-Malcev decomposition which is valid in this setting. The proof hereof is constructive.

Theorem 4.3.2. *[GJJ17] Let $A = \bigoplus_{i,j} A_{ij}$ be a finite dimensional S-graded F-algebra over a field F such that $AJ(A)A = 0$ and $S = \mathcal{M}(\{e\}^0, n, m, P)$. Then, there exist orthogonal idempotents $f_1, \ldots, f_m$ and orthogonal idempotents $f'_1, \ldots, f'_n$ (some of them could be zero) such that*

$$B = \bigoplus_{i,j} f'_i A f_j = \bigoplus_{i,j} (B \cap A_{ij})$$

is an S-graded maximal semisimple subalgebra of A, $f'_i \in B \cap R_i$ for $1 \le i \le n$, $f_j \in B \cap L_j$ for $1 \le j \le m$, $\sum_{i=1}^{n} f'_i = \sum_{j=1}^{m} f_j = 1_B$, and $A = B \oplus J(A)$ (direct sum of subspaces).

So, at this point we see $A = B \oplus J(A)$ with B a graded subalgebra which can be constructed concretely. For the decomposition of the radical we have to introduce some more notations. Consider for $1 \le i \le n$ and $1 \le j \le m$ the subspaces

$$J_{ij}^{10} := f_i' L_j (1 - 1_B) \quad \text{and} \quad J_{ij}^{01} := (1 - 1_B) R_i f_j.$$

Also, put

$$J_{*j}^{10} := \sum_{1 \le i \le n} J_{ij}^{10} = 1_B L_j (1 - 1_B) \quad \text{and} \quad J_{i*}^{01} := \sum_{1 \le j \le m} J_{ij}^{01} = (1 - 1_B) R_i 1_B.$$

These subspaces form the building blocks of $J(A)$.

Theorem 4.3.7. *[GJJ17] Let A be a finite dimensional S-graded-simple F-algebra. Let B and let $f_1, \ldots, f_m$, $f_1', \ldots, f_n'$ be, respectively, a graded subalgebra and orthogonal idempotents from Theorem 4.3.2.*

Then each J_{*j}^{10} is a left B-submodule of $J(A)$ and $J_{*j}^{10} = \bigoplus_{i=1}^n J_{ij}^{10}$. Also each J_{i*}^{01} is a right B-submodule of $J(A)$ and $J_{i*}^{01} = \bigoplus_{j=1}^m J_{ij}^{01}$. Moreover,

$$J(A) = \bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10} \oplus J(A)^2 \quad \text{and} \quad J(A)^2 = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{i*}^{01} J_{*j}^{10},$$

direct sums of subspaces.

In addition, there exists an F-linear map

$$\varphi \colon \bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10} \to B$$

behaving 'very nicely'. Furthermore,

$$\begin{aligned} A_{ij} = f_i' B f_j \oplus & \left\{ \varphi(v) + v \mid v \in J_{ij}^{10} \oplus J_{ij}^{01} \right\} \\ & \oplus span_F \left\{ \varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10} \right\} \end{aligned} \tag{2}$$

is a direct sum of subspaces, for all $1 \le i \le n$, $1 \le j \le m$.

Lastly, $B \cong M_k(D)$ for some $k \in \mathbb{N}$ and a division algebra D satisfying

$$\dim_F \bigoplus_{i=1}^{n} J_{i*}^{01} \le (n-1) \dim_F B = (n-1) k^2 \dim_F D, \tag{3}$$

$$\dim_F \bigoplus_{j=1}^{m} J_{*j}^{10} \le (m-1) \dim_F B = (m-1) k^2 \dim_F D, \tag{4}$$

$$\dim_F J(A) \le (nm-1) \dim_F B = (|S|-1) \dim_F B = (|S|-1) k^2 \dim_F D. \tag{5}$$

Of course in the actual statement of Theorem 4.3.7 we describe what we mean by that φ behaving 'very nicely'.

In Chapter 5 we first work in an associative setting and then in a non-associative one. In the former part we investigate the graded codimensions and their exponential growth of an infinite subfamily of above classified algebras. However, we will only use very little information from the classification. Due to this, both chapters can be read independently of each other, even though some intuition behind the algebras under consideration may be lost.

The associative part of the chapter is split into two cases depending on certain properties of the grading. Both cases show a very different behaviour. In the first case, the result is analogous to the group-graded case.

Theorem 5.4.5. *[GJJ17] Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 5.4. Suppose also that (5.13) holds or $T_0 = \varnothing$. Then there exist $C > 0$, $D \in \mathbb{R}$, such that*

$$Cn^D(\dim_F A)^n \leq c_n^{T\text{-gr}}(A) \leq (\dim_F A)^{n+1}.$$

In particular, $\operatorname{PIexp}^{T\text{-gr}}(A) = \dim_F A$.

In the second case, however, we obtain irrational graded PI-exponents. For example any number $1 + \sqrt{m} + m$ can be realized.

Theorem 5.5.5. *[GJJ17] Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 5.4. Suppose also that $|\bar{t}_0| > \frac{|T_0|}{2}$ for some $\bar{t}_0 \in T_0/\sim$. Then,*

$$\exp^{T\text{-gr}}(A) = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < 2|T_0| + 4|T_1| = \dim A.$$

To end this thesis we consider non-associative algebras, more precisely Lie algebras. Amongst other things we construct the first example of a finite dimensional semigroup-graded Lie algebra with non-integer graded PI-exponent. Along the way, we also prove a semigroup-graded version of Ado's Theorem asserting that a finite dimensional Lie algebra has a finite dimensional faithful representation. In contrast to the associative case, this algebra can not be simple with respect to the grading.

Theorem 5.7.1. *Let L be the Lie algebra with $(\mathbb{Z}_2, \cdot)$-grading from the beginning of section 5.7. Then* $\exp^{\mathbb{Z}_2}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{\mathbb{Z}_2}(L)} = 2 + 2\sqrt{2}$.

Guideline

In Chapter 1 we give a glimpse of the background needed to understand and get intuition for the research performed during this project. First, we start by providing in section 1.1 all basic definitions and we introduce codimensions of an algebra defined over any principal ideal domain. If an algebra A is defined over different principal ideal domains (in particular fields) one can associate different codimension sequences to A. The connection between them is also described in this section.

Next, in order to compute codimensions we use representation theory of the symmetric group. This is a very rich theory. The essentials are recalled in section 1.2. Hereby we opted to keep the exposition as independent as possible from the ground ring in order to better emphasize where in the classical asymptotic theory of polynomial identities one needs A to be an F-algebra with $\operatorname{char}(F) = 0$. In addition to that, it is also of use in Chapter 3 where we only work over $\mathbb{Z}$ whose codimension theory, we believe, has more resemblances with the $\operatorname{char}(F) = p \neq 0$ setting, then with the $\operatorname{char}(F) = 0$ case.

With S_n-representation theory at hand, we survey in section 1.3 the main results on Regev's conjecture and, moreover, we explain how one has to proceed to compute the PI-exponent of an F-algebra with $\operatorname{char}(F) = 0$. Finally, before moving further on our journey in the world of invariants, we introduce in section 1.4 the main features of Kemer's theory.

Chapter 2 focuses on the proof of Giambruno's conjecture which describes a value for the polynomial part of basic algebras. This is based on joint work with Aljadeff and Karasik [AJK17]. The chapter makes ample use of the theory reviewed in section 1.4 and consists of two sections, upper and lower bound.

Next, in Chapter 3 we leave the setting of algebras defined over fields of characteristic 0 and we investigate $\mathbb{Z}$-algebras, i.e. rings. Among other things, we prove a variant of the Amitsur and Regev conjecture for (unital) torsion-free rings and discuss the existence of a $\mathbb{Z}S_n$-filtration of $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \operatorname{Id}(A)}$ that may serve as a substitute for the direct sum decomposition in the case of a field F of characteristic 0. In section 3.2 the problem of its existence is reduced to proper polynomial identities. The existence and the strength of such a filtration is afterwards proven in section 3.4 and in section 3.5, respectively in the case of a generalized upper-triangular matrix ring and in the case of the Grassmann algebra. This chapter is based on joint work with Gordienko [GJ13].

Chapter 4 leaves for the first time the ungraded setting and considers algebras with a semigroup gradation. The chapter addresses the problem of classifying all finite dimensional T-graded simple algebras for an arbitrary semigroup T. In section 4.1 we reduce the problem at first to three types of semigroups. Afterwards, the classification problem for the semigroups $\mathcal{M}(\{e\}^0, n, m, P)$ is solved in section 4.3 and in section 4.4.

Finally, in Chapter 5 we consider the algebras classified earlier. For all these we give an upper-bound on the exponential growth of their graded codimensions in section 5.3. Next, we compute in section 5.4 and in section 5.5 the exact value of the graded PI-exponent for an infinite subfamily of these semigroup graded-simple algebras. These results yield arbitrarily large irrational graded PI-exponents. Chapter 4 and the just mentioned result are based on joint work with Gordienko and Jespers [GJJ17]. Finally, we also consider Lie algebras and produce in section 5.7 the first example of a graded Lie algebra with non-integer graded PI-exponent.

In appendix A we give a survey of some of the research performed during the ph.d. on topics not explicitly connected to codimensions.

1 A glimpse into the theory of Polynomial Identities

As Grothendieck taught us,
objects aren't of great importance;
It is the relations between them that are.
Jean-Pierre Serre

In this chapter we review the different building blocks for the investigations in polynomial identities of algebras. We will start by sketching what polynomials and asymptotic methods have in common and how this fits in getting ring theoretical information. This will be in terms of a sequence of numbers, called codimensions. In order to extract structural information from this we will need to take some S_n action into account. This will enable us to use S_n-representation theory as a main tool. Therefore, in the next section, we review the necessary representation theory over a field of arbitrary characteristic. With this at hand, we explain in the third section how to compute the exponential growth rate of the codimension sequence in an explicit way. In contrast to the first two sections, this will be done over a field of characteristic zero. Besides the exponential part also the polynomial component of the growth rate contains interesting data. This is the content of Chapter 2. However, in order to tackle the later problem, we need methods from Kemer's solution to the Specht problem, which he solved by proving that relatively free algebras are representable over some field extension. More precisely, we need the so-called basic algebras and Kemer polynomials. In the fourth section we recall these concepts and illustrate them with some important examples.

1.1 From Polynomials to Asymptotics

In this section we recall basic definitions and introduce codimensions for algebras over principal ideal domains. For the former the reader will find a complete survey in the books [DF04, GZ05, KBKR16]. Codimensions over fields are explained in a very clear way in [GZ05]. Over $\mathbb{Z}$ these sequences have been introduced in [GJ13] and the results here are direct generalisations of Proposition 2.1-2.4 therein.

1.1.1 Basic definitions

By A we denote an associative, not necessarily unital, F-algebra over a unital integral domain F. Let $X = \{x_i \mid i \in \mathbb{N}\}$ be a countable set of non-commuting variables and let $F\langle X \rangle$ denote the free unital associative algebra in these variables. A polynomial $f(x_1, \dots, x_n) \in F\langle X \rangle$ is called a *polynomial identity* of A, denoted $f \equiv_A 0$, if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. We also say that A satisfies the polynomial f.

Example 1.1.1.

1. An algebra A is commutative if and only if $xy - yx \equiv_A 0$
2. Also by definition an algebra A is nilpotent of degree d if and only if $x_1 \dots x_d \equiv_A 0$.
3. The polynomial $[[x, y]^2, z]$ is a polynomial identity of $M_2(F)$ due to the Cayley-Hamilton theorem. Indeed for a 2-by-2 matrix P its characteristic equation takes the form $x^2 - \operatorname{tr}(P)x + \det(P)$, consequently $[P, Q]^2 = -\det([P, Q])I_2$ is a central matrix for any pair of matrices P and Q.
4. Suppose F is a field such that $\operatorname{char}(F) \neq 2$ and $\dim_F A = d < \infty$. Then A satisfies the *Standard polynomial* St_{d+1} *of degree* $d + 1$:

$$St_{d+1}(x_1, \dots, x_{d+1}) := \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma) x_{\sigma(1)} \dots x_{\sigma(d+1)},$$

   where S_{d+1} denotes the symmetric group on $d + 1$ letters. This is an identity of A due to the combination of two easy but crucial remarks that will be recurrent in this thesis. First St_{d+1} is *multilinear* in each variable (i.e. each variable has as exponent exactly one in each monomial) and due to this when checking whether it is a polynomial identity its suffices to do so on basis elements. Secondly, St_{d+1} is *alternating in its variables*, i.e. for $\pi \in S_{d+1}$,

$$\pi St_{d+1}(x_1, \dots, x_n) = St_{d+1}(x_{\pi(1)}, \dots, x_{\pi(d+1)}) = \operatorname{sign}(\pi) St_{d+1}(x_1, \dots, x_n).$$

Due to this property, if the same element is twice substituted in St_{d+1} the result of the evaluation is zero.

5. By the previous example any finite dimensional algebra satisfies a polynomial identity. More generally any F-algebra A that is a finite module over its centre $\mathcal{Z}(A)$, say generated by d elements over $\mathcal{Z}(A)$, satisfies some polynomial identity over F. To be more precise, similarly to before, it satisfies St_{d+1} which has coefficients ± 1, in particular coefficients in F.

An F-algebra does not have to satisfy a polynomial identity, however if it does one calls A a *PI-algebra over F*. For example the Weyl algebra $\mathbb{A}_1 = \langle x, y \mid xy - yx = 1 \rangle$ over $\mathbb{C}$ is not PI. The reason for this follows from [MR01, Theorem 13.10.3] where it is explained that PI-algebras have finite dimensional irreducible representations. Now, $\mathbb{A}_1$ does not have such representations for otherwise we would have $0 = \mathrm{tr}(\rho(xy - yx)) = \mathrm{tr}(1) \neq 0$ in some representations ρ.

Given an algebra A we define

$$\mathrm{Id}(A, F) = \{f \in F\langle X \rangle \mid f \equiv_A 0\}$$

the set of polynomial identities of A over F. Usually the field F is understood and then we simply write $\mathrm{Id}(A)$. Clearly $\mathrm{Id}(A)$ is an ideal of $F\langle X \rangle$. Moreover if $f(x_1, \dots, x_n) \in \mathrm{Id}(A)$ then also $f(g_1, \dots, g_n) \in \mathrm{Id}(A)$ for all $g_i \in F\langle X \rangle$. In more sophisticated words, $\mathrm{Id}(A)$ is invariant under endomorphisms $\phi \in \mathrm{End}(F\langle X \rangle)$. Such ideals are called *T-ideals*.

Definition 1.1.2. An ideal I of $F\langle X \rangle$ is a T-ideal if $\phi(I) \subseteq I$ for all $\phi \in \mathrm{End}(F\langle X \rangle)$. For $J \subseteq F\langle X \rangle$ we denote by $(J)_T$ the T-ideal generated by the polynomials in J.

In this thesis we will be interested in the information on A determined by $\mathrm{Id}(A)$. Surely, an algebra is not determined by its T-ideal of polynomial identities. Therefore, one rather investigates the so called PI-equivalence classes, where two algebras A and B are said to be *PI-equivalent*, written $A \sim_{PI} B$, if $\mathrm{Id}(A) = \mathrm{Id}(B)$.

The equivalence class of A contains a distinguished object, namely $F\langle X \rangle / \mathrm{Id}(A)$. This algebra is the free object in the category consisting of all algebras B such that $\mathrm{Id}(A) \subseteq \mathrm{Id}(B)$. To avoid confusion these distinguished objects are called *relatively free algebras*.

For a given set of polynomials $I \subseteq F\langle X \rangle$ there is an associated PI-algebra $F\langle X \rangle / (I)_T$. Note that $\mathrm{Id}\left(F\langle X \rangle / (I)_T\right) = (I)_T$. In particular, each T-ideal appears as the set of polynomial identities of an algebra.

So we have seen that any algebra determines a T-ideal and vice versa. On the other hand many algebras correspond to the same T-ideal. This is best expressed in the language of varieties.

Definition 1.1.3. Let I be a non-empty set in $F\langle X\rangle$. Then $\mathcal{V} = \mathcal{V}(I) = \{A \in \underline{Alg}_F \mid I \subseteq \mathrm{Id}(A)\}$ which consists of all F-algebras containing I in their T-ideal of polynomial identities is called the *variety determined by I.*

For example if $I = \{xy - yx\}$ then $\mathcal{V}(I)$ is the class of commutative F-algebras. The class $\mathcal{V}(I)$ is a variety in the sense of Birkhoff. Note that $\mathcal{V}(I) = \mathcal{V}((I)_T)$ and $(I)_T = \bigcap_{A\in\mathcal{V}} \mathrm{Id}(A)$. So any variety determines a T-ideal and vice versa. Remark also that $A \sim_{PI} B$ if and only if $\mathcal{V}(\mathrm{Id}(A)) = \mathcal{V}(\mathrm{Id}(B))$.

Thus, in the sequel we will focus on understanding the T-ideal structure of $\mathrm{Id}(A)$, or equivalently the class structure of varieties $\mathcal{V}(I)$, which in turn results in understanding the algebra structure of relatively free algebras. Each viewpoint will have its own, not only intuitive, benefit at a certain moment. For example relatively free algebras play a key role in Kemer's solution to the Specht problem which asserts that any T-ideal is finitely generated as T-ideal, see [Kem91, AKBK16].

In Examples 1.1.1 we already emphasized the importance of multilinear polynomials (actually all examples are multilinear). This is not a coincidence since, over a field of characteristic 0, they generate T-ideals. In the next paragraphs we formalise this.

Definition 1.1.4. Let $|X| = k$, with k a finite cardinal. Then

$$F\langle X\rangle = \bigoplus_{n\in\mathbb{N}} F\langle X\rangle^{(n)}$$

is the natural $\mathbb{N}$-grading on $F\langle X\rangle$, i.e. the grading induced by setting $\deg(x) = 1$ for all $x \in X$. So $F\langle X\rangle^{(n)}$ is the F-subspace generated by all monomials of total degree n. One can further decompose

$$F\langle X\rangle^{(n)} = \bigoplus_{\substack{(i_1,\dots,i_k)\\ i_1+\dots+i_k=n}} F\langle X\rangle^{(i_1,\dots,i_k)},$$

where $F\langle X\rangle^{(i_1,\dots,i_k)}$ is the subspace spanned by all monomials of degree i_j in x_j for all $j \leq k$. This yields a $\mathbb{N}^k$ grading of $F\langle X\rangle$.

The polynomials $g \in F\langle X\rangle^{(n)}$ are called *homogeneous of degree n*. If $g \in F\langle X\rangle^{(1,\dots,1)}$ we say g is multilinear. Thus, for $n \leq k$,

$$P_n(F) := \mathrm{span}_F\{x_{\sigma(1)}\dots x_{\sigma(n)} \mid \sigma \in S_n\}$$

are the multilinear polynomials of degree n.

The previous definitions clearly generalise to countable cardinals, in particular we can consider $P_n(F)$ for any n. If $f \in F\langle X \rangle$, then one can decompose f according to above $\mathbb{N}^k$-grading, called the *multihomogeneous decomposition*, $f(x_1, \dots, x_k) = \sum f^{(i_1,\dots,i_k)}$. Then, a typical Vandermonde argument yields the following result, see [GZ05, Theorem 1.3.2] for a proof.

Theorem 1.1.5. *Suppose* $|F| > \deg f$. *If* $f \in \mathrm{Id}(A)$, *then* $f^{(i_1,\dots,i_k)} \in \mathrm{Id}(A)$ *for all* $(i_1, \dots, i_k)$.

Thus $\mathrm{Id}(A)$ is generated by multihomogeneous polynomials if for example F is an infinite field. Actually the proof of the above result also holds for algebras over integral domains. Suppose now $f(x_1, \dots, x_n)$ is a polynomial identity of A which is not multilinear. Then there exists a variable, say x_1, in which f has degree $d > 1$. Then one defines the linearization of f in x_1 to be the polynomial

$$g(z_1, z_2, x_2, \dots, x_n) = f(z_1 + z_2, x_2, \dots, x_n) - f(z_1, x_2, \dots, x_n) - f(z_2, x_2, \dots, x_n),$$

where z_1 and z_2 are new variables. Then g is still a polynomial identity which has a non-zero homogeneous component of degree $d-1$ but has no homogeneous component of degree d in z_1 or z_2. Note also that if $\mathrm{char}(F) > \deg f$, then $(g)_T = (f)_T$. This sketches a proof for the following result. See [DF04, Proposition 1.2.8] for more details.

Proposition 1.1.6. *Let A be a PI-algebra over an integral domain R, then A satisfies a multilinear polynomial. If $R = F$ a field with char$(F) = 0$, then* $\mathrm{Id}(A)$ *is generated as a T-ideal by multilinear polynomials.*

Thus, over a field of characteristic zero, in order to describe $\mathrm{Id}(A)$ it is enough to understand $\bigcup_{n \in \mathbb{N}} P_n(F) \cap \mathrm{Id}(A)$. However, we will see later that over a field of characteristic p multilinear polynomials still retain a lot of structural information. In the next section we associate to the multilinear polynomials a sequence of numbers, called codimensions. This sequence is the main protagonist of this thesis.

1.1.2 Codimensions

Over Fields

Except when stated otherwise, A will be a F-algebra with F a field of *any characteristic*. We are interested in $\bigcup_{n \in \mathbb{N}} P_n(F) \cap \mathrm{Id}(A)$. Unfortunately, only for very few examples generators for $\mathrm{Id}(A)$, as T-ideal, are known. Therefore, we will rather study some numbers invariant under PI-equivalence associated to multilinear polynomials. A first naive idea would be to consider $\dim_F P_n(F) \cap \mathrm{Id}(A)$, for all n. However this is, without surprise, also only known for few examples and on the other hand asymptotically $\dim_F P_n(F) \cap \mathrm{Id}(A) \simeq n! = \dim_F P_n(F)$, i.e. it yields no information. Since the latter grows factorially one might expect that the following sequence has a slower, more interesting, growth.

Definition 1.1.7. For all $0 \leq n \in \mathbb{N}$,

$$c_n(A, F) := \dim_F \frac{P_n(F)}{P_n(F) \cap \mathrm{Id}(A, F)}$$

is called *the n-th codimension of A* and $(c_n(A, F))_n$ its codimension sequence. If F is understood we write $c_n(A)$.

Suppose A is finite dimensional over F with F-basis $\mathcal{B} = \{b_1, \dots, b_d\}$ and $f = \sum_{\sigma \in S_n} a_\sigma x_{\sigma(1)} \dots x_{\sigma(n)} \in P_n(F)$. Then

$$f(x_1, \dots, x_n) \in \mathrm{Id}(A) \text{ if and only if } f(b_{i_1}, \dots, b_{i_n}) = 0 \text{ for all basis elements } b_{i_j} \in \mathcal{B}.$$

This yields a system of d^{n+1} equations, since any choice of b_{i_j}'s yield d equations, in $n!$ variables (namely the coefficients a_σ). Clearly the dimension of the solution set of this system is equal to $\dim_F P_n(F) \cap \mathrm{Id}(A)$ on the one hand and $n! - r$ with $r (\leq d^{n+1})$ the rank of the system on the other hand. In particular $c_n(A) = r \leq d^n$ is exponentially bounded. In [Reg72], Regev proved this fact for any PI-algebra. At that time it served as a tool in order to show that the tensor product of two PI-algebras is still a PI-algebra (see [GZ05, Theorem 4.2.4] for a simplified proof).

Theorem 1.1.8 (Regev)**.** *Suppose the algebra A satisfies an identity of degree $d \geq 1$, then $c_n(A) \leq (d-1)^{2n}$.*

Example 1.1.9.

1. Suppose A is commutative or in other words $xy - yx \in \mathrm{Id}(A)$. If moreover A is non-nilpotent, then $\frac{P_n(F)}{P_n(F) \cap \mathrm{Id}(A)} = \mathrm{span}_F\{x_0 \cdots x_n\}$ and $c_n(A) = 1$ for all n.
2. If A is a nilpotent of degree d, then $c_n(A) = 0$ for all $n \geq d$.

Note also that, by Theorem 1.1.8, $\limsup_{n \to \infty} \sqrt[n]{c_n(A)}$ and $\liminf_{n \to \infty} \sqrt[n]{c_n(A)}$ exist. The goal of this thesis is to understand $c_n(A)$ asymptotically. In this spirit, Regev made the following conjecture that will be a leitmotif in this text.

Conjecture 1 (Regev). *Let A be a PI-algebra over a field F of characteristic 0. Then, there exist constants $t \in \frac{\mathbb{Z}}{2}, d \in \mathbb{N}$ and $C \in \mathbb{Q}(\sqrt{2\pi}, \sqrt{b})$ for some positive integer b such that*

$$c_n(A) \simeq Cn^t d^n,$$

where $f \simeq g$ if $\lim\limits_{n\to\infty} \frac{f}{g} = 1$

A consequence of this conjecture is that $d = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ would exist and it is an integer. The latter was first conjectured by Amitsur and was proven in 1998 by Giambruno and Zaicev in their breakthrough papers [GZ98, GZ99].

In Section 1.3 we will review the current status of this conjecture and show how to prove the existence and integrality of $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$. We already reveal that this will be achieved through the smart use of the representation theory of S_n. Namely

$$\pi \cdot \sum_{\sigma\in S_n} a_\sigma x_{\sigma(1)} \dots x_{\sigma(n)} := \sum_{\sigma\in S_n} a_\sigma x_{\pi(\sigma(1))} \dots x_{\pi(\sigma(n))}$$

yields an FS_n-module structure on $P_n(F)$ that leaves $P_n(F) \cap \mathrm{Id}(A)$ invariant since $\mathrm{Id}(A)$ is a T-ideal. In particular, $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A)}$ is an FS_n-module. In Section 1.2 we give an account of the needed S_n-representation theory. However, first we discuss another problem. Namely, if we consider A over different fields, or even principal ideal domains, how do the different codimensions relate to each other?

Varying the ground ring

Let R be a commutative ring. If R contains a unit element 1_R we denote by $\mathrm{char}(R)$ its characteristic, i.e. the smallest non-zero positive integer n such that $n1_R = 0$. As usual, if $n1_R \neq 0$ for all n, then $\mathrm{char}(R) = 0$.

Let $P_n(R)$ be the free R-module generated by the multilinear monomials $x_{\sigma(1)} \dots x_{\sigma(n)}$ of degree n for all $\sigma \in S_n$ and A an R-algebra. Clearly $P_n(R) \cap \mathrm{Id}(A, R)$ is an R-submodule of $P_n(R)$ and thus we can consider the R-module $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ of non-polynomial identities of A. In case R is a principal ideal domain, it can be decomposed as R-module into primary submodules.

Definition 1.1.10. Let

$$\frac{P_n(R)}{P_n(R) \cap \mathrm{Id}(A, R)} \cong \underbrace{R \oplus \cdots \oplus R}_{c_n(A,R,0)} \oplus \bigoplus_{1\leq i\leq t} \underbrace{\Big(R/(p_i)^{k_i} \oplus \cdots \oplus R/(p_i)^{k_i}\Big)}_{c_n(A,R,p_i^{k_i})},$$

where $p_1, \dots, p_t$ are, possibly equal, irreducible elements of R. Then the $c_n(A, p_i^{k_i})$ are called the *R-codimensions* of polynomial identities of A.

Note that the symmetric group S_n again acts on $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ by permutations of variables, i.e., $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ is an RS_n-module. We refer to $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ as the *RS_n-module of ordinary multilinear polynomial functions* on A.

In case $R = F$ is a field then only $c_n(A, F, 0)$ is non-zero, since then $(p_i) = F$, and thus we recover the classical definition of codimensions. In this case, we will write $c_n(A, F)$ and $\mathrm{Id}(A, F)$ or even $c_n(A)$ and $\mathrm{Id}(A)$ if the ground field F is understood.

In case $R = \mathbb{Z}$, the above decomposition is simply the decomposition of $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ as a finitely generated abelian group and the p_i's are prime numbers. In Chapter 3 we will investigate this further.

In the sequel of this section we will denote by R a principal ideal domain, K its field of fractions and F some field extension of K. We will not distinguish R and its natural image inside K. If A is an F-algebra, than we can also treat it as an algebra over K or R and that way this give rise to different codimensions. Fortunately, these behave nicely with respect to each other.

Proposition 1.1.11. *Let A be an algebra over a field F and $R \subseteq K \subseteq F$ as before. Then $c_n(A, R, q) = 0 = c_n(A, F, q)$ for $q \neq 0$, a non-zero prime power. Also if* $\mathrm{char}\, F = p$ *then $c_n(A, \mathbb{Z}, q) = 0$ for all n whenever $q \neq p$.*

Proof. Let $f \in P_n(\mathbb{Z})$. Note that $(\mathrm{char}\, F)f \in \mathrm{Id}(A, \mathbb{Z})$ for all $f \in \mathbb{Z}\langle X\rangle$. Hence $\mathrm{char}\, F = p > 0$ implies $c_n(A, \mathbb{Z}, q) = 0$ for all $n \in \mathbb{N}$ and $q \neq p$, proving the second statement.

Clearly $c_n(A, F, q) = 0$ for $0 \neq q \in F$. Let now p be a non-zero irreducible element of R, $q = p^k$ and suppose $c_n(A, R, q) \neq 0$. Then there exists an $f \in P_n(R)$, but not in $P_n(R) \cap \mathrm{Id}(A, R)$, such that $q.f \in \mathrm{Id}(A, R)$. Since we view R as a subring of its field of fractions, there exists an $r \in K$ such that $rq = 1$. Now $f = r.q.f \in \mathrm{Id}(A, K) \cap P_n(R) = \mathrm{Id}(A, R) \cap P_n(R)$ a contradiction. So indeed $c_n(A, R, q) = 0$. ∎

The only non-zero codimensions behave as follows with respect to each other.

Proposition 1.1.12. *Let A be an algebra over a field F such that $R \subseteq K \subseteq F$. Then $c_n(A, F, 0) \leq c_n(A, K, 0) = c_n(A, R, 0)$ for all $n \in \mathbb{N}$.*

Proof. By Proposition 1.1.11, $\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)}$ is a free R-module of finite rank. Let $f_1,\ldots,f_s$ be the preimages of its free generators in $P_n(R)$. Note that $P_n(R)\subseteq P_n(K)\subseteq P_n(F)$ and, for every $\sigma\in S_n$, the monomial $x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$ can be expressed as an R-linear combination of $f_1,\ldots,f_s$ plus an element of $P_n(R)\cap \mathrm{Id}(A,R)$. Hence, the images of $f_1,\ldots,f_s$ generate $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A,F)}$ and $c_n(A,F,0)\le c_n(A,R,0)=s$. The same argument, replace R into K, shows that also $c_n(A,F,0)\le c_n(A,K,0)$.

Suppose $f_1,\ldots,f_s$ are K-linearly dependent modulo $\mathrm{Id}(A,K)$. In this case, $q_1^{-1}r_1f_1+\cdots+q_s^{-1}r_sf_s\in \mathrm{Id}(A,K)$ for some $q_i,r_i\in R$ and $q_i\ne 0$. Thus

$$\left(\prod_{i=2}^{s} q_i\right) r_1f_1+q_1\left(\prod_{i=3}^{s} q_i\right) r_2f_2+\cdots+\left(\prod_{i=1}^{s-1} q_i\right) r_sf_s\in \mathrm{Id}(A,K)\cap P_n(R)=\mathrm{Id}(A,R)\cap P_n(R).$$

However, as the f_i are linearly independent modulo $\mathrm{Id}(A,R)$, all $r_i=0$. Therefore, the images of $f_1,\ldots,f_s$ form a K-basis of $\frac{P_n(K)}{P_n(K)\cap \mathrm{Id}(A,K)}$ and $c_n(A,K,0)=c_n(A,R,0)=s$. ∎

In general it is possible that $c_n(A,F,0)\ne c_n(A,K,0)$ for $F\supsetneqq K$. The arguments below show that this can already happen in the easiest case, i.e. for $R=\mathbb{Z}, K=\mathbb{Q}$ and F a degree 2 number field. If the reader is not familiar with Specht modules $S^F(\lambda)$ we refer to the next section for the definitions.

Example 1.1.13. Note that $P_3(\mathbb{Q})\cong \mathbb{Q}S_3\cong S^{\mathbb{Q}}(3)\oplus S^{\mathbb{Q}}(2,1)\oplus S^{\mathbb{Q}}(2,1)\oplus S^{\mathbb{Q}}(1^3)$. Let $a\in\mathbb{Q}S_3$ be such that $S^{\mathbb{Q}}(2,1)=\mathbb{Q}S_3a$. Denote by f_1 and f_2 the polynomials that correspond to a in the copies of $S^{\mathbb{Q}}(2,1)$ in $P_3(\mathbb{Q})$. Let $F=\mathbb{Q}(\sqrt{2})$. Consider the T-ideal I of $F\langle X\rangle$ generated by $(f_1+\sqrt{2}f_2)$. We claim that $c_3(F\langle X\rangle/I,F)=4<c_3(F\langle X\rangle/I,\mathbb{Q})=6$.

Proof. First we notice that $P_3(F)\cap \mathrm{Id}(F\langle X\rangle/I,F)=FS_3\cdot(f_1+\sqrt{2}f_2)\cong S^F(2,1)$. Hence by the Hook formula, Theorem 1.2.15, $c_3(F\langle X\rangle/I,F)=6-2=4$. Note that, in this low-dimensional case, one can also compute it directly, without use of the non-trivial Hook formula. However, $P_3(\mathbb{Q})\cap \mathrm{Id}(F\langle X\rangle/I,\mathbb{Q})=P_3(\mathbb{Q})\cap FS_3(f_1+\sqrt{2}f_2)=0$. Indeed, suppose $f=b(f_1+\sqrt{2}f_2)\in P_3(\mathbb{Q})$ for some $b\in FS_3$. Note that $b=b_1+\sqrt{2}b_2$ where $b_1,b_2\in\mathbb{Q}S_3$. Therefore, $f=(b_1+\sqrt{2}b_2)(f_1+\sqrt{2}f_2)=(b_1f_1+2b_2f_2)+\sqrt{2}(b_1f_2+b_2f_1)$ and $f\in P_3(\mathbb{Q})$ implies $b_1f_2+b_2f_1=0$. Recall that $\mathbb{Q}S_3f_1\oplus\mathbb{Q}S_3f_2$ is the direct sum of $\mathbb{Q}S_3$-submodules. Hence $b_1f_2=b_2f_1=0$. However, $\mathbb{Q}S_3f_1\cong\mathbb{Q}S_3f_2$. Thus $b_1f_1=b_2f_2=0$ too, $f=0$, $P_3(\mathbb{Q})\cap \mathrm{Id}(F\langle X\rangle/I,\mathbb{Q})=0$ and $c_3(F\langle X\rangle/I,\mathbb{Q})=6$. ∎

At this point it is important to note that in general $F\langle X\rangle/I \otimes_{\mathbb{Q}} F$ is a larger algebra than $F\langle X\rangle/I$. Due to this, codimensions remain equal under extension of scalars, see [GZ98, Remark 1].

Theorem 1.1.14. *Let A be an F-algebra with F any field and $F \subseteq K$. Then $c_n(A \otimes_F K, K) = c_n(A, F)$ for all n.*

Finally, in case A is an F-algebra with $\operatorname{char} F = p > 0$ one can also consider A as a ring, i.e. as $\mathbb{Z}$-algebra. By Proposition 1.1.12, $c_n(A, F) \leq c_n(A, \mathbb{F}_p)$, however the proposition doesn't say how the $\mathbb{F}_p$-codimensions and the $\mathbb{Z}$-codimensions connect to each other.

Proposition 1.1.15. *Let A be an algebra over a field F, $\operatorname{char} F = p$. Then $c_n(A, \mathbb{F}_p) = c_n(A, \mathbb{Z}, p)$ for all $n \in \mathbb{N}$.*

Proof. Suppose $c_n(A, \mathbb{Z}, p) = u$ and $c_n(A, \mathbb{F}_p) = v$. So, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(A,\mathbb{Z})} \cong (\mathbb{Z}/p\mathbb{Z})^u \cong \mathbb{F}_p^u$ and $\frac{P_n(\mathbb{F}_p)}{P_n(\mathbb{F}_p) \cap \operatorname{Id}(A,\mathbb{F}_p)} \cong \mathbb{F}_p^v$.

Let $f = \sum_{\sigma \in S_n} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \in P_n(\mathbb{Z})$ and write $a_\sigma = t_\sigma + pk_\sigma$ with $0 \leq t_\sigma < p$ and $t_\sigma, k_\sigma \in \mathbb{Z}$. Then, $f_l = \sum_{\sigma \in S_n} (t_\sigma + pl_\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)} \in P_n(\mathbb{Z})$ for any choice of $l_\sigma \in \mathbb{Z}$. Note that $f - f_l \in \operatorname{Id}(A, \mathbb{Z}) \cap P_n(\mathbb{Z})$. On the other hand, if $\overline{f} = \sum_{\sigma \in S_n} \overline{a_\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)} \in P_n(\mathbb{F}_p)$, then we define $f_z := \sum_{\sigma \in S_n} a_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} \in P_n(\mathbb{Z})$, where $a_\sigma \in \mathbb{Z}$ is such that $a_\sigma \bmod p = \overline{a_\sigma}$.

Consider now the canonical map of abelian groups $\pi : \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(A,\mathbb{Z})} \to \frac{P_n(\mathbb{F}_p)}{P_n(\mathbb{F}_p) \cap \operatorname{Id}(A,\mathbb{F}_p)}$. Then clearly $\phi : \frac{P_n(\mathbb{F}_p)}{P_n(\mathbb{F}_p) \cap \operatorname{Id}(A,\mathbb{F}_p)} \to \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(A,\mathbb{Z})} : \overline{f} \mapsto f_z$ is well defined because $f - f_l \in \operatorname{Id}(A, \mathbb{Z})$. It clearly is an isomorphism of abelian groups. This proves that indeed $u = v$. ∎

Denote by $\Gamma_n(R)$ the R-submodule of $P_n(R)$ generated by the product of long commutators

$$\{[x_{\sigma(1)}, \ldots, x_{\sigma(i_1)}].[x_{\sigma(i_1+1)}, \ldots, x_{\sigma(i_2)}] \ldots [x_{\sigma(i_l+1)}, \ldots, x_{\sigma(n)}] \mid 2 \leq i_1 < \cdots < i_l < n, \sigma \in S_n\}$$

and all long commutators are left normed, e.g. $[x, y, z, t] := [[[x, y], z], t]$. The elements of $\Gamma_n(R)$ are called the *proper polynomials*. By definition $\Gamma_1(R) = 0$.

Again, $\Gamma_n(R)$ is an RS_n-module with submodule $\Gamma_n(R) \cap \operatorname{Id}(A, R)$ for any R-algebra A over a principal ideal domain R. Therefore, analogously as before, we define the

R-codimensions $\gamma_n(A,R,q)$ of proper polynomial identities, i.e.

$$\frac{\Gamma_n(R)}{\Gamma_n(R)\cap \mathrm{Id}(A,R)} \cong \underbrace{R\oplus\dots\oplus R}_{\gamma_n(A,R,0)} \oplus \bigoplus_{1\le i\le t}\Big(\underbrace{R/(p_i)^{k_i}\oplus\dots\oplus R/(p_i)^{k_i}}_{\gamma_n(A,R,p_i^{k_i})}\Big)$$

where $p_1,\dots,p_t$ are irreducible elements of R. If A has a unit element 1_A, then, by definition, $\gamma_0(A,R,q)$ is the number of $R/(q)$ in the decomposition of the cyclic R-submodule of A generated by 1_A.

First, we describe the relation between proper and ordinary codimensions.

Theorem 1.1.16. *Let A be a unital R-algebra over a principal ideal domain R. Then $c_n(A,R,q)=\sum_{j=0}^{n}\binom{n}{j}\gamma_j(A,R,q)$ for every $n\in\mathbb{N}$ and $q\in\{p_1^{k_1},\dots,p_t^{k_t}\}\cup\{0\}$ as in the definition.*

Proof. First, we notice that

$$P_n(R)=\bigoplus_{k=0}^{n}\ \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}\,\sigma_{i_1,\dots,i_k}\Gamma_{n-k}(R) \qquad (1.1)$$

is a direct sum of R-modules and where $\Gamma_0(R):=R$ and $\sigma_{i_1,\dots,i_k}\in S_n$ is any permutation such that $\sigma((n-k)+j)=i_j$ for all $1\le j\le k$ (so $\sigma_{i_1,\dots,i_k}(\{1,\dots,n-k\})\cap\{i_1,\dots,i_k\}=\emptyset$).

We show this explicitly in the spirit of Specht [Spe50]. Using the equalities $yx=[y,x]+xy$ and $[\dots,\dots]x=x[\dots,\dots]+[[\dots,\dots],x]$, we can present every polynomial from $P_n(R)$ as a sum of polynomials $x_{i_1}x_{i_2}\dots x_{i_k}f$ where $1\le i_1<i_2<\dots<i_k\le n$, $2\le k\le n$ and f a proper multilinear polynomial of degree $(n-k)$ in the variables from the set $\{x_1,x_2,\dots,x_n\}\backslash\{x_{i_1},x_{i_2},\dots,x_{i_k}\}$. In other words, $f\in\sigma_{i_1,\dots,i_k}\Gamma_{n-k}(R)$. In order to check that the sum in (1.1) is direct, assume that a sum of polynomials $x_{i_1}x_{i_2}\dots x_{i_k}\sigma_{i_1,\dots,i_k}f$ where $f\in\Gamma_{n-k}(R)$, for different k and i_j is zero. Now choose one term $g:=x_{i_1}x_{i_2}\dots x_{i_k}\sigma_{i_1,\dots,i_k}f$ with the greatest k among the terms with a nonzero coefficient. Then also the sum, where we set the variables $x_{i_1}=x_{i_2}=\dots=x_{i_k}=1$ and $x_j=x_j$ for the rest of the variables, is zero. (We assume that we are working in the free ring with 1 on the set $X=\{x_1,x_2,\dots\}$.) Due to our choice of g, all terms except the term g vanish. It follows that $f=0$. Therefore, the sum is direct and (1.1) holds.

Let k be as before and substitute $x_{i_1}=x_{i_2}=\dots=x_{i_k}=1_A$ and arbitrary elements of A for the other x_j's, we obtain

$$P_n(R) \cap \mathrm{Id}(A,R) = (\mathrm{char}\, A)Rx_1x_2\dots x_n \oplus \bigoplus_{k=0}^{n-2} \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} x_{i_1}x_{i_2}\dots x_{i_k}\, \sigma_{i_1,\dots,i_k}(\mathrm{Id}(A,R)\cap \Gamma_{n-k}(R)). \tag{1.2}$$

Combining (1.1) and (1.2), we get the direct sum of R-modules

$$\frac{P_n(R)}{P_n(R)\cap \mathrm{Id}(A,R)} \cong \bigoplus_{k=0}^{n} \bigoplus_{1\le i_1<i_2<\dots<i_k\le n} \frac{\Gamma_{n-k}(R)}{\Gamma_{n-k}(R)\cap \mathrm{Id}(A,R)}$$

where by definition $\frac{\Gamma_0(R)}{\Gamma_0(R)\cap\mathrm{Id}(A,R)} := \langle 1_A\rangle_R \subseteq A$. Hence the result follows. ∎

Corollary 1.1.17. *Let A be a unital R-algebra over a principal ideal domain R. Then, all multilinear polynomial identities of A over R are consequences of proper multilinear polynomial identities of A over R and the identity* $(\mathrm{char}\, A)x \equiv 0$.

Proof. This follows from (1.2) since $\mathrm{Id}(A,R)$ is a T-ideal and $\sigma_{i_1,\dots,i_k}$ can be seen as an element of $\mathrm{End}(R\langle X\rangle)$. ∎

Remark 1.1.18. The proofs of Propositions 1.1.11, 1.1.12 and 1.1.15 are also valid for proper codimensions. So, these results with $c_n(A,R,q)$ replaced by $\gamma_n(A,R,q)$ remain valid.

1.2 Representation theory of S_n

In this section we will review all needed background on the representation theory of the symmetric group S_n over a field of arbitrary characteristic. Along the way we will emphasize which parts of the theory rather hold, and should be considered, over $\mathbb{Z}$. In a first instance we will work towards Theorem 1.2.8 which gives a description of the (absolutely) irreducible representations of S_n. More precisely, these will be parametrized by certain partitions, called p-regular, and to each corresponds the irreducible representation $D^F(\lambda) = S^F(\lambda)/\mathrm{Rad}(S^F(\lambda))$, where $S^F(\lambda)$ is called a Specht module. In contrast to $D^F(\lambda)$ Specht modules $S^F(\lambda)$ are well understood. That's why, in a next part, we are interested in the problem when a Specht module is actually already irreducible over F. Over a field of characteristic zero this will always be the case, but interestingly some of them also are irreducible when $\text{char}(F) = p$. Necessary and sufficient conditions on λ have been proven by Fayers, see Theorem 1.2.12 and 1.2.13.

As mentioned in the previous section, we will need the representation theory of S_n in order to understand, asymptotically, the FS_n-module of multilinear polynomial functions and its dimension. In particular, it will be crucial to have a good knowledge about $\dim_F D^F(\lambda)$ and $\dim_F S^F(\lambda)$. The main known results are described in Subsection 1.2.3. Next, since FS_n is not semisimple if $\text{char}(F) \mid n$, we will also discuss composition series of the classes of modules considered in this thesis. However, over $\mathbb{Z}$ the group ring $\mathbb{Z}S_n$ is not even Artinian and thus does not even possess a composition series. Consequently in our study of polynomial identities of rings in Chapter 3 we need a substitute for this. This role will be played by so called Specht series whose existence and description is finally obtained in Theorem 1.2.30.

The main reference for this section will the book of James [Jam78b]. However, also more recent results are mentioned and the corresponding references will be cited. We also try to cite as much as possible the original papers.

1.2.1 Specht modules

Let F be a field of characteristic $p \geq 0$. We are interested in describing the absolutely irreducible F-representations of S_n. In case $p \nmid n$, the group algebra FS_n is semisimple and, consequently, any representation is a direct sum of irreducible F-representations and there are only a finite number of these. On the other hand, when $p \mid n$, FS_n is usually of wild representation type. Nevertheless, it is possible to draw a common story for both cases. Therefore we will always remain as general as possible and be more specific when and how the theory simplifies in the semisimple setting. We begin by recalling the following result, [CR62][Theorem 82.6 and 83.5].

Theorem 1.2.1. *Let F be a field of $\text{char}\, F = p \geq 0$ and let G be a finite group. The number of non-isomorphic absolutely irreducible FG-modules is less than or equal to the number of p-regular conjugacy classes of G. If, moreover, F is a splitting field for G, then the number of inequivalent irreducible FG-modules is equal to the number of p-regular conjugacy classes of G.*

Recall that a conjugacy class of G is called *p-regular* if for any element in that class its order is coprime to p. In particular, if $p = 0$, then these are simply all the conjugacy classes.

In case $G = S_n$, the conjugacy classes have a nice description. For this purpose one must decompose a given $\sigma \in S_n$ into a product of disjoint cycles $\sigma = \sigma_1 \dots \sigma_l$. Since disjoint cycles commute we can assume that $\text{length}(\sigma_i) \geq \text{length}(\sigma_{i+1})$ and then the tuple $(\text{length}(\sigma_1), \dots, \text{length}(\sigma_l))$ is called the *cycle type* of S_n. It is well known that the conjugacy class $\mathcal{C}_{S_n}(\sigma) = \{\tau^{-1}\sigma\tau := \sigma^\tau \mid \tau \in S_n\}$ of σ is determined by its cycle type.

Thus conjugacy classes of S_n correspond to tuples of positive integers $\lambda = (\lambda_1, \cdots, \lambda_l)$ such that $\lambda_i \geq \lambda_{i+1} > 0$ and $\sum \lambda_i = n$. The tuple λ is called a *(proper or ordered) partition* of n and is denoted $\lambda \vdash n$. The numbers λ_i are called *the parts* of λ and l the *height* of λ. For our convenience, we assume $\lambda_i = 0$ for all $i > l$. If consecutive parts are equal we use the exponent notation, instead of repeating these entries. For example (1^n) is the partition of n with n parts equal to 1 or for example $(3, 2^2, 1)$ denotes the partition $(3, 2, 2, 1)$.

By decomposing σ into disjoint cycles, we see that its order is coprime to p if and only if no cycle has length divisible by p. Thus p-regular classes correspond to partitions λ where no part λ_i is divisible by p. Such partitions are called *p-class regular*. Thus in a next step we want to construct for any such partition, up to isomorphism, a unique absolutely irreducible F-representation. However, it turns out that there is an other (bijective) set of partitions, called p-regular partitions, that is more naturally suitable for this purpose.

A partition $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ is called *p-singular* if $\lambda_{i+1} = \cdots = \lambda_{i+p} > 0$ for some i, otherwise λ is called *p-regular*.

Proposition 1.2.2 ([Ols86]). *The set of p-regular partitions is bijective to the set of p-class regular partitions.*

Proof.[Sketch] Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a p-regular partition. Write $\lambda_i = p^{n_i} m_i$ with $(m_i, p) = 1$. Let $\mu(i) = (m_i, \dots, m_i)$ be the $(p^{n_i} \times m_i)$-rectangle partition, i.e. μ_i has p^{n_i} parts of size m_i. Suppose now that $m_{j_1} \geq \ldots \geq m_{j_l}$. We define $\mu_\lambda = (\mu(j_1), \dots, \mu(j_l))$ which is clearly a p-class regular partition of n. One can prove quite easily that $\varphi : \lambda \mapsto \mu_\lambda$ is a well-defined bijection. ∎

We will now associate to any p-regular partition a module that will turn out to be absolutely irreducible. In particular, in characteristic zero, we have to do so for any partition. Therefore, let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ and $S_{\lambda_1} \times \cdots \times S_{\lambda_l}$ the Young subgroup corresponding to it, i.e. S_{λ_i} is the symmetric group on the set $\{\lambda_{i-1} + 1, \dots, \lambda_{i-1} + \lambda_i\}$

where $\lambda_0 = 0$ by definition.

Definition 1.2.3. Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. Then $M^{\mathbb{Z}}(\lambda)$ is the (left) $\mathbb{Z}S_n$-module induced by the left multiplication action of S_n on the left cosets of $S_{\lambda_1} \times \cdots \times S_{\lambda_l}$ inside S_n.

Thus $M^{\mathbb{Z}}(\lambda)$ is the permutation module of S_n on $S_{\lambda_1} \times \cdots \times S_{\lambda_l}$. Equivalently it can be defined using the language of Young tableaux.

Definition 1.2.4. Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. Then the Young diagram associated to λ is the finite subset of $\mathbb{N} \times \mathbb{N}$ defined as $D_\lambda = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq l, 1 \leq j \leq \lambda_i\}$. Any of the $n!$ filling of the boxes of D_λ using all the integers $\{1, \ldots, n\}$ is called a Young tableau of shape λ, denoted T_λ.

We will picture a Young diagram using boxes where we shall adopt the convention that the array of boxes representing D_λ is such that the second coordinate j increases from left to right and the coordinate i increases from top to bottom. For example $D_{(4,2,1)}$ is represented by

$$D_{(4,2,1)} = \ydiagram{4,2,1}.$$

Further we write $T_\lambda = D_\lambda(a_{ij})$, where a_{ij} is the integer in the (i, j) box. From now on we let S_n act naturally on a Young tableau T_λ with $\lambda \vdash n$, i.e. $\sigma T_\lambda = D_\lambda(\sigma(a_{ij}))$. For example consider $T_{(4,2,1)} = \begin{ytableau} 1 & 2 & 4 \\ 5 & 3 \\ 6 \end{ytableau}$ and $\sigma = (123)(46)$, then $\sigma T_{(4,2,1)} = \begin{ytableau} 2 & 3 & 6 \\ 5 & 1 \\ 4 \end{ytableau}$.

Definition 1.2.5. The row stabilizer R_{T_λ} of T_λ is the subgroup of S_n keeping the rows of T_λ fixed setwise, i.e. $R_{T_\lambda} = \mathrm{Sym}(\{a_{11}, \ldots, a_{1\lambda_1}\}) \times \cdots \times \mathrm{Sym}(\{a_{l1}, \ldots, a_{l\lambda_l}\})$. The column stabilizer C_{T_λ} of T_λ is defined similarly using the columns.

We call two Young tableaux T_λ and T'_λ equivalent if $T_\lambda = \sigma T'_\lambda$ for some $\sigma \in R_{T_\lambda}$. Its equivalence class, denoted $[T_\lambda]$, is called a λ-*tabloid*.

Define now $N^{\mathbb{Z}}(\lambda)$ to be the (finitely generated) free abelian group generated by the tabloids $[T_\lambda]$ of shape λ. Extending linearly the above defined action of S_n on the tableaux turns $N^{\mathbb{Z}}(\lambda)$ into a $\mathbb{Z}S_n$-module. The following proposition is easy to see.

Proposition 1.2.6. *The modules $N^{\mathbb{Z}}(\lambda)$ and $M^{\mathbb{Z}}(\lambda)$ are isomorphic as $\mathbb{Z}S_n$-modules.*

In the sequel we will no longer use the notation $N^{\mathbb{Z}}(\lambda)$ and simply switch between these two descriptions without further notice. Define now $M^R(\lambda) = R \otimes_{\mathbb{Z}} M^{\mathbb{Z}}(\lambda)$ for any (unital) ring R. In other words $M^R(\lambda)$ is the free R-module with formal basis consisting of all λ-tabloids.

Definition 1.2.7. For a given tableau T_λ we define $e_{T_\lambda} = a_{T_\lambda} b_{T_\lambda}$ and $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$, where $a_{T_\lambda} = \sum_{\pi \in R_{T_\lambda}} \pi$ and $b_{T_\lambda} = \sum_{\sigma \in C_{T_\lambda}} \mathrm{sign}(\sigma)\sigma$ are the so called *Young symmetrizers.* Then the RS_n-submodule $S^R(\lambda) := RS_n b_{T_\lambda} M^R(\lambda)$ of $M^R(\lambda)$ is called the *Specht module* corresponding to λ.

Remark. We see that $S^R(\lambda)$ is the submodule of $M^R(\lambda)$ spanned by the elements $b_{T_\lambda}[T_\lambda]$, called *polytabloids.* Later it will sometimes be convenient to rather work inside the group algebra RS_n, i.e. with left ideals instead of left modules. For this note that $\varphi : \sigma a_{T_\lambda} \mapsto \sigma[T_\lambda]$ defines a well defined RS_n-isomorphism from $RS_n a_{T_\lambda}$ to $M^R(\lambda)$ and the restriction of φ to the left ideal $RS_n e^*_{T_\lambda} = RS_n b_{T_\lambda} a_{T_\lambda}$ a RS_n-isomorphism with $S^R(\lambda)$.

Note also that the definition of $S^R(\lambda)$ through polytabloids only depends on λ, while the one above could seem to depend rather on the Young tableau T_λ. However, all choices of a tableau will yield isomorphic RS_n-modules.

Finally note that in the definition of $M^R(\lambda)$ we never used the fact that λ was an ordered partition, i.e. that $\lambda_i \geq \lambda_{i+1}$. Therefore, one considers more generally the so called *unordered partitions* μ of n which are tuples $(\mu_1, \ldots, \mu_s) \in \mathbb{N}^s$ such that $\sum_{i=1}^s \mu_i = n$. In this case we write $\mu \vDash n$. Again, for our convenience, we assume $\mu_i = 0$ for all $i > s$. Similarly as before one defines $M^R(\mu)$. In case μ is unordered one calls D_μ a generalized Young diagram D_μ.

For a proof of the following result we refer to James' book [Jam78b, Theorem 4.12 and 11.5] or the original paper [Jam76].

Theorem 1.2.8 (James)**.** *Let F be any field with $char(F) = p \geq 0$. Then*

$$\{S^F(\lambda)/Rad(S^F(\lambda)) \mid \lambda \vdash n \text{ is } p\text{-regular } \}$$

is a complete set of absolutely irreducible inequivalent FS_n-representations. In particular $\mathbb{Q}$ and $\mathbb{F}_p$ are splitting fields of S_n in characteristic 0, respectively p. Moreover if $char(F) = p \nmid n$, then $Rad(S^F(\lambda)) = 0$.

We denote the FS_n-module $S^F(\lambda)/\mathrm{Rad}(S^F(\lambda))$ by $D^F(\lambda)$. Recall that $\mathrm{Rad}(M) = \bigcap\{N \leq M \mid N \text{ is a maximal submodule of } M\}$ and that, moreover, if M is a finitely generated module over an Artinian ring R then $\mathrm{Rad}(M) = \mathrm{Jac}(R)M$. If $char(F) \nmid n$ the group algebra is semisimple and thus $\mathrm{Jac}(FS_n) = 0$ which explains the second part of Theorem 1.2.8.

Actually $\mathrm{Rad}(S^R(\lambda))$ has a nice description (which unfortunately is hard to use in practice). For this we consider the unique R-bilinear form

$$\langle \cdot, \cdot \rangle : M^R(\lambda) \times M^R(\lambda) \to R \tag{1.3}$$

for which $\langle [T_\lambda], [T'_\lambda] \rangle = 1$ if $[T_\lambda] = [T'_\lambda]$ and 0 otherwise. Clearly this form is non-degenerate, symmetric and S_n-invariant. With this at hand we can consider $I^\perp = \{x \in M^R(\lambda) \mid \langle u, x \rangle = 0 \text{ for all } u \in I\}$ for any subset I of $M^R(\lambda)$. If R is a field the following important result, called the *Submodule Theorem*, holds. We refer to [Jam78b, Theorem 4.8] for a proof or the original paper [Jam76].

Theorem 1.2.9 (James)**.** *Let $\lambda \vdash n$ and U a FS_n-submodule of $M^F(\lambda)$. Then either $S^F(\lambda) \subseteq U$ or $U \subseteq S^F(\lambda)^\perp$.*

From this follows that either $S^F(\lambda) \cap S^F(\lambda)^\perp = S^F(\lambda)$ or $S^F(\lambda) \cap S^F(\lambda)^\perp$ is the unique maximal submodule of $S^F(\lambda)$. Now, by [Jam78b, Theorem 11.1], the former precisely occurs when λ is p-singular. So, for p-regular partitions, which parametrize the absolutely irreducible FS_n-modules, we have that $\mathrm{Rad}(S^F(\lambda)) = S^F(\lambda) \cap S^F(\lambda)^\perp$.

For a given tableau T_λ of λ the Submodule Theorem also implies that the FS_n-submodule

$$\sum\{X \text{ an } FS_n\text{-submodule of } S^F(\lambda) \ : \ b_{T_\lambda} X = 0\}$$

of $S^F(\lambda)$ is contained in $S^F(\lambda) \cap S^F(\lambda)^\perp$. The converse also is true over any integral domain R, because $\langle b_{T_\lambda} u, v \rangle = \langle u, b_{T_\lambda} v \rangle$ for any $u, v \in M^R(\lambda)$ due to the bilinearity and S_n-invariance of $\langle \cdot, \cdot \rangle$. So, over any field, we have one more description of $\mathrm{Rad}(S^F(\lambda))$.

Remark 1.2.10. The description $\mathrm{Rad}(S^F(\lambda)) = \sum\{X \leq S^F(\lambda) \ : \ b_{T_\lambda} X = 0\}$ is not so much a consequence of the Submodule Theorem but rather of the description $S^F(\lambda) = FS_n b_{T_\lambda} M^F(\lambda)$ of the Specht module and the crucial fact that $b_{T_\lambda} M^F(\lambda)$ is 1-dimensional over F. More precisely, let $T_\lambda, T'_\lambda \in M^F(\lambda)$, if $b_{T_\lambda}[T'_\mu] \neq 0$ then $\lambda \trianglerighteq \mu$ and if $\lambda = \mu$ then $b_{T_\lambda}[T'_\lambda] = \pm b_{T_\lambda}[T_\lambda]$, see [Jam78b, Lemma 4.6]. Recall that we say that λ *dominates* μ, denoted $\lambda \trianglerighteq \mu$, if $\sum_i \lambda_i \geq \sum_i \mu_i$ for any i. The previous also implies that for a submodule X of $M^F(\lambda)$ either $b_{T_\lambda} X = 0$ or $S^F(\lambda) \subseteq X$.

1.2.2 p-irreducibility of Specht modules

Unfortunately, in practice, when $\mathrm{char}(F) = p > 0$, due to the nature of above descriptions, it is hard to describe concretely $\mathrm{Rad}(S(\lambda))$ and thus the irreducible modular representations of S_n. Therefore, it is of interest to know when a Specht module $S(\lambda)$ over $\mathbb{Q}$, which is always irreducible, remains irreducible after reducing modulo p which in case of S_n amounts to considering irreducibility of $S^{\mathbb{F}_p}(\lambda)$. James and Mathas made in [Mat99] a conjecture concerning this, which was finally proved in 2005 by Fayers in [Fay04, Fay05]. Before formulating this result we have to introduce some notations.

Definition 1.2.11. Let $\lambda \vdash n$. Then one defines the *hook number* at the position (i, j) of λ, written $h_\lambda(i, j)$, as the number of boxes of D_λ to the right and below (i, j), and the box (i, j) itself. That is, $h_\lambda(i, j) = \lambda_i + \lambda'_j - i - j + 1$, where $\lambda' = (\lambda'_1, \ldots, \lambda'_l)$ is the *conjugate partition* of λ (i.e. rows and columns interchanged).
Further, let $v_p(m)$ be the highest power of p which divides m with $m \in \mathbb{N}$.

Now we can state necessary and sufficient conditions for $S^{\mathbb{F}_p}(\lambda)$ to be irreducible.

Theorem 1.2.12 (Fayers)**.** *Let p be an odd prime and λ a partition of $n \in \mathbb{N}$. Then the Specht module $S^{\mathbb{F}_p}(\lambda)$ is reducible if and only if D_λ has positions $(i, j), (i, y)$ and (x, j) such that*

$$v_p(h_\lambda(i, j)) > 0, \quad v_p(h_\lambda(i, j)) \neq v_p(h_\lambda(i, y)) \text{ and } v_p(h_\lambda(i, j)) \neq v_p(h_\lambda(x, j)).$$

A partition not satisfying the above condition is called a *JM-partition* or also *p-irreducible*. The case $p = 2$ was solved separately in [JM99] by James and Mathas in 1999.

Theorem 1.2.13 (James and Mathas)**.** *Let $n \neq 4$ and $\lambda \vdash n$. Then $S^{\mathbb{F}_2}(\lambda)$ is irreducible if and only if λ is 2-regular, a JM-partition and λ' is 2-regular. If $n = 4$, then $S^{\mathbb{F}_2}(\lambda)$ for $\lambda = (2, 2)$ is also irreducible.*

A partition λ such that its conjugate partition λ' is p-regular is called *p-restricted*. Note that by Theorems 1.2.12 and 1.2.13 $S^{\mathbb{F}_p}(\lambda)$ is irreducible if and only if $S^{\mathbb{F}_p}(\lambda')$ is irreducible. Conditions for the irreducibility of $S^{\mathbb{F}_p}(\lambda)$ for p-regular partitions were *conjectured* much earlier already by *Carter* and were proven in [Jam78a, JM79]. The latter states that λ is p-regular and $S^{\mathbb{F}_p}(\lambda)$ irreducible if and only if $v_p(h_\lambda(i, j)) = v_p(h_\lambda(k, j))$ for all (i, j) and (k, j).

Remark. 1. The property of being p-irreducible is really a property of the prime p rather then the partition. More precisely, Kleshchev and Premet proved in [KP00] that if $\lambda \neq (n), (1^n)$, then there exists a prime p such that $S^F(\lambda)$ is reducible for any field of characteristic p. As follows from Theorem 1.2.15 the condition $\lambda \neq (n), (1^n)$ is the same as saying that $S(\lambda)$ is not 1-dimensional.

2. Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. From Theorem 1.2.12 and 1.2.13 follows that if $S^{\mathbb{F}_p}(\lambda)$ is irreducible, then the same holds for the partition $\mu \vdash n - \lambda_1$ that is formed from λ by deleting its first part.

Now we give some examples of JM-partitions.

Example 1.2.14. 1. It is easy to see that the partitions $\lambda = (n), \lambda = (1^n)$ and in case $p > 2$, $\lambda = (2p-1, p)$ satisfy the necessary conditions.

2. If p does not divide $h_\lambda(i,j)$ for all i and j then the conditions are trivially satisfied. This is for example the case for the *'wide staircase'* $\lambda^u = ((p-1)u, (p-1)(u-1), \ldots, (p-1)2, p-1)$ for any positive integer $u \geq 2$. Indeed, to see this write $j = (p-1)j' + j''$ with $1 \leq j'' \leq p-1$ and $0 \leq j' \leq u-1$, then

$$\begin{aligned} h_{\lambda^u}(i,j) &= \lambda_i^u + (\lambda_j^u)' - i - j + 1 \\ &= (p-1)(u-i+1) + u - j' - i - j + 1 \\ &= p(u-i+1) - j' - j \\ &= p(u-i+1-j') - j'' \equiv -j'' \mod p \end{aligned}$$

So, as claimed, $v_p(h_{\lambda^u}(i,j)) = 0$ for all places (i,j). Note also that λ^u is a p-regular partition of $n = \sum \lambda_i^u = (p-1)\binom{u+1}{2}$.

1.2.3 Dimension of Specht modules

For our applications to polynomial identities it will be crucial to have control over the dimensions of the irreducible representations of S_n. Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$ and F an arbitrary field of characteristic $p \geq 0$. Note that

$$\mathbb{Z} - \text{rank } M^{\mathbb{Z}}(\lambda) = \frac{n!}{\lambda_1! \ldots \lambda_l!} = [S_n \ : \ S_{\lambda_1} \times \cdots \times S_{\lambda_l}] = \dim_F M^F(\lambda)$$

by definition of $M^R(\lambda)$ as permutation module. Thus the R-rank of the Young permutation modules $M^R(\lambda)$ are independent of the ground ring. More surprisingly, also the dimension of $S^F(\lambda)$ is independent of F and moreover equals the $\mathbb{Z}$-rank of $S^{\mathbb{Z}}(\lambda)$. A very convenient formula for this is the following result due to Frame-Robinson-Thrall. See [FRT54] or [Jam78b, Theorem 20.1] for a proof. Note that in both references the result is stated for (arbitrary) fields. However, a quick look at the proof shows that the statement also holds for $\mathbb{Z}$.

Theorem 1.2.15 (Hook formula)**.**

$$\dim_F S^F(\lambda) = \frac{n!}{\prod_{i,j} h_\lambda(i,j)} = \mathbb{Z} - rank\ S^{\mathbb{Z}}(\lambda),$$

where the product runs over all boxes of D_λ.

There also exist other formulae such as the Young-Frobenius formula, see [GZ05, Proposition 2.2.9]. The above formula for $\dim_F S^F(\lambda)$ became possible due to the description of a standard basis of the Specht modules.

Definition 1.2.16. A tableau T_λ of shape λ is *standard* if the integers in the rows increase from left to right and the integers in each column increase from top to bottom. Further, $[T_\lambda]$ is a *standard tabloid* if there is a standard tableau in the equivalence class $[T_\lambda]$ and moreover $b_{T_\lambda}[T_\lambda]$ is called a *standard polytabloid.*

Note that a standard tabloid contains a unique standard tableau, even though a standard polytabloid may involve several. The latter form a basis of $S^F(\lambda)$ over any field F, see [Jam78b, Theorem 8.4].

Theorem 1.2.17. *The module $S^{\mathbb{Z}}(\lambda)$ is free as $\mathbb{Z}$-module with basis the set*

$$\{b_{T_\lambda}[T_\lambda] \mid T_\lambda \text{ is a standard tableau } \}.$$

In particular this set forms a F-basis of $S^F(\lambda)$ for any field F, called the standard basis.

It is important to remark that in the proof of the above theorem one first proves that the standard tableaux span $S^{\mathbb{Q}}(\lambda)$ in such a way that each polytabloid in $S^{\mathbb{Q}}(\lambda)$ is an integral linear combination of the standard polytabloids. Due to this one can extract the spanning property for any field of any characteristic. This also explains why reducing modulo p the representations $S^{\mathbb{Q}}(\lambda)$, in the sense of Brauer, amounts to considering $S^{\mathbb{F}_p}(\lambda)$. It also justifies $M^F(\lambda) := F \otimes_{\mathbb{Z}} M^{\mathbb{Z}}(\lambda)$.

Corollary 1.2.18. *Let λ be a p-regular partition and G_λ the Gram matrix of $S^F(\lambda)$ with respect to the standard basis and the bilinear form defined in (1.3). Then*

$$\dim_F D^F(\lambda) = rank(G_\lambda) \geq \dim_F S^F(\lambda) - v_p(\det(G_\lambda)).$$

Moreover, the rank of the Gram matrix only depends on char F.

Proof. [Sketch] For simplicity of notation, let $\{e_i \mid i \in I\}$ denote the standard basis of $S^F(\lambda)$. By definition, the *Gram matrix is the matrix* $(\langle e_i, e_j\rangle)_{i,j}$.

The statement $D^F(\lambda) = \mathrm{rank}(G_\lambda)$ is a specific case of a more general result in linear algebra, see [Jam78b, Theorem 1.6]. In short, let $S^F(\lambda)^*$ be the vector space dual to $S^F(\lambda)$ and define the F-linear map $\varphi : S^F(\lambda) \to S^F(\lambda)^* : w \mapsto (\varphi_w : S^F(\lambda) \to F : v \mapsto \langle w, v\rangle)$. By using the dual of the standard basis it is easy to see that $\dim_F(S^F(\lambda)/S^F(\lambda)\cap S^F(\lambda)^\perp) = \dim_F \mathrm{im}\,\varphi = \mathrm{rank}(\langle e_i, e_j\rangle_{i,j})$. Now recall that $D^F(\lambda) = S^F(\lambda)/\mathrm{Rad}(S^F(\lambda))$ and $\mathrm{Rad}(S^F(\lambda)) = S^F(\lambda) \cap S^F(\lambda)^\perp$.

For the inequality $\dim_F D^F(\lambda) \geq \dim_F S^F(\lambda) - v_p(\det(G_\lambda))$ we refer to [JM79].

Finally the fact that the rank of the Gram matrix only depends on char F follows from the fact that any tabloid occuring in $b_{T_\lambda}[T_\lambda]$ has coefficient ± 1, therefore $\langle b_{T_\lambda}[T_\lambda], b_{T'_\lambda}[T'_\lambda]\rangle \in K$ for any pair of standard polytabloids where K is the prime field of F. ∎

It is still an open problem to find a generic formula, such as the Hook formula, for $\dim_F D^F(\lambda)$. However, Corollary 1.2.18 gives an under bound in terms of the determinant of the Gram matrix and $\dim S(\lambda)$. In [JM79], James and Murphy proved an expression for the determinant in function of the fractions $\frac{h_\lambda(a,c)}{h_\lambda(b,c)}$. In particular if $v_p(h_\lambda(a,c)) = v_p(h_\lambda(b,c))$ for each column, their result, combined with Corollary 1.2.18, implies the sufficiency of Carter's conjecture concerning p-irreducibility of partitions. Before stating the result we need to introduce one notation.

Definition 1.2.19. Let $\beta_1, \ldots, \beta_l$ be integers. If all β_i are positive and $\beta_i \neq \beta_j$ for all i and j, then $d(\beta_1, \ldots, \beta_l) = \dim S(\mu).\mathrm{sign}(\sigma)$, with $\mu = (\mu_1, \ldots, \mu_l)$ the partition having the numbers $\beta_1, \ldots, \beta_l$ as hook numbers in the first column, i.e. $\{\beta_i \mid 1 \leq i \leq l\} = \{h_\mu(i, 1) \mid 1 \leq i \leq l\}$ and $\sigma \in S_l$ the permutation such that $\beta_{\sigma(1)} > \cdots > \beta_{\sigma(l)}$. In the other cases we set $d(\beta_1, \ldots, \beta_l) = 0$.

Theorem 1.2.20 (James-Murphy)**.** *Let $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. Then*

$$\det(G_\lambda) = \prod_{1\le a<b\le l} \prod_{c=1}^{\lambda_b} \frac{h_\lambda(a,c)}{h_\lambda(b,c)}^{d(h_\lambda(1,1),h_\lambda(2,1),\ldots,h_\lambda(a,1)+h_\lambda(b,c),\ldots,h_\lambda(b,1)-h_\lambda(b,c),\ldots,h_\lambda(l,1))}.$$

Remark. The exponent of $\frac{h_\lambda(a,c)}{h_\lambda(b,c)}$ in Theorem 1.2.20 can be interpreted in a more intuitive way. Namely, the partition μ in Definition 1.2.19 consists in wrapping the (b,c)-rim hook of λ, i.e. all nodes (i,j) with $b \le i$ and $c \le j$ on the rim of D_λ, and unwrapping it at the end of the a-th row. Moreover, $\operatorname{sign}(\sigma)$ is $+1$ if and only if the sum of the leg length of the unwrapped and wrapped rim hook is even.

Now we compute $\dim S(\lambda)$ for a specific partition, this example will be surprisingly recurrent in this thesis and therefore deserves emphasizes. In the sequel we will often use, without mentioning, *Stirling's formula* asserting that

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$$

for all $n \in \mathbb{N}$. In particular $n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

Example 1.2.21. Let $\lambda = (u^d)$ be the rectangle with d rows of length u. Then, for $u \to \infty$, $\dim_F S^F(\lambda) \simeq C u^{\frac{1-d^2}{2}} d^{du}$, where $C = (d-1)!(d-2)!\cdots 1!\sqrt{2\pi}^{1-d}\sqrt{d}$.

Proof. To start we determine the hook numbers of each row,

$$\prod_j h_\lambda(1,j) = d\cdots(d+u-1) = \frac{(d+u-1)!}{(d-1)!},$$
$$\prod_j h_\lambda(2,j) = (d-1)\cdots(d+u-2) = \frac{(d+u-2)!}{(d-2)!}$$

and so on till $\prod_j h_\lambda(d,j) = u!$. Altogether,

$$\prod h_\lambda(i,j) = \frac{(u!)^d(u+1)^{d-1}(u+2)^{d-2}\cdots(u+d-1)}{(d-1)!(d-2)!\cdots 1!}.$$

Put $K = (d-1)!(d-2)!\cdots 1!$. Then by the Hook and Stirling formulae

$$\begin{aligned}\dim_F S^F(\lambda) &\simeq \frac{K.\sqrt{2\pi d}.u^{1/2}.\left(\frac{du}{e}\right)^{du}}{(\sqrt{2\pi u})^d\left(\frac{u^u}{e^u}\right)^d(u+1)^{d-1}(u+2)^{d-2}\cdots(u+d-1)}\\ &= \frac{Cd^{ud}}{u^{\frac{d-1}{2}}(u+1)^{d-1}(u+2)^{d-2}\cdots(u+d-1)},\end{aligned}$$

where $C = (d-1)!(d-2)!\cdots 1!\sqrt{2\pi}^{1-d}.\sqrt{d}$. When using that $(u+1)^{d-1}(u+2)^{d-2}\cdots(u+d-1)$ grows as $u^{(d-1)+\cdots+1} = u^{\frac{d^2-d}{2}}$ for $u \to \infty$, the result follows. ∎

Without surprise, the dimension for different Specht modules $S(\lambda)$ and $S(\mu)$ for $D_\mu \subseteq D_\lambda$ are connected, see [GZ05, lemma 6.2.4] for a proof in characteristic 0. Since we found no reference stating Lemma 1.2.22 for arbitrary characteristic we give a sketch of the proof.

Lemma 1.2.22. *Let $\lambda \vdash n$ and $\mu \vdash m$ be such that $\mu_i \leq \lambda_i$ for all i. Then* $\dim S(\lambda) \leq \dim S(\mu) \leq n^{n-m} \dim S(\lambda)$.

Proof. Since $\mu_i \leq \lambda_i$ for all i, we can consider D_μ as a subdiagram of D_λ. Now we can get from D_λ to D_μ by taking $n-m$ times one box out of D_λ. This procedure is the subject of the Branching Theorem, see [Jam78b, Theorem 9.3] or Theorem 1.2.32, which yields the lower bound. The upper bound follows immediately from the fact that $\prod_{i,j} h_\mu(i,j) \leq \prod_{i,j} h_\lambda(i,j)$ and thus $\dim S(\lambda) = \frac{n!}{\prod_{i,j} h_\lambda(i,j)} \leq \frac{n!}{\prod_{i,j} h_\mu(i,j)} < n^{n-m} \dim S(\mu)$. ∎

1.2.4 Composition Series

Let F be a field of characteristic $p \geq 0$. The group algebra FS_n is finite dimensional over F, in particular left Artinian, and therefore by the theorem of Hopkins-Levitzki any finitely generated FS_n-module M possesses a composition series. By Theorem 1.2.8 this means that there exists a chain of FS_n-submodules $\{0\} = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_t = M$ such that $M_i/M_{i-1} \cong D^F(\lambda)$ for some p-regular partition λ. In case $\mathrm{char}(F) = 0$, by the theorem of Jordan-Hölder and Maschke, this composition series actually delivers the direct sum decomposition into simple modules of M.

More precisely, by the Hook formula 1.2.15, Theorem 1.2.8 and a first course in classical representation theory we know that $FS_n = \bigoplus_{\lambda \vdash n} \frac{n!}{\prod_{i,j} h_\lambda(i,j)} S^F(\lambda)$ in case F is algebraically closed and has $\mathrm{char}(F) = 0$. By Theorem 1.2.23 below this decomposition actually holds for any field of characteristic 0.

Recall that the regular module FS_n is isomorphic, as FS_n-module, to the permutation module $M^F((1^n))$. In characteristic zero, the direct summands of $M^F(\lambda)$ for any partition λ are also known, as proven in [Jam78b, Theorem 13.13 and 14.1]. In order to state the latter results *we allow now tableaux to have repeated entries.* Then a tableau T_μ for some partition μ is said to be *of type* λ if the number of i in the tableau equals λ_i, the i-th part of λ. Also we say that T_μ is *semistandard* if the entries are non-decreasing from left to right and strictly increasing from top to bottom. So if $\lambda = (1^n)$, where $\mu \vdash n$, then the semistandard tableaux coincide with the standard tableaux.

Theorem 1.2.23. *Let $\lambda \vdash n$ and char$(F) = 0$. Then the multiplicity of $S^F(\mu)$ as a direct summand of $M^F(\lambda)$ equals $\dim_F \operatorname{Hom}_{FS_n}(S^F(\mu), M^F(\lambda))$ which is the number of semistandard μ-tableaux of type λ. In particular $[M^F(\lambda) : S^F(\lambda)] \neq 0$ if and only if $\mu \trianglerighteq \lambda$.*

Remark. Actually for any field F, $\dim_F \operatorname{Hom}_{FS_n}(S^F(\lambda), M^F(\mu))$ equals the number of semistandard μ-tableaux of type λ, by Corollary 13.14 in [Jam78b]. Thus if $\operatorname{char}(F) = p > 0$ the latter result describes a so-called Specht series of $M^F(\mu)$. In the next section we will focus on this kind of chains, instead of composition series.

For a field of characteristic $p \neq 0$, the problem of describing the composition series of the modules $M^F(\lambda)$ and $S^F(\lambda)$ is still surprisingly open. Actually this problem is equivalent with the problem of determining $\dim_F D^F(\lambda)$, see Kleshchev's paper [Kle98] for a nice survey. Now we review briefly some known results. We start by [Jam78b, Lemma 11.3 and Lemma 11.4].

Lemma 1.2.24. *Consider a $\mu \vdash n$, a p-regular $\lambda \vdash n$ and a submodule U of $M^F(\lambda)$. If there exists a $0 \neq \phi \in \operatorname{Hom}_{FS_n}(S^F(\lambda), M^F(\mu))$ then $\mu \trianglerighteq \lambda$. Moreover for such ϕ if $\lambda = \mu$, then im$(\phi) \subseteq (S^F(\mu) + U)/U$. In particular if $\operatorname{Hom}_{FS_n}(D^F(\lambda), M^F(\mu)) \neq 0$, then $\mu \trianglerighteq \lambda$ and $\lambda \neq \mu$ if $S^F(\lambda) \subseteq U$.*

In contrast to characteristic 0, where $D^F(\mu) = S^F(\mu)$, not all $D^F(\mu)$ with $\mu \trianglerighteq \lambda$ occur as composition factor of $M^F(\lambda)$. The previous lemma immediately implies the following result, in which we use '$\leftrightarrow$' to abbreviate "has the same composition factors as".

Theorem 1.2.25. *For any partition $\lambda \vdash n$ and any field F,*

$$\mathit{Rad}(S^F(\lambda)) \leftrightarrow \sum_{\mu \triangleright \lambda} a_{\lambda\mu} D^F(\mu),$$

for some $a_{\lambda\mu} \in \mathbb{N}$. In particular, if λ is p-singular then the same holds for $S^F(\lambda) = \mathit{Rad}(S^F(\lambda))$ and otherwise $S^F(\lambda) \leftrightarrow D^F(\lambda) + \sum_{\mu \triangleright \lambda} a_{\lambda\mu} D^F(\mu)$.

Remark 1.2.26. Since FS_n is Artinian the modules $M^F(\lambda)$ and $S^F(\lambda)$ also have a direct sum decomposition into indecomposable modules. Due to Remark 1.2.10, or the Submodule Theorem 1.2.9, if $M^F(\lambda) = Y_1 \oplus Y_2$ is a direct sum as FS_n-modules, then for precisely one i we have that $b_{T_\lambda} Y_i \neq 0$ and, for that i, $S^F(\lambda) \subseteq Y_i$. Consequently we can choose an FS_n-module $Y^F(\lambda)$, called *Young module*, minimal with respect to $S^F(\lambda) \subseteq Y^F(\lambda) \subseteq M^F(\lambda)$ and $Y^F(\lambda) \oplus X = M^F(\lambda)$ for some FS_n-module X. This module will be indecomposable. It is known – see [Jam83] for the original proof using Shur algebras or [Erd01] for a proof using merely S_n – that all indecomposable summands of $M^F(\lambda)$ are isomorphic to a Young module $Y^F(\mu)$ with $\mu \trianglerighteq \lambda$. As proven in [Erd01] the summands of $FS_n = M^F((1^n))$, which are the projective indecomposables, are all the Young modules $Y^F(\lambda)$ with λ a p-restricted partition.

The reader may wonder now why we will use composition and Specht series instead of the direct sum decomposition $M^F(\lambda) = \sum_{\mu \trianglerighteq \lambda} c_{\lambda\mu} Y^F(\mu)$ which a priori looks handier. One reason for this is that nothing generic about $\dim_F Y^F(\mu)$ seems to be known. Secondly, in the next chapters we will construct partitions of bounded height but with arbitrary large rows, in particular these partitions will not be p-restricted, appearing in a certain quotient of the regular module. So, by above result of Erdmann, Young modules do not seem to be the natural context for us.

1.2.5 Specht Series

Let M be a finitely generated RS_n-module for some ring R. In case R is a field the module M has a composition series, which we described in the previous section for certain relevant modules. However if $R = \mathbb{Z}$ the group ring $\mathbb{Z}S_n$ is not Artinian and thus does not posses a composition series. Nevertheless, we will see in this section that it still has a very convenient filtration, namely a so called Specht series. This is a series $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$ such that M_{i+1}/M_i is a Specht module. In Chapter 3 we will use this series as a substitute for the composition series. Note that by Lemma 1.2.24 we already know that

$$M^F(\lambda) \leftrightarrow S^F(\lambda) + \sum_{\mu \triangleright \lambda} b_{\lambda\mu} S(\mu),$$

for some $b_{\lambda\mu} \in \mathbb{N}$.

The main goal of this section is Theorem 1.2.30. In this result we construct a Specht series for certain RS_n-modules $S^R(\lambda;\mu)$ with $R = F$ or $R = \mathbb{Z}$ which are generalisations of the permutation and Specht modules. For fields this result was achieved by James in

[Jam77] and a very clear account of this can be found in Section 15 and Section 17 of [Jam78b]. In joint work with Gordienko [GJ13] we showed how the case $R = \mathbb{Z}$ follows from his results over F by adding some torsion-freeness assertions. The remains of this section is mainly an account of this.

We start by defining the protagonists $S^{\mathbb{Z}}(\lambda;\mu)$.

Definition 1.2.27. Let $\mu \vDash n$, $\lambda \vdash n'$, $n' \leq n$ and $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. By T_λ we mean the subtableau of T_μ defined by the partition λ (i.e. the tableau consisting of the λ_i-first boxes in each row i). Then

$$S^{\mathbb{Z}}(\lambda;\mu) := \operatorname{span}_{\mathbb{Z}}\{e^{\lambda,\mu}_{T_\mu} \mid T_\mu \text{ is a tableau of shape } \mu\}$$

a subspace of $M^{\mathbb{Z}}(\mu)$ and $e^{\lambda,\mu}_{T_\mu} := \sum_{\sigma\in C_{T_\lambda}}(\operatorname{sign}\sigma)\sigma[T_\mu]$.

In previous definition we used $C_{T_\lambda} \leq C_{T_\mu} \leq S_n$ to denote the subgroup that leaves the numbers outside of T_λ invariant and sends every number from each column of T_λ to the same column. Note that $S^{\mathbb{Z}}(\mu;\mu) \cong S^{\mathbb{Z}}(\mu)$ and $S^{\mathbb{Z}}(0;\mu) \cong M^{\mathbb{Z}}(\mu)$. We assume that $S(0;0) = 0$ and define $S^F(\lambda;\mu)$ as the subspace in $M^F(\mu)$ generated by $S^{\mathbb{Z}}(\lambda;\mu) \otimes_{\mathbb{Z}} 1$.

From now on we always assume that in a pair $(\lambda;\mu)$ we have $\lambda_1 = \mu_1$. Also we write $M(\mu)$ and $S(\lambda;\mu)$ instead of $M^{\mathbb{Z}}(\mu)$ and $S^{\mathbb{Z}}(\lambda;\mu)$.

Recall that an element r is called *torsion* if there exists some non-zero $m \in \mathbb{Z}$ such that $mr = 0$.

Lemma 1.2.28. *Let $\mu \vDash n$, $\lambda \vdash n'$, $n' \leq n$. Suppose $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Then $M(\mu)/S(\lambda;\mu)$ is torsion-free.*

Proof. Recall that $M(\mu)$ is a finitely generated free Abelian group. So, it is torsion-free, and $S(\lambda;\mu)$ is a subgroup of $M(\mu)$. Hence, we can choose a basis $a_1, a_2, \dots, a_t$ in $M(\mu)$ such that $m_1a_1, m_2a_2, \dots, m_ka_k$ is a basis as $\mathbb{Z}$-module of $S(\lambda;\mu)$ for some $m_i \in \mathbb{N}$ and $1 \leq k \leq t$. We claim that all $m_i = 1$. First, we notice that $a_1 \otimes 1, a_2 \otimes 1, \dots, a_t \otimes 1$ form a basis of $M^F(\mu)$ and $m_1a_1 \otimes 1, m_2a_2 \otimes 1, \dots, m_ka_k \otimes 1$ generate $S^F(\lambda;\mu)$ for any field F. Thus $\dim_F S^F(\lambda;\mu) = k$ for $\operatorname{char} F = 0$ and $\dim_F S^F(\lambda;\mu) < k$ if $\operatorname{char} F \mid m_i$ for at least one m_i. However, by [Jam78b, Theorem 17.13 (III)], $\dim_F S^F(\lambda;\mu)$ does not depend on the field F. Therefore, all $m_i = 1$ and $M(\mu)/S(\lambda;\mu)$ is a free Abelian group. ∎

Remark. It follows from the proof of lemma 1.2.28 that $\dim_F S^F(\lambda;\mu)$ not only is independent of the field F but it is, moreover, also equal to the $\mathbb{Z}$-rank of $S(\lambda;\mu)$.

Let λ and μ be as in Definition 1.2.27 and let $c \geq 2$ be a natural number satisfying the following conditions: $\mu_{c-1} = \lambda_{c-1}$ and $\mu_c > \lambda_c$. Then we define on pairs $(\lambda;\mu)$ *the operators A_c ("adding") and R_c ("raising")* in the following way:

1. if $\lambda_c = \lambda_{c-1}$, then $A_c(\lambda;\mu) = (0;0)$ where 0 represents the zero partition $0 \vdash 0$, otherwise $A_c(\lambda;\mu) = (\tilde\lambda;\mu)$ where $\tilde\lambda_i = \lambda_i$ for $i \neq c$ and $\tilde\lambda_c = \lambda_c + 1$.
2. $R_c(\lambda;\mu) = (\tilde\lambda;\tilde\mu)$ where $\tilde\mu_i = \mu_i$ for $i \neq c-1, c$; $\tilde\mu_c = \lambda_c$, $\tilde\mu_{c-1} = \mu_{c-1} + (\mu_c - \lambda_c)$, $\tilde\lambda_1 = \tilde\mu_1$ and $\tilde\lambda_i = \lambda_i$ for $i > 1$.

Fix $i \in \mathbb{N}$ and $0 \leq v \leq \mu_{i+1}$ for a given $\mu \vdash n$. Let $\nu \vDash n$, $\nu_j = \mu_j$ for $j \notin \{i, i+1\}$, $\nu_i = \mu_i + \mu_{i+1} - v$, $\nu_{i+1} = v$. Then we define $\psi_{i,v} \in \mathrm{Hom}_{\mathbb{Z}S_n}(M(\mu), M(\nu))$ in the following way: $\psi_{i,v}[T_\mu] = \sum[T_\nu]$ where the summation runs over the set of all tabloids $[T_\nu]$ such that $[T_\nu]$ agrees with $[T_\mu]$ in all the rows except the i-th and the $(i+1)$-th, and the $(i+1)$-th is a subset of size v of the $(i+1)$-th row in $[T_\mu]$. Analogously, we define $\psi^F_{i,v} \in \mathrm{Hom}_{FS_n}(M^F(\mu), M^F(\nu))$ for any field F.

Lemma 1.2.29.

1. $\psi_{c-1,\lambda_c} S(\lambda;\mu) = S(R_c(\lambda;\mu))$*;*
2. $\ker \psi_{c-1,\lambda_c} \cap S(\lambda;\mu) = S(A_c(\lambda;\mu))$*.*

In particular, $S(\lambda;\mu)/S(A_c(\lambda;\mu)) \cong S(R_c(\lambda;\mu))$ as $\mathbb{Z}S_n$-modules.

Proof. The proof of the first part of the lemma and of the embedding $\ker \psi_{c-1,\lambda_c} \supseteq S(A_c(\lambda;\mu))$ is completely analogous to [Jam78b, Lemma 17.12]. Now recall that there exists a natural embedding $M(\lambda)\otimes 1 \subset M^{\mathbb{Q}}(\lambda)$. By [Jam78b, Theorem 17.13], $\ker \psi^{\mathbb{Q}}_{c-1,\lambda_c} \cap S^{\mathbb{Q}}(\lambda;\mu) = S^{\mathbb{Q}}(A_c(\lambda;\mu))$. Thus if $\psi_{c-1,\lambda_c} a = 0$ for some $a \in S(\lambda;\mu)$, then there exists a positive integer m such that $ma \in S(A_c(\lambda;\mu))$. Hence $a \in S(A_c(\lambda;\mu)) \subseteq M(\mu)$, since $M(\mu)/S(A_c(\lambda;\mu))$ is torsion-free by Lemma 1.2.28. ∎

Theorem 1.2.30. *Let $n \in \mathbb{N}$, $\lambda \vdash n'$, $\mu \vDash n$, $n' \leq n$, $\lambda_i \leq \mu_i$ for all $i \in \mathbb{N}$. Then $S(\lambda;\mu)$ has a chain of submodules*

$$S(\lambda;\mu) = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_t = 0,$$

with factors M_i/M_{i+1} isomorphic to Specht modules $S^{\mathbb{Z}}(\nu)$. Moreover, $S(\lambda;\mu)/M_i$ is torsion-free for any i.

Proof. If $\mu = \lambda$, then $S(\lambda;\mu) = S(\lambda)$ and there is nothing to prove. If $\mu \neq \lambda$, then we find $c \in \mathbb{N}$ such that $\lambda_i = \mu_i$ for all $1 \leq i \leq c-1$ and $\lambda_c < \mu_c$. Since by agreement $\lambda_1 = \mu_1$, we have $c \geq 2$. Now we add a box in the c-th row. By Lemma 1.2.29 we get a chain $S(\lambda;\mu) \supseteq S(A_c(\lambda;\mu))$ with factor $S(R_c(\lambda;\mu))$. If $\lambda_c \neq \lambda_{c-1}$ then $\tilde{\lambda}_c > \lambda_c$ and R_c moves boxes from the c-th row of D_μ upper. Now we can simply do the same for these two modules. If $\lambda_c = \lambda_{c-1}$ then $S((\lambda;\mu)) \cong S(R_c(\lambda;\mu))$ and we continue with raising boxes till we reach height d such that $\lambda_d \neq \lambda_{d-1}$ and then as before we add a box in row d. We see that in both cases by induction we get the first part of the Theorem.

Suppose $S(\lambda;\mu)/M_i$ is not torsion-free and $ma \in M_i$ for some $a \in S(\lambda;\mu)$, $a \notin M_i$, and $m \in \mathbb{N}$. Then we can find an index $0 \leq k < i$ such that $a \in M_k$, $a \notin M_{k+1}$. However $ma \in M_i \subseteq M_{k+1}$. i.e., the Specht module M_k/M_{k+1} is not torsion-free either. We get a contradiction since all Specht modules are subgroups in finitely generated free Abelian groups. ∎

Note that by tensoring this chain with F we get a Specht series for $S^F(\lambda;\mu)$ with as factors Specht modules $S^F(\nu)$ over F. Actually, by invoking [Jam78b, lemma 16.3] we recover Theorem 1.2.23 from the proof of Theorem 1.2.30.

Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash t \leq n$. Remark then that $(S(\lambda) \otimes_{\mathbb{Z}} S((n-t))) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} (S(\lambda) \otimes_{\mathbb{Z}} S((n-t)))$ is isomorphic as $\mathbb{Z}S_n$-module to $S(\lambda;\mu)$ with $\mu = (\lambda_1, \dots, \lambda_l, n-t)$. The proof of Theorem 1.2.30 also yields the $\mathbb{Z}$-analogue of the particular case of the Littlewood- Richardson rule that sometimes is referred to as Young's rule [GZ05, Theorem 2.3.3], [Dre00, Theorem 12.5.2] and sometimes as Pieri's formula [FH91, (A.7)]. Recall that λ' denotes the conjugate partition of λ.

Corollary 1.2.31 (Young's rule)**.** *Let $\lambda \vdash t$ and $t \leq n \in \mathbb{N}$. Then,*

$$(S(\lambda) \otimes_{\mathbb{Z}} S((n-t))) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} (S(\lambda) \otimes_{\mathbb{Z}} S((n-t)))$$

has a series of submodules with factors $S(\nu)$, where ν runs over the set of all partitions $\nu \vdash n$ such that $\lambda_i \leq \nu_i$ and $\nu'_i \leq \lambda'_i + 1$. Moreover, each factor occurs exactly once.

Remark that, as FS_n-modules, $M^F(\lambda) \cong \left(S^F((\lambda_1)) \otimes \cdots \otimes S^F((\lambda_l))\right) \uparrow S_n$. So the previous results mainly concerned Specht series of certain inductions of tensor products of Specht modules. However, one also needs information about restriction. The following result describes the factors of a Specht series of the restriction of the FS_n Specht module $S^F(\lambda)$ to FS_{n-1}, denoted $S^F(\lambda) \downarrow S_{n-1}$, see [Jam78b, Theorem 9.3] for a proof.

Theorem 1.2.32 (Branching Theorem)**.** *Let be $\lambda \vdash n$ and F a field of arbitrary characteristic. Then $S^F(\lambda) \downarrow S_{n-1}$ has a Specht series whose factors are the Specht modules $S^F(\mu)$ with μ a partition of $n-1$ obtained from λ by taking one box of D_λ away. Moreover $S^F(\mu_1)$ occurs above $S^F(\mu_2)$ in the series if $\mu_1 \trianglerighteq \mu_2$.*

1.3 Behind the Asymptotics

In Section 1.1 we saw that over a field F of characteristic 0 multilinear polynomials generate as a T-ideal $\mathrm{Id}(A)$ for a given F-algebra A. Further we associated with A a sequence $(c_n(A))_n$, called the codimension sequence, which following conjecture 1, made by Regev, grows asymptotically as the function $\psi(n) = cn^td^n$ with the constants c, t and d having nice properties. In this section we start with a short survey of the current status of research concerning this conjecture. Then we draw the main lines needed to prove that in a first instance the exponential growth rate 'd' exists and in a second instance it is an integer which is moreover computable.

Till the end of the chapter all fields will be assumed to have characteristic 0.

1.3.1 Amitsur and Regev Conjecture

Let A be a finitely generated PI-algebra over a field F. One also says that A is an affine PI-algebra over F. We know that A is PI-equivalent with the relatively free algebra $F\langle X\rangle / \mathrm{Id}(A)$, where X is a finite set. A fundamental result of Kemer [Kem91] asserts that the latter is representable, i.e. it can be embedded in a matrix algebra over some field extension. This enabled Kemer to reduce the study of A, up to PI-equivalence, to a finite dimensional algebra.

Theorem 1.3.1 (Representability Theorem of Kemer)**.** *Let A be an affine PI F-algebra. Then there exists a field extension L of F and a finite dimensional L-algebra B such that $B \sim_{PI} A \otimes_F L$.*

In particular, by Theorem 1.1.14, $c_n(A) = c_n(B)$ for all n. The only drawback is that using the Representability theorem one loses a concrete interpretation in terms of the structure of A, since Kemer's result is not constructive. In Section 1.4 we will return in greater detail to the Representability Theorem and the techniques involved in its proof.

So, we may assume that A is finite dimensional over F. In this case Giambruno and Zaicev proved that $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is determined by the F-dimension of a suitable semisimple subalgebra. To state their result more precisely, we need Wedderburn-Malcev's theorem, sometimes also called Wedderburn's Principal Theorem [Row88, Theorem 2.5.37].

Theorem 1.3.2 (Wedderburn-Malcev)**.** *Suppose A is a finite dimensional algebra over a perfect field F, e.g. char$(F) = 0$. Then there exists a maximal semisimple F-subalgebra B of A such that $A = B \oplus J(A)$, a direct sum of F-vector spaces, where $J(A)$ denotes the Jacobson radical of A. Moreover B is unique up to inner automorphisms.*

From now on we fix a decomposition of A as above. In order to compute $c_n(A)$ we may by Theorem 1.1.14 assume that F is algebraically closed. Then, by Wedderburn-Artin's Theorem, $B \cong A_1 \oplus \cdots \oplus A_q$ is a direct sum of simple rings with $A_i \cong M_{d_i}(F)$. With this notation, Giambruno and Zaicev proved the following result in [GZ98].

Theorem 1.3.3 (Giambruno-Zaicev)**.** *Let A be a finite dimensional F-algebra with $\overline{F} = F$. Then there exist constants $d \in \mathbb{Z}, n_0 \in \mathbb{N}$ and $c_1, c_2, t_1, t_2 \in \mathbb{R}$ such that $C_1 n^{t_1} d^n \leq c_n(A) \leq C_2 n^{t_2} d^n$ for all $n \geq n_0$. Moreover,*

$$d = \max\{\dim_F(A_{i_1} \oplus \cdots \oplus A_{i_r}) \mid A_{i_1} J(A) A_{i_2} \cdots J(A) A_{i_r} \neq 0 \textit{ with } i_j \neq i_k \textit{ for } j \neq k\}.$$

The constant d that represents the exponential growth rate of $(c_n(A))_n$ is called the *PI-exponent* of A and denoted $\exp(A)$. Further, as mentioned before, by invoking Kemer's Representability Theorem, the first part of Theorem 1.3.3 also holds for affine algebras over arbitrary fields of characteristic 0. Note also that the integrality of $\exp(A) = \lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ of course implies that the codimension sequence can not have intermediate exponential growth, which shows the great contrast with for example the word growth of finitely generated groups. This growth dichotomy was already proven in 1978 by Kemer in [Kem78].

Example 1.3.4.

1. Suppose $A = M_d(F)$, then $\exp(A) = d^2$.
2. Let $A = UT(d_1, \ldots, d_q)$ be the subalgebra of $M_{d_1+\ldots+d_q}(F)$ consisting of the ma-

trices

$$\begin{pmatrix} M_{d_1}(F) & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & M_{d_q}(F) \end{pmatrix}.$$

It is easy to see that the Wedderburn-Malcev decomposition of $UT(d_1,\dots,d_q)$ is

$$M_{d_1}(F)\oplus\cdots\oplus M_{d_q}(F)\oplus\begin{pmatrix} 0 & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & 0 \end{pmatrix} \text{ and } \exp(UT(d_1,\dots,d_q)) = d_1^2+\ldots+d_q^2.$$

In 2008 a weakened version of Regev's conjecture was confirmed by Berele and Regev in [BR08] for affine algebras (or more generally, PI-algebras satisfying a Capelli polynomial).

Theorem 1.3.5 (Berele-Regev)**.** *Let A be a PI-algebra satisfying a Capelli polynomial such that $(c_n(A))_n$ is monotonic nondecreasing for large enough n. Then $(c_n(A))_n$ is asymptotically bounded by the functions*

$$c_1 n^t(\exp(A))^n \lesssim c_n(A) \lesssim c_2 n^t(\exp(A))^n,$$

for some constants $c_1, c_2 \in \mathbb{R}$ and $t \in \frac{1}{2}\mathbb{Z}$. If, furthermore, A is unital then $c_1 = c_2$.

If a sequence is monotonic nondecreasing for large enough n, one says that it is eventually nondecreasing. Note that this is the case if A is unital. Moreover, in this case the authors also express the constant $c_1 = c_2$ as a sum of Selberg-type integrals. However, they are not able to compute it explicitly and consequently not able to confirm that $c_1 \in \mathbb{Q}(\sqrt{2\pi}, \sqrt{d})$ for some non-zero $d \in \mathbb{N}$.

Recently, in [GZ14], Giambruno and Zaicev proved that *the sequence of codimensions is indeed eventually nondecreasing,* showing the asymptotic inequality above holds for any affine PI-algebra. We refer to $t = t(A)$ as *the polynomial part* of the codimension sequence of A.

At this point it is important to emphasize that the proof of Berele and Regev for the existence of the parameter $t(A)$ does not give a formula for its calculation. In joint work with Aljadeff and Karasik [AJK17] we present an interpretation, à la Giambruno and Zaicev, of the polynomial part of the codimension growth for any finite dimensional

F-algebra A. More precisely, we give an explicit formula for so-called *'basic algebras'*, which may be viewed as the building blocks when describing algebras to PI-equivalence. Chapter 2 will be a detailed account of this article. In the next paragraphs we take a shortcut to the definition of a basic algebra in order to already state our interpretation of $t(A)$.

Let A be a finite dimensional algebra, A_{ss} a maximal semisimple subalgebra of A and let

$$\mathrm{Par}(A) = (\dim_F A_{ss}, \mathrm{nildeg}(J(A)) - 1)$$

be its *parameter* (nildeg$(J(A))$ denotes the nilpotency degree of the Jacobson radical). Such an algebra A is said to be basic (or fundamental) if it is not PI equivalent to $B_1 \oplus \cdots \oplus B_l$ where $\mathrm{Par}(B_i) < \mathrm{Par}(A)$ for every $1 \leq i \leq l$. By induction one can easily get the following.

Theorem 1.3.6. *Let A be a finite dimensional algebra over F. Then there exist basic algebras $B_1, \ldots, B_l$ such that A is PI-equivalent to $B_1 \oplus \cdots \oplus B_l$.*

This can be used to reduce the problem of interpreting the polynomial part (from arbitrary finite dimensional algebras) to basic algebras.

Corollary 1.3.7. *With the above notation,* $\exp(A) = \max_{1 \leq i \leq l} \exp(B_i)$ *and*

$$t(A) = \max_j \{t(B_j) : \exp(B_j) = \exp(A)\}.$$

Remark. The original definition of a *basic algebra* used in the proof of the Representability Theorem 1.3.1 for affine PI-algebras is different, yet equivalent, to the one we presented above. The decomposition of finite dimensional algebras into the direct sum of basic algebras up to PI equivalence (Theorem 1.3.6 above) using the other definition is a key and nontrivial step in the original work of Kemer.

For basic algebras Giambruno made the following conjecture.

Conjecture 2 (Giambruno)**.** *Let A be a basic algebra with Wedderburn-Malcev decomposition $A \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_q}(F) \oplus J(A)$. Then*

$$t(A) = \frac{q-d}{2} + s,$$

where $d = d_1^2 + \cdots + d_q^2$ and $s+1$ is the nilpotency degree of $J(A)$.

In case $A = M_d(F)$ this was established by Regev [Reg84]. The conjecture is also known to hold for the algebra of upper-block triangular matrices $UT(d_1, \dots, d_q)$ [GZ03]. This was proved by Giambruno and Zaicev. In their proof they used Lewin's Theorem [Lew74] and Berele and Regev's result [BR98b].

Applying Regev's result for matrix algebras, Giambruno's conjecture can be re-stated as follows. Let A be a basic algebra over F and $A \cong A_1 \oplus \cdots \oplus A_q \oplus J(A)$ be its Wedderburn-Malcev decomposition. Then

$$t(A) = t(A_1) + \cdots + t(A_q) + (\text{nildeg}(J(A)) - 1).$$

In [AJK17], see also Chapter 2, we prove this conjecture for an arbitrary basic algebra.

Remark.

- Kemer proved more generally that any PI-algebra is PI-equivalent with the Grassmann envelope of a finite dimensional superalgebra. Using Kemer's machinery, Giambruno and Zaicev were able to confirm in [GZ99] Amitsur's conjecture, asserting the existence and integrality of $d(A)$, also for non-affine PI-algebras. The unital case of Theorem 1.3.5 was generalized by Berele in [Ber08] to arbitrary unital PI-algebra's. We do not go in more details concerning these results because in this thesis we will only consider affine PI-algebras.
- Over a field of characteristic zero, also many generalisations of Amitsur and Regev's conjecture have been heavily investigated, for example considering non-associative algebras or including some (semi)group gradation or some Hopf algebra action. In Chapter 5 we do so as well.

1.3.2 On computing the Exponential Growth Rate

In this section we explain how one can prove the existence and/or integrality of $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$. The methods given are a combination of the original paper [GZ98] combined with ideas in [GMZ08, MVZ11].

If $\text{char}(F) = 0$, since $\frac{P_n(F)}{P_n(F)\cap \text{Id}(A)}$ is an FS_n-module, it is isomorphic, by Theorem 1.2.8, to the direct sum of Specht modules

$$\frac{P_n(F)}{P_n(F) \cap \text{Id}(A)} \cong \bigoplus_{\lambda \vdash n} m_\lambda S^F(\lambda),$$

where m_λ is the multiplicity of $S^F(\lambda)$. In particular, $c_n(A) = \sum_{\lambda \vdash n} m_\lambda \dim_F S^F(\lambda)$ for all n. Of course, if all multiplicities are known we are finished, however this is not doable.

Therefore, we will have to make asymptotic estimates. This is done by first bounding by above and then by below.

Note that $f \in \mathrm{Id}(A)$ if and only if $FS_n f \subseteq \mathrm{Id}(A)$. Using Theorem 1.2.23 we can write $FS_n = \bigoplus_{\substack{\lambda \vdash n, \\ T_\lambda \text{ standard}}} FS_n e^*_{T_\lambda}$ and see that $FS_n f \subseteq \mathrm{Id}(A)$ if and only if $e^*_{T_\lambda} f \in \mathrm{Id}(A)$ for all $\lambda \vdash n$ and standard Young tableaux T_λ. In particular, $m_\lambda \neq 0$ precisely when there exists a polynomial $f \in P_n(F)$ and tableau T_λ such that $e^*_{T_\lambda} f \notin \mathrm{Id}(A)$.

Upper bound

Now we explain how to bound $(c_n(A))_n$ from above. A first crucial result in this direction is that the multiplicities do not contribute to the exponential part. Recall that $l_n(A) = \sum_{\lambda \vdash n} m_\lambda$ is called the n-th colength of A.

Theorem 1.3.8 (Berele-Regev, [BR83]). *Let A be a PI-algebra over F. Then its sequence of colengths is polynomially bounded, i.e. there exist constants C and k such that*

$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda \leq Cn^k,$$

for all n.

So it is enough to estimate from above $\sum_{\lambda \vdash n, m_\lambda \neq 0} \dim_F S^F(\lambda)$. Write $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ for a sufficiently large n. Then, by the Hook formula 1.2.15 and because of the Stirling formula, we have

$$\begin{aligned} \dim_F S^F(\lambda) &= \frac{n!}{\prod_{i,j} h_\lambda(i,j)} \leq \frac{n!}{\lambda_1! \cdots \lambda_l!} \\ &\simeq \frac{\sqrt{2\pi}^{1-l} \sqrt{n} (\frac{n}{e})^n}{\sqrt{\lambda_1 \cdots \lambda_l} (\frac{\lambda_1}{e})^{\lambda_1} \cdots (\frac{\lambda_q}{e})^{\lambda_l}} \\ &= \frac{\sqrt{2\pi}^{1-l} \sqrt{n}}{\sqrt{\lambda_1 \cdots \lambda_l}} \left(\frac{1}{(\frac{\lambda_1}{n})^{\frac{\lambda_1}{n}} \cdots (\frac{\lambda_l}{n})^{\frac{\lambda_l}{n}}} \right)^n, \end{aligned} \tag{1.4}$$

for any partition λ of n. Now Theorem 1.3.8 and (1.4) imply that

$$\limsup_{n \to \infty} \sqrt[n]{c_n(A))} \leq \sup_{\substack{\lambda \vdash n, \\ m_\lambda \neq 0}} \Phi \left(\frac{\lambda_1}{n_1}, \cdots, \frac{\lambda_l}{n_l} \right), \tag{1.5}$$

where $\Phi(x_1, \cdots, x_l) = \frac{1}{x_1^{x_1} \cdots x_l^{x_l}}$ is a function on $\mathbb{R}^l$ that becomes continuous in the region $x_1, \cdots, x_l \geq 0$ if we define $0^0 = 1$. However, since we are interested in partitions and

$\dim_F A < \infty$, we can restrict Φ to the compact region

$$\Omega := \left\{ (\alpha_1, \cdots, \alpha_{\dim A}) \in \mathbb{R}^{\dim A} \mid \sum_{1 \leq i \leq \dim A} \alpha_i = 1,\ \alpha_1 \geq \ldots \geq \alpha_{\dim A} \geq 0 \right\}.$$

Indeed if $l(\lambda) > \dim_F(A)$, then $m_\lambda = 0$ because $e^*_{T_\lambda} f \in \mathrm{Id}(A)$ for any $f \in P_n(F)$ and any Young tableau T_λ. This can be understood by observing that $e^*_{T_\lambda} f$ is a multilinear polynomial alternating in the variables whose numbers are in the first column of T_λ. In particular, as in Example 1.1.1, if we substitute the same basis element twice in a variable with a number in the first column the evaluation is zero, which necessarily happens if $l(\lambda) > \dim_F(A)$. Equation (1.5), due to the continuity of Φ, now yields the upper bound:

$$\limsup_{n\to\infty} \sqrt[n]{c_n(A))} \leq \max_{(\alpha_1, \cdots, \alpha_{\dim A}) \in \Omega} \Phi(\alpha_1, \cdots, \alpha_{\dim A}).$$

Furthermore, if $\Omega_0 \subseteq \Omega$ is a subregion such that $m_\lambda = 0$ for all partitions with $\frac{\lambda}{n} := (\frac{\lambda_1}{n}, \cdots, \frac{\lambda_l}{n}) \notin \Omega_0$, then $\limsup_{n\to\infty} \sqrt[n]{c_n(A))} \leq \max_{\Omega_0} \Phi$. It is interesting to remark that, for Ω as above, $\max_\Omega \Phi = \dim_F A$ (e.g. see Lemma 5.3.3). We had already obtained this upper bound using linear algebra arguments in Section 1.1.2. However, except for central simple algebras, $\dim_F A$ is not a tight upper bound. Hence, in a first instance, one has to further reduce the region Ω by proving general conditions on the parts of a partition that forces $m_\lambda = 0$. For finite dimensional associative algebras over an algebraically closed field this was done, using other terminology, in [GZ98] and in this form, e.g., in [Gor13a, Lemma 7].

Proposition 1.3.9. *Let d be the constant from Theorem 1.3.3 and $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$. If $\sum\limits_{i=d+1}^{l} \lambda_i \geq$ nildeg$(J(A))$ or $\lambda_{\dim A+1} > 0$, then $m_\lambda = 0$.*

As $\sum\limits_{i=d+1}^{l} \lambda_i$ is bounded by a number independent of n we see that the last $l - d$ parts of $\frac{\lambda}{n}$ become arbitrarily small and thus do not contribute to the maximum due to the definition of Φ. So, for our purposes, we may restrict Φ to the polyhedron $\Omega := \left\{ (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d \mid \sum_{1 \leq i \leq d} \alpha_i = 1,\ \alpha_1 \geq \ldots \geq \alpha_d \geq 0 \right\}$ and $\limsup \sqrt[n]{c_n(A)} \leq d$.

Summarized: Using S_n-representation theory, $c_n(A) = \sum m_\lambda \dim_F S^F(\lambda)$. However, the multiplicities are polynomially bounded. So we may focus on $\sum_{m_\lambda \neq 0} \dim_F S^F(\lambda)$, which we bound by first giving a uniform bound on the length of the partitions with $m_\lambda \neq 0$ and then giving an exponential function bounding all $\dim_F S^F(\lambda)$ for the just obtained set of partitions. The latter function can be found by considering the function Φ on a sufficiently precise compact region.

Lower bound

Since $c_n(A) \geq \dim_F S^F(\lambda)$ for all Specht modules appearing in the decomposition of $\frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A)}$, it is sufficient to find a partition $\mu = (\mu_1, \dots, \mu_d)$ such that $\mu_d > 0,\ m_\mu \neq 0$ and

$$\begin{aligned}\dim_F S^F(\mu) &\geq \frac{n!}{(\mu_1 + d - 1)! \dots (\mu_d + d - 1)!}\\ &\geq \frac{n!}{n^{d(d-1)}\mu_1! \dots \mu_d!} \simeq Cn^B \left(\frac{1}{(\frac{\mu_1}{n})^{\frac{\mu_1}{n}} \cdots (\frac{\mu_d}{n})^{\frac{\mu_d}{n}}} \right)^n \\ &\simeq Cn^B d^n,\end{aligned} \tag{1.6}$$

for some constants $B, C \in \mathbb{R}$ in order to get the needed lower bound. Let $(\alpha_1, \dots, \alpha_d)$ be an extremal point of Φ on Ω. Then a natural candidate for a partition satisfying the right asymptotic behaviour is $\mu = (\mu_1, \dots, \mu_d)$ with

$$\begin{cases} \mu_i = \lfloor \alpha_i n \rfloor & \text{for } 2 \leq i \leq d, \\ \mu_1 = n - \sum_{i=2}^d \mu_i. \end{cases}$$

Indeed $(\frac{\mu_1}{n}, \dots, \frac{\mu_d}{n})$ converges to $(\alpha_1, \dots, \alpha_d)$. Thus for every $\epsilon > 0$ there exists a n_0 such that $\Phi(\frac{\mu_1}{n}, \dots, \frac{\mu_d}{n}) \geq d - \epsilon$ for all $n \geq n_0$. In view of (1.6) this shows that $S^F(\mu)$ has indeed the right dimension. Unfortunately in general it is hard to prove that $m_\mu \neq 0$.

In this thesis we will use two very different methods to find the necessary partition fitting in the above story. A *first one* is used in Section 5.6 and focuses on proving that previously defined partition μ actually satisfies $m_\mu \neq 0$. In this method we reduce the polyhedron Ω to a region Ω_0 having the properties

$$\max_{\Omega} \Phi = \max_{\Omega_0} \Phi \text{ and if } \lambda \vdash n \text{ such that } (\frac{\lambda_1}{n}, \dots, \frac{\lambda_d}{n}) \in \Omega_0 \text{ then } m_\lambda \neq 0.$$

This clearly ensures that above construction of μ from an extremal point will satisfy all we need. For concrete examples, as in section 5.6, this path will be feasible and is almost necessary in order to construct concrete examples with non-integer PI-exponent. On the other hand for general classes this is not feasible.

Therefore, we also sketch a *second method* where the focus is less on the precise form of the partitions and rather on constructing in a generic way, i.e. for all n sufficiently large, non-polynomial identities with 'many' alternating sets and only a finite number (independent of n) of variables outside these alternating sets. The needed partitions will then come as a by-product. In order to state this more precisely we first formulate the exact definition of alternating.

Definition 1.3.10. Let $f(X,Y)$ be a multilinear polynomial over F in non-commuting variables with $X = \{x_1,\dots,x_n\}$ and Y an arbitrary finite set. Then one says f *is alternating on* X if a substitution by x_j in any variable x_i with $i \neq j$ produces the zero polynomial.

If $\operatorname{char}(F) \neq 2$, f is alternating in X if and only if there exists a multilinear polynomial $h(X,Y)$ such that

$$f(x_1,\dots,x_n,Y) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma) h(x_{\sigma(1)},\dots,x_{\sigma(n)},Y) = \sum_{\sigma\in S_n} \operatorname{sign}(\sigma)\sigma h.$$

To come back to the second method, to be more precise, the goal is to prove the existence of a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have polynomials satisfying the assumptions of the following proposition.

Proposition 1.3.11. *Let $d \neq 0$. Suppose that there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ there exist disjoint subsets $X_1,\dots,X_{2k} \subseteq \{x_1,\dots,x_n\}$, $k = \lfloor \frac{n-n_0}{2d} \rfloor$, $|X_1| = \cdots = |X_{2k}| = d$ and a polynomial $f_k \in P_n(F) \setminus \operatorname{Id}(A)$ alternating in the variables of each set X_j. Then for all $n \geq n_0$ there exists a partition $\lambda = (\lambda_1,\dots,\lambda_s) \vdash n$ such that $\lambda_i > 2k - nildeg(J(A))$ for every $1 \leq i \leq d$ and $m_\lambda \neq 0$*

Proof. As explained at the beginning of the section, since $f_k \notin \operatorname{Id}(A)$ there exists a standard tableau T_λ corresponding to some partition $\lambda \vdash n$ such that $e^*_{T_\lambda} f_k \notin \operatorname{Id}(A)$. This λ has the desired shape. It is sufficient to prove that $\lambda_d > 2k - \operatorname{nildeg}(J(A))$. This follows from the facts that $e^*_{T_\lambda} = b_{T_\lambda} a_{T_\lambda}$ where a_{T_λ} is symmetrizing set-wise the variables with number in the same row of T_λ and f is alternating in the sets X_i. In particular, each row of T_λ may contain at most one variable from each set X_i, since otherwise $a_{T_\lambda} f_k = 0$. Thus $\sum_{i=1}^{d-1} \lambda_i \leq 2k(d-1) + (n-2kd) = n-2k$. Combined with the restricted region Ω we are working with, i.e. $\sum\limits_{i=d+1}^{l} \lambda_i < \operatorname{nildeg}(J(A))$ by Proposition 1.3.9, we indeed get that $\lambda_d > 2k - \operatorname{nildeg}(J(A))$. ∎

Now as $m_\lambda \neq 0$, with λ as above, we know that $c_n(A) \geq \dim_F S^F(\lambda)$. Further, by Corollary 1.2.22 and Example 1.2.21, $\dim_F S^F(\lambda) \geq \dim_F S^F((2k)^d) \simeq Ck^{\frac{1-d^2}{d}} d^{2kd}$ for some constant $C \in \mathbb{R}$ as $k \to \infty$. As $k = \lfloor \frac{n-n_0}{2d} \rfloor$, the existence of the above polynomials would indeed yield the lower bound. Note that we really used that the cardinality of $\{x_1, \ldots, x_n\} \setminus (X_1 \cup \ldots \cup X_{2k})$ is uniformly bounded (for all n) and due to this the asymptotics of the partitions are 'governed by a rectangle'.

In the proof of Theorem 1.3.3 the polynomials f_k were constructed by first doing so on the simple components of A and then by glueing them together using the definition of d. More precisely, assume that $M_{d_1}(F)J(A)\ldots J(A)M_{d_r}(F) \neq 0$ such that $d = d_1^2 + \cdots + d_r^2$. Suppose $a_1c_1a_2c_2\ldots c_{r-1}a_r \neq 0$ for some elements $c_j \in J(A)$ and $a_j \in M_{d_j}(F)$. Then

$$f_k := \mathrm{Alt}_{X^{(1)}} \cdots \mathrm{Alt}_{X^{(k)}} \mathrm{Alt}_{Y^{(1)}} \cdots \mathrm{Alt}_{Y^{(k)}} \prod_{j=1}^{k} \mathrm{Reg}^{(j)}_{d_1^2}.\ u_1 z_1 \ldots u_{r-1} z_{r-1}. \prod_{j=1}^{k} \mathrm{Reg}^{(j)}_{d_r^2}.\ u_r,$$

where

$$\begin{aligned} \mathrm{Reg}^{(j)}_{d_i^2} &:= \mathrm{Reg}^{(j)}_{d_i^2}(x^{(j)}_{1,i}, \ldots, x^{(j)}_{d_i^2,i}, y^{(j)}_{1,i}, \ldots, y^{(j)}_{d_i^2,i}) \\ &:= \sum_{\sigma,\tau \in S_{d_i^2}} \mathrm{sign}(\sigma\tau) x^{(j)}_{\sigma(1)} y^{(j)}_{\tau(1)} x^{(j)}_{\sigma(2)} x^{(j)}_{\sigma(3)} x^{(j)}_{\sigma(4)} y^{(j)}_{\tau(2)} y^{(j)}_{\tau(3)} y^{(j)}_{\tau(4)} \\ &\qquad \ldots x^{(j)}_{\sigma(d_i^2-2d_i+2)} \cdots x^{(j)}_{\sigma(d_i^2)} y^{(j)}_{\tau(d_i^2-2d_i+2)} \cdots y^{(j)}_{\tau(d_i^2)} \end{aligned} \tag{1.7}$$

is Regev's polynomial, $X^{(j)} := \bigcup_{i=1}^{r} \{x^{(j)}_{1,i}, \ldots, x^{(j)}_{d_i^2,i}\}$ and $\mathrm{Alt}_{X^{(j)}} := \sum_{\sigma \in S_d} \mathrm{sign}(\sigma)\sigma$ the operator that makes the polynomial alternating in the variable from the set $X^{(j)}$. It was proven by Formanek [For87] that, over a field of characteristic 0, the Regev polynomial is a proper central polynomial, i.e. there exists a non-zero evaluation but any evaluation yields a central element of $M_{d_i}(F)$. One can prove that there exists an evaluation of f_k whose output is a non-zero multiple of $a_1c_1a_2c_2\ldots c_{r-1}a_r \neq 0$.

We want to emphasize that there exists no universal method for constructing the polynomials f_r in the sense that for, e.g., non-associative algebras or for algebras over a field of characteristic p, the above polynomials do not work (in general).

Remark. In Chapter 5 we consider finite dimensional associative and Lie algebras over a field of characteristic 0 endowed with a semigroup grading. We will see that, unlike the case of group gradings, the region Ω will be different to the one above. More precisely, the lengths of the partitions will still be bounded uniformly, but the sum of certain parts will also be bounding each other (e.g. $\lambda_d + \lambda_{d-1} \leq \lambda_1$). So in these cases the partitions are rather 'governed by staircases' then 'rectangles'. Due to this, $\max_\Omega \Phi$ will not always be an integer. However, the PI-exponent still exists in these cases. The first example of a non-associative algebra A with $\liminf \sqrt[n]{c_n(A)} \neq \limsup \sqrt[n]{c_n(A)}$ has been discovered by Zaicev in [Zai14].

1.3.3 Some further remarks

In the previous section we discussed how to compute the PI-exponent. From the proof of Giambruno-Zaicev's Theorem one can actually extract more. We here list two such things.

1. A polynomial $f(x_1, \dots, x_n) \in F\langle X\rangle$ is central if $\operatorname{im}(f) \in \mathcal{Z}(A)$ where we consider f as a function from A^n to A. As in [Reg16], we denote by $\mathrm{Id}^z(A)$ the T-ideal generated by the central polynomials of A. Note that $\mathrm{Id}(A) \subseteq \mathrm{Id}^z(A)$. Similarly to the classical situation one defines the *central codimensions* $c_n^z(A) := \dim_F \frac{P_n(F)}{P_n(F)\cap \mathrm{Id}^z(A)}$. From a diagonal look on the proof of Proposition 1.3.9 and the outline of the proof of Giambruno-Zaicev's Theorem sketched in the previous section it is not hard to grasp that the following 'central codimensions version' of Amitsur's Conjecture holds.

   Theorem 1.3.12. *Let A be a finite dimensional F-algebra with $\overline{F} = F$ and $A = A_1 \oplus \dots \oplus A_q \oplus J(A)$ its Wedderburn-Malcev decomposition. Then $d := \lim_{n\to\infty} \sqrt[n]{c_n^z(A)}$ exists and is equal to*

   $$d = \max\{dim_F(A_{i_1} \oplus \dots \oplus A_{i_r}) \mid A_{i_1} J A_{i_2} \cdots J A_{i_r} \nsubseteq \mathcal{Z}(A) \text{ with } i_j \neq i_k \text{ for } j \neq k\}.$$

   We have no reference for this result, however it is known by the experts. Unfortunately, no Representability Theorem for $\mathrm{Id}^z(\cdot)$ is known. In particular, we can not generalise the existence of $\lim_{n\to\infty} \sqrt[n]{c_n^z(A)}$ to affine algebras.

2. Suppose $A = B_1 \oplus \dots \oplus B_l$ a direct sum of subalgebras. Taking as F-basis of A the union of an F-basis of the different B_i's we remark that $P_n(F) \cap \mathrm{Id}(A) = P_n(F) \cap \bigcap \mathrm{Id}(B_i)$. Considering the canonical map it is easy to see that $c_n(B_i) \leq c_n(F\langle X\rangle / (\mathrm{Id}(B_1) \cap \dots \cap \mathrm{Id}(B_l))) \leq \sum_{i=1}^{l} c_n(B_i)$. Hence,

   $$\exp(A) = \max_{1\leq i\leq l} \exp(B_i).$$

   This could also be obtained by using Proposition 1.3.9. Since we know by Theorem 1.3.5 that for any PI-algebra, for sufficiently large n, $c_1 n^t d^n \leq c_n(A) \leq c_2 n^t d^n$, it is not so hard to deduce that

   $$t(A) = \max\{t(B_i) \mid \exp(B_i) = \exp(A)\}.$$

1.4 Kemer Theory and Representability

In this section we discuss some parts of the innovative theory introduced by Kemer in order to prove his Representability Theorem 1.3.1. A special role is played by the basic algebras introduced in Section 1.3.1. These algebras have the advantage of possessing different viewpoints, algebraic, combinatorial and geometric ones, each of which has its own benefits. To start, we explain in Section 1.4.1 that the parameter $\mathrm{Par}(A) = (\dim_F A_{ss}, \mathrm{nildeg}(J(A)) - 1)$ has implications on the existence of certain types of non-identities for A. More precisely, on the size and number of alternating sets of non-polynomial identities. Based on this, we recall the definition of a Kemer index which is a tuple of two numbers recording how big and how much of such alternations can exist before forcing a multilinear polynomial to be an identity. Thereupon such extremal polynomials can be formalised in the notion of Kemer polynomials. Then we recall the main results that tell us why basic algebras as in Section 1.3.1 can serve as minimal models for a given Kemer index.

Next in Section 1.4.2 we discuss several examples of basic algebras of high importance in this thesis.

The materials from these sections are mainly based on the especially nicely written paper [AKBK16], but also on Section 2 of [AJK17] which is joint work with Aljadeff and Karasik. Chapter 2 is a complete account of the latter article. See also [Kem91] for the original reference by Kemer.

1.4.1 Basic algebras and Kemer polynomials

Let A be a finite dimensional algebra over a field F of characteristic 0 and $A \cong A_{ss} \oplus J(A)$, with $\dim_F A_{ss} = n$ and $s = \mathrm{nildeg}(J(A))$ the *nilpotency degree* of $J(A)$, i.e. the smallest positive integer such that $J(A)^{s-1} \neq 0$ but $J(A)^s = 0$. Suppose $f(X_1, \dots, X_r, Y)$ is a multilinear polynomial alternating in each set of variables X_i, where $|X_i| = m$ for all $1 \leq i \leq r$ (cf. Definition 1.3.10). A substitution with an element from A_{ss} is called a *semisimple evaluation* and one with an element from $J(A)$ a *radical evaluation*. Since we are interested in multilinear polynomials and because $A \cong A_{ss} \oplus J(A)$ we may assume that any substituted element either is semisimple or is in $J(A)$. In the sequel, by a non-identity of A is meant a non-polynomial identity.

Clearly in a non-zero evaluation of f, if $m > n$, at least one variable from each alternating set will assume a radical evaluation. Hence if $r \geq s$ we will have at least s radical evaluations and so the polynomial vanishes with any evaluation in this case. In other words, if we know that for $r \geq s$ there is a non-identity then $m \leq n$. With this simple remark in mind it makes sense to define numbers recording how big r and m can actually be for non-identities of A. In particular do values of m exist such that r can be arbitrarily large?

Definition 1.4.1 (Kemer index)**.** For any $\nu \in \mathbb{N}$, let

$$\Delta_\nu = \{r \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \mathrm{Id}(A) \text{ alternating in } \nu \text{ disjoint sets of size r}\}.$$

Clearly if $\nu \leq \gamma$, then $\max \Delta_\nu \geq \max \Delta_\gamma$. Let $d(A) = \lim_\nu \max(\Delta_\nu)$.

Next, we let

$$\begin{aligned} S_\nu^{d(A)} = \{j \in \mathbb{N} \cup \{0\} \mid \exists p(X) \notin \mathrm{Id}(A) &\text{ alternating in } \nu \text{ disjoint sets of size d(A)} \\ &\text{and alternating in } j \text{ disjoint sets of size } d+1\}. \end{aligned}$$

Also here $\max S_\nu^{d(A)} \geq \max S_\gamma^{d(A)}$ if $\nu \leq \gamma$ and we set $s(A) = \lim_\nu \max S_\nu^{d(A)}$.

The tuple $\kappa_A = (d(A), s(A))$ is called the *Kemer index of A*. We also write $(d,s) := (d(A), s(A))$ if A is clear from the context.

Remark.

- Note that d and s are finite. Indeed, d is finite because $\max \Delta_\nu \leq \dim_F A$ for any $\nu > 0$ and s is finite due to the definition of d. More generally, one can associate a Kemer index to any PI-algebra satisfying a Capelli identity Cap_n since then $\max \Delta_\nu \leq n < \infty$. In particular by [AKBK16, Theorem 3.1.] or [GZ05, Theorem 1.12.2] any affine PI-algebra has a Kemer index.

- The disjoint alternating sets of size d are called *small sets* and the disjoint sets of size $d+1$ are called *big sets*. In other words, there exist non-identities in an arbitrary number of alternating small sets, but only a finite number of big sets (actually at most s big sets).

- So any affine F-algebra W determines a point $\kappa_W = (d,s)$ in the set $\Omega = \mathbb{N} \times \mathbb{N}$ which we equip with the left lexicographic ordering. With this convention, using the arguments mentioned just before Definition 1.4.1, we have that

$$\kappa_A = (d,s) \leq (\dim_F A_{ss}, \mathrm{nildeg}(J(A))) \tag{1.8}$$

  for a finite dimensional F-algebra A. Further, if W_1 and W_2 are two affine algebras such that $\mathrm{Id}(W_1) \subseteq \mathrm{Id}(W_2)$, then $\kappa_{W_1} = (d(W_1), s(W_1)) \geq (d(W_2), s(W_2)) = \kappa_{W_2}$.

- Regev's central polynomial (1.7) takes the identity matrix I_n as value on $A = M_n(F)$. Therefore the product of any number of copies of Regev's central polynomial, each in other sets of indeterminates, is a non-identity showing that $\kappa_A = (n^2, 0)$. Actually $\kappa_A = (\dim_F A, 0)$ if and only if A is a central simple algebra over F. At the other end of the spectrum, it is easy to see that A is nilpotent with nilpotency index l if and only if $\kappa_A = (0, l-1)$.

Now we consider extremal polynomials which are not in $\mathrm{Id}(A)$ and whose alternations realize the Kemer index κ_A. These will play a key role later on.

Definition 1.4.2. (Kemer polynomials) Let ν_0 be a number where $\max \Delta_{\nu_0} = d$ and $\max S^d_{\nu_0} = s$. Then a multilinear polynomial f is called a *Kemer polynomial* of A if $f \notin \mathrm{Id}(A)$ has at least ν_0 small sets (cardinality d) and exactly s big sets (cardinality $d+1$).

As noted before, $\kappa_A = (\dim_F A, 0) = \mathrm{Par}(A)$ if $A = M_n(F)$. Consider now $A^r := A \times \cdots \times A$, r times, then $\mathrm{Id}(A) = \mathrm{Id}(A^r)$, in particular $\kappa_A = \kappa_{A^r}$, but $\mathrm{Par}(A) = (n^2, 0) < (rn^2, 0) = \mathrm{Par}(A^r)$. So, this example shows that the Kemer index of a finite dimensional algebra may be far from its parameter. Consequently, in order to establish a framework where one has a precise interplay between combinatorial tools and algebraic tools, coming respectively from the polynomial identities and the parameter of the algebra, one needs appropriate finite dimensional algebras which serve as minimal models for a given Kemer index. An amazing result of Kemer, see [Kem91] or ([AKBK16, Prop. 5.13]), says that the basic algebras introduced in Section 1.3.1 fulfill this role.

Theorem 1.4.3 (Kemer)**.** *A finite dimensional algebra A over a field F of characteristic 0 is basic if and only if $\kappa_A = \mathrm{Par}(A)$.*

It is not hard to see that if A is non-basic then, $\kappa_A < \mathrm{Par}(A)$. Indeed in this case, by definition, $A \cong B_1 \oplus \cdots \oplus B_r$ for subalgebras B_i with $\mathrm{Par}(B_i) < \mathrm{Par}(A)$ for all i. So, either $\dim_F(B_i)_{ss} < \dim_F A_{ss}$ or $\mathrm{nildeg}(J(B_i)) < \mathrm{nildeg}(J(A))$. Suppose now that $\kappa_A = \mathrm{Par}(A)$, i.e. A has Kemer polynomials with $\dim_F A_{ss}$ small sets and $\mathrm{nildeg}(J(A)) - 1$ big sets. However, in both cases, such Kemer polynomials with at least $\mu \geq \max_i \dim_F B_i$ small sets vanish on each B_i, hence also on A and thus do not exist.

Also, an important consequence of this theorem is that Kemer polynomials of basic algebras do not vanish only for evaluations where all simple components A_i are represented among the substitutions and precisely $J(A) - 1$ variables are substituted by radical elements.

For the proof of the fundamental Theorem 1.4.3 two properties of finite dimensional algebras, named *'full'* and *'property K'*, were introduced. It was then shown that basic algebras satisfy both conditions and that an algebra satisfying both also satisfies $\kappa_A = \mathrm{Par}(A)$. These steps are the content of the so called 'Kemer Lemma 1' and 'Kemer Lemma 2', see [AKBK16, pages 10-20]. So, the expanded version of the above theorem is the following.

Theorem 1.4.4 (Kemer)**.** *Let A be a finite dimensional algebra over a field F of characteristic 0. Then the following conditions are equivalent:*

1. *A is basic,*
2. *A is full and satisfies property K,*
3. $\kappa_A = \mathrm{Par}(A)$.

We now recall briefly the definition of 'full' and 'property K', see [AKBK16, definitions 5.14, 7.3].

Definition 1.4.5. Let A be a finite dimensional algebra. If there exists a multilinear non-identity f such that on *every non-vanishing evaluation* of f on A one must substitute at least one element from every simple component then A is said to be *full*.

Definition 1.4.6. Let A be a finite dimensional algebra. Then A has *property K* if there exists a multilinear non-identity f such that f vanishes on any evaluation on A with less than $\mathrm{nildeg}(J(A)) - 1$ radical substitutions.

Note that the first part of Theorem 1.4.4 says that a basic algebra has a non-identity f satisfying property full and a non-identity h satisfying property K, but the next part tells that actually there is one polynomial satisfying both properties at the same time. The latter is the content of the so-called 'Kemer Lemma 2'.

In the next paragraphs we discuss some striking properties of basic algebras. Afterwards we exhibit examples of basic algebras, all of which will be used later on.

To start, remark that from Kemer's Theorem 1.4.3 and Giambruno-Zaicev's Theorem 1.3.3 we immediately get the value of the PI-exponent of a basic algebra A.

Proposition 1.4.7. *Let A be a basic algebra with Kemer index $\kappa_A = (d, s)$. Then* $\exp(A) = d = \dim_F A_{ss}$.

Proof. To start, we write $A = A_1 \oplus \cdots \oplus A_q \oplus J(A)$ with $A_{ss} = A_1 \oplus \cdots \oplus A_q$ a maximal semisimple subalgebra. By Theorem 1.4.3, $\kappa_A = (d, s) = (\dim_F(A_{ss}), \mathrm{nildeg}(J(A)) - 1)$. So, by definition of the Kemer index, there exists for any $n = dm + (d+1)s + c \geq d\nu_0 + (d+1)s$ a Kemer polynomial $f(X_1, \dots, X_m, Y_1, \dots, Y_s, Z)$ for A. So f is a multilinear polynomial alternating in the small sets X_i and the big sets Y_i, where $|X_i| = \dim_F A_{ss}, |Y_i| = \dim_F A_{ss} + 1$ and $|Z| = c$.

Since f is alternating in de $Y_i's$ and $|Y_i| = d + 1$, in order to have a non-zero evaluation, one must substitute at least one radical basis element in each Y_i. Since $s = \mathrm{nildeg}(J(A)) - 1$ we may also neither substitute two radical basis elements in a Y_i nor a radical element in a X_i or Z. Thus, indeed, in each non-zero evaluation of f each simple component is represented and exact s radical elements are used. In other words, since $A_i \cdot A_j = 0$ for any $i \neq j$, if $f(a_1, \dots, a_n)$ is a non-zero evaluation, which exists, then necessarily each non-zero term must be an element from $A_{\sigma(1)} J(A) A_{\sigma(2)} J(A) \dots J(A) A_{\sigma(q)}$ for some $\sigma \in S_q$. Therefore, by Theorem 1.3.3, we have that $\exp(A) = \dim_F(A_{\sigma(1)} \oplus \cdots \oplus A_{\sigma(q)}) = \dim_F A_{ss}$ as claimed. ∎

In particular we also get the following.

Corollary 1.4.8. *Let $A = A_1 \oplus \cdots \oplus A_q \oplus J(A)$ be a basic algebra with simple components A_i. Then $q \leq s + 1 =$ nildeg$(J(A))$. If B is another basic algebra PI-equivalent to A, then* $\dim_F A_{ss} = \dim_F B_{ss}$.

The last statement was considerably strengthened by Procesi [Pro16, Corollary 3.10].

Theorem 1.4.9 (Procesi)**.** *Let A and B be basic algebras. If A and B are PI-equivalent, then* $A_{ss} \cong B_{ss}$.

Remark. The previous results can be generalized to the context of finite group-graded PI-algebras and even to PI-algebras endowed with an action by a finite dimensional semisimple Hopf algebra, see [AKB10, Kar16].

1.4.2 Examples

At the moment only few examples of basic algebras are known. In this section we first discuss classical examples, i.e. $M_d(F)$ and $UT(d_1, \dots, d_l)$, and then we explain how one can associate a new basic algebra $\mathcal{A}$ to any given basic algebra A.

Upper-triangular Matrix algebras

To start, consider the matrix algebra $M_d(F)$. Using Capelli polynomials it is readily proven that $M_d(F)$ has Kemer index $(d^2, 0)$ and thus is basic. To see this, note that any multilinear polynomial alternating on d^2+1 variables vanishes on $M_d(F)$. Now recall the nth Capelli polynomial

$$\mathrm{cap}_n(X;Y) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) y_1 x_{\sigma(1)} \cdots y_n x_{\sigma(n)} y_{n+1},$$

with $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_{n+1}\}$.

It is well known that $\mathrm{cap}_{d^2}(X;Y) \notin \mathrm{Id}(M_d(F))$ and, moreover, all diagonal elementary matrices can be realized as a nonzero evaluation. For example, e_{11} can be realized by substituting for $\{x_1, \ldots, x_{d^2}\}$ all the e_{ij} ordered according to the left lexicographic order of the indices, the variables y_1, y_{d^2+1} by respectively e_{11}, e_{d1} and for all other y_i we do the unique substitution making $y_1 x_1 y_2 x_2 \ldots x_{d^2} y_{d^2+1}$ the only monomial with non-zero evaluation.

Therefore, for any $\mu \in \mathbb{N}$, the polynomial

$$\mathrm{Cap}_{d^2}(X_1, \ldots, X_\mu; Y_1, \ldots, Y_\mu) = \prod_{i=1}^{\mu} \mathrm{cap}_{d^2}(X_i, Y_i),$$

where $X_i = \{x_{i,1}, \ldots, x_{i,d^2}\}$, is a Kemer polynomial of $M_d(F)$, proving that indeed $\kappa_{M_d(F)} = (d^2, 0)$.

The next natural and important example is the algebra of upper block triangular matrices $UT(d_1, \ldots, d_q)$ for positive integers $d_1, \ldots, d_q$ introduced in Example 1.3.4. Recall that this is the subalgebra of $M_{d_1+\cdots+d_q}(F)$ consisting of the matrices

$$\begin{pmatrix} M_{d_1}(F) & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \cdots & 0 & M_{d_q}(F) \end{pmatrix}.$$

We claim the following.

Proposition 1.4.10. *The algebra $UT(d_1, \ldots, d_q)$ is a basic algebra with Kemer index $(d, q-1)$, where $d = d_1^2 + \cdots + d_q^2$.*

First, as mentioned in Example 1.3.4, we have that $UT(d_1, \ldots, d_q)$ has PI-exponent d, the dimension of a maximal semisimple subalgebra. Hence, its Kemer index has the form $\kappa = (d, s)$. Moreover, since the nilpotency degree of $UT(d_1, \ldots, d_q)$ is $q - 1$, we have that $s \leq q - 1$ by (1.8).

In order to complete the proof of Proposition 1.4.10, it is enough to construct for an arbitrary μ a (Kemer) polynomial with μ small sets of cardinality d and precisely $q - 1$ sets of cardinality $d + 1$. We start with the construction of polynomials with arbitrarily many small sets of cardinality d. For each simple component $M_{d_i}(F)$, $i = 1, \ldots, q$, we consider the polynomial $\mathrm{Cap}_{d_i^2}(X_{i,1}, \ldots, X_{i,\mu}; Y_{i,1}, \ldots, Y_{i,\mu})$ and their product bridged by the variables $w_1, \ldots, w_{q-1}$:

$$\mathrm{Cap}_{d_1^2}(X_{1,1}, \ldots, X_{1,\mu}; Y_{1,1}, \ldots, Y_{1,\mu}) \times w_1 \cdots w_{q-1} \times \mathrm{Cap}_{d_q^2}(X_{q,1}, \ldots, X_{q,\mu}; Y_{q,1}, \ldots, Y_{q,\mu}).$$

We denote this polynomial by $\mathrm{Cap}_{d_1^2,\ldots,d_q^2}(X_{i,j}; Y_{i,j}, i = 1, \ldots, q, j = 1, \ldots, \mu, W)$ or in short $\mathrm{Cap}_{d_1^2,\ldots,d_q^2}(X_{i,j}; Y_{i,j}, W)$.

Now we alternate in the polynomial above the sets $X_{1,j}, \ldots, X_{q,j}$ for every $j = 1, \ldots, \mu$, i.e. we consider its image under the action by

$$\mathrm{Alt}_{X_1} \cdots \mathrm{Alt}_{X_\mu} = \sum_{\sigma_1 \in \mathrm{Sym}(X_1)} \mathrm{sign}(\sigma_1)\sigma_1 \quad \cdots \sum_{\sigma_\mu \in \mathrm{Sym}(X_\mu)} \mathrm{sign}(\sigma_\mu)\sigma_\mu,$$

with $X_j = X_{1,j} \cup \cdots \cup X_{q,j}$. Denote by $f_{1,\mu}(X_{i,j}; Y_{i,j}, W)$ the polynomial obtained in this way. Next we construct the needed $q - 1$ big sets by alternating w_j with the set $X_j = X_{1,j} \cup \cdots \cup X_{q,j}$, for $j = 1, \ldots, q - 1$. The result is a polynomial $f_{2,\mu}$ which alternates on $\mu - (q - 1)$ small sets of cardinality d and precisely $q - 1$ big sets of cardinality $d + 1$. Using the non-zero evaluation from the example $A = M_d(F)$ for each simple component of $UT(d_1, \ldots, d_q)$ and appropriate radical substitutions for the variables of W, it is not hard to show that $f_{1,\mu}$ and $f_{2,\mu}$ are nonidentities of $UT(d_1, \ldots, d_q)$. Our construction of $f_{2,\mu}$ shows that κ, the Kemer index of $UT(d_1, \ldots, d_q)$, satisfies $\kappa_{UT(d_1,\ldots,d_q)} \geq (d, q - 1)$. On the other hand $\kappa_{UT(d_1,\ldots,d_q)} \leq \mathrm{Par}(UT(d_1, \ldots, d_q)) = (d, q - 1)$ and hence $\kappa_{UT(d_1,\ldots,d_q)} = \mathrm{Par}(UT(d_1, \ldots, d_q))$. This shows $UT(d_1, \ldots, d_q)$ is basic and f_2 is a Kemer polynomial.

The associated basic algebra

Let now $A \cong A_{ss} \oplus J(A) \cong A_1 \oplus \cdots \oplus A_q \oplus J(A)$ be a, not necessarily basic, finite dimensional algebra and let $r \geq \dim_F(J(A))$. Denote by $A_{ss} * F\langle b_1, \ldots, b_r\rangle$ the free product of the F-algebras A_{ss} and $F\langle b_1, \ldots, b_r\rangle$. Then, for any $u \in \mathbb{N}$, we consider the associated algebra

$$\mathcal{A}_u := \frac{A_{ss} * F\langle b_1, \ldots, b_r\rangle}{(b_1, \ldots, b_r)^{u+1}_{A_{ss} * F\langle b_1, \ldots, b_r\rangle}}.$$

One advantage of this algebra is that we have better control on the nilpotency degree of $J(A)$ and the multiplication between the radical and the semisimple part.

Proposition 1.4.11. *Let $r \geq \dim(J(A))$. Then the algebra $\mathcal{A}_u$ satisfies following properties.*

(i) $\mathcal{A}_u$ is finite dimensional

(ii) nildeg$(J(\mathcal{A}_u)) = u + 1$

(iii) A is an epimorphic image of $\mathcal{A}_u$.

Proof. Choose a basis $\Phi = \{a_1, \ldots, a_d\}$ of A_{ss} (e.g. the elementary matrices of the simple components A_j). Consider non-zero monomials in $A_{ss} * F\langle b_1, \ldots, b_r\rangle$. These are words of the form

$$a_{i_1} b_{i_1} a_{i_2} b_{i_2} \cdots a_{i_k} b_{i_k} a_{i_{k+1}},$$

where $k \geq 0$, $a_{i_j} \in \Phi, b_{i_j} \in \{b_1, \ldots, b_r\}$ and we also allow consecutive b_{i_j}'s. By definition of $\mathcal{A}$, monomials are zero in $\mathcal{A}$ if $k > u$ and hence their number is finite. This proves the first part of the lemma.

The second statement is clear by the definition of $\mathcal{A}_u$.

For the third statement define a map $\phi : \mathcal{A}_u \to A$ as the identity on A_{ss} and sending $\{b_1, \ldots, b_r\}$ surjectively on a basis of $J(A)$. The extension on whole $\mathcal{A}_u$ yields the necessary epimorphism. ∎

In case $u = \text{nildeg}(J(A)) - 1$ and $r = \dim_F J(A)$ we write $\mathcal{A} = \mathcal{A}_{\text{nildeg}(J(A))-1}$ instead. Moreover, if A is basic, then so is $\mathcal{A}$ as shown below. The algebra $\mathcal{A}$ will play a key role in Chapter 2 and more precisely in the proof of the upper-bound of Giambruno's Conjecture 2.

Proposition 1.4.12. *If the algebra A is basic then $\mathcal{A}$ is also basic.*

Proof. Note that A_{ss} is a maximal semisimple subalgebra which supplements the radical $J(\mathcal{A})$ of $\mathcal{A}$. By Proposition 1.4.11 the radical is generated by the variables b_i and its nilpotency degree equals nildeg$(J(A))$. It follows that $\text{Par}(\mathcal{A}) = (d, \text{nildeg}(J(A)) - 1)$. But the algebra A is a quotient of $\mathcal{A}$ and hence its Kemer index is at least the Kemer index of A. This implies the Kemer index of $\mathcal{A}$ equals $\text{Par}(\mathcal{A}) = (d, \text{nildeg}(J(A)) - 1)$ and the result follows. ∎

2 The Polynomial Growth Rate of Codimensions

> The real satisfaction from
> mathematics is in learning from
> others and sharing with others.
>
> *William Thurston*

Let A be a a finitely generated, not necessarily unital, PI algebra over an algebraically closed field of characteristic 0. By the Berele-Regev Theorem 1.3.5, and [GZ14], we know that there exist $c_1, c_2 \in \mathbb{R}$ and $t \in \frac{1}{2}\mathbb{Z}$ such that

$$c_1 n^t (\exp(A))^n \lesssim c_n(A) \lesssim c_2 n^t (\exp(A))^n$$

for large enough n and where $t(A) := t$ is called the polynomial part of A.

As explained in Section 1.3.1, in order to compute $\exp(A)$ one first has to invoke Kemer's Representability Theorem 1.3.1 to switch over to a finite dimensional representant B of the PI-equivalence class of A. Then Giambruno-Zaicev's Theorem 1.3.3 yields a concrete formula for $\exp(A) = \exp(B)$. More precisely, it says that the PI-exponent equals the F-dimension of a semisimple subalgebra B' of B. The algebra B' is connected to how many different Wedderburn-Artin components of B can be bridged together by radical elements. So this gives, at least for finite dimensional algebras, a nice interpretation of the exponential growth rate, $\exp(A)$, of the codimension sequence $c_n(A)$.

By Berele-Regev's Theorem, we know that the polynomial growth rate t is a half-integer, however from their proof, even for finite dimensional algebras, no interpretation for t can be extracted. In this Chapter we address this problem.

As for the PI-exponent, at first instance we have to reduce the problem to 'nicer' algebras. Of course, by the Representability Theorem, we may assume A to be finite dimensional. Then, by Theorem 1.3.6, A is PI-equivalent to a direct sum $B_1 \oplus \cdots \oplus B_l$ of basic algebras. Due to Corollary 1.3.7, one of the B_i's satisfies $\exp(B_i) = \exp(A)$ and $t(B_i) = t(A)$. Thus, it is enough to find an interpretation for the polynomial part of a basic algebra. As mentioned in Section 1.3.1, a formula for this is the content of Giambruno's Conjecture 2:

$$t(A) = \frac{q-d}{2} + s,$$

where $B_i \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_q}(F) \oplus J(B_i)$ is the Wedderburn-Malcev decomposition of the basic algebra B_i and $s = \mathrm{nildeg}(J(B_i)) - 1$. Remark that, in order to find an interpretation in terms of A, one must also be able to describe the algebra B_i in terms of A, which unfortunately is still an open problem. For example, by Proposition 1.4.7, we know that $\exp(A) = \exp(B_i) = d_1^2 + \ldots + d_q^2$ is an integer, i.e. Amitsur's conjecture holds, but in this way we do not get the origin of the suitable semisimple subalgebra as is the case of Giambruno-Zaicev's formula.

In this chapter we solve Giambruno's conjecture. The proof is subdivided in proving separately the upper and lower bound. The techniques for both parts are totally different. For the upper bound we associate to A the basic algebra $\mathcal{A} := \mathcal{A}_{\mathrm{nildeg}(A)-1}$ from Proposition 1.4.12 and then in Section 2.1 we prove that $t(\mathcal{A}) \leq \frac{q-d}{2} + s$ with $s = \mathrm{nildeg}(\mathcal{A}) - 1 = \mathrm{nildeg}(A) - 1$. Recall that $t(A) \leq t(\mathcal{A})$ by Proposition 1.4.11. Next, in Section 2.2, we prove the lower bound by using the Kemer polynomials of A in order to create enough linearly independent sets whose size we can control.

Till the end of the chapter we denote by A a basic algebra with Wedderburn-Malcev decomposition $A = A_{ss} \oplus J(A) = A_1 \oplus \ldots \oplus A_q \oplus J(A)$ with $A_i \cong M_{d_i}(F)$.

2.1 Upper bound

Consider the algebra $\mathcal{A}$ defined as

$$\mathcal{A} = \frac{A_{ss} * F\{b_1, \ldots, b_{\dim_F J(A)}\}}{\left\langle b_1, \ldots, b_{\dim_F J(A)} \right\rangle^{s+1}},$$

where $\langle \cdot \rangle$ denotes the ideal generated by $\{b_1, \ldots, b_{\dim_F J(A)}\}$. As observed in Proposition 1.4.11, the algebra A is an epimorphic image of $\mathcal{A}$ and so $\mathrm{id}(\mathcal{A}) \subseteq \mathrm{id}(A)$. Also, the algebras A and $\mathcal{A}$ have the same Kemer index and moreover have isomorphic semisimple subalgebras supplementing the corresponding radicals. In particular, they have the same exponent, namely $\dim_F A_{ss}$. It follows that $t(A) \leq t(\mathcal{A})$ and hence it is, as claimed before, sufficient to show $t(\mathcal{A}) \leq \frac{q-d}{2} + s$.

For this we have to investigate $P_n(\mathcal{A}) := \frac{P_n(F)}{P_n(F)\cap \mathrm{id}(\mathcal{A})}$, for sufficiently large n. In order to check whether a multilinear polynomial is a polynomial identity it is enough to do so on basis elements of $\mathcal{A}$. For this reason it is important to fix a convenient basis for $\mathcal{A}$, which we do now.

Recall that $A_{ss} = A_1 \oplus \ldots \oplus A_q$. For $1 \leq l \leq q$, we denote the matrix units of A_l by $e_{j_1,j_2}(A_l)$ and $e_{j_1}(A_l) = e_{j_1,j_1}(A_l)$, $1 \leq j_1, j_2 \leq d_l$. Next, for $1 \leq k, k' \leq q$, $1 \leq i \leq d_k$ and $1 \leq j \leq d_{k'}$, let

$$W_{i,j}(A_k, A_{k'}) = \{e_{i,j_0}(A_k)b_{l_0}e_{i_1,j_1}(A_{k_1})b_{l_1}\cdots e_{i_{s'},j_{s'}}(A_{k_{s'}})b_{l_{s'}}e_{i_{s'+1},j}(A_{k'}) \mid 0 \leq s' \leq s\}.$$

Note that in the expression above, the indices j_0 and $i_{s'+1}$ run over the sets $\{1, \ldots, d_k\}$ and $\{1, \ldots, d_{k'}\}$ respectively, the indices i_p, j_p run over the set $\{1, \ldots, d_{k_p}\}$, $p = 1, \ldots, s'$, and l_ν, $\nu = 0, \ldots, s'$, runs over the set $\{1, \ldots, \dim_F J(A)\}$.

We denote by W the union of the sets $W_{i,j}(A_{k_1}, A_{k_2})$, $1 \leq k_1, k_2 \leq q$, $1 \leq i \leq d_{k_1}$, $1 \leq j \leq d_{k_2}$. Thus, **a basis for $\mathcal{A}$ is the set**

$$\{e_{j_1,j_2}(A_l) \,|\, 1 \leq l \leq q,\, 1 \leq j_1, j_2 \leq d_l\} \cup W.$$

Throughout this section we fix this basis.

2.1.1 Some reductions

Now we focus on decomposing the F-vector space $P_n(\mathcal{A})$ into a sum of smaller subspaces in such a way that on the one hand the number of summands is bounded above by a constant independent of n and on the other hand the dimensions of the subspaces are more tractable. This will happen in several stages, but first we pass over to the algebra of generic elements of $\mathcal{A}$.

More precisely, since $\mathcal{A}$ is a finite dimensional algebra, we can identify its relatively free algebra $F\langle X_i \mid i \in \mathbb{N}\rangle/\,\mathrm{Id}(\mathcal{A})$ with a subalgebra of

$$\mathcal{A}_K = \mathcal{A} \otimes_F K,$$

where K is the (commutative) polynomial ring

$$K = F\left[\theta^{(i)}_{j_1,j_2}(A_l), \theta^{(i)}(w) \,|\, i \in \mathbb{N},\, 1 \leq l \leq q,\, 1 \leq j_1, j_2 \leq d_l,\, w \in W\right].$$

This was first proven by Procesi in [Pro67] and we refer to [GZ05, Theorem 1.4.4] for a proof fitting in the notations of this thesis.

Let us make this identification explicit for the algebra $\mathcal{A}$ and its basis introduced before. Write,

$$X_i(A_k) = \sum_{a_1,a_2} \theta^{(i)}_{a_1,a_2}(A_k) e_{a_1,a_2}(A_k).$$

Then the variable $X_i + \mathrm{Id}(\mathcal{A})$ of the relatively free algebra of $\mathcal{A}$ is identified with

$$\sum_{k=1}^{q} X_i(A_k) + \sum_{w\in W} \underbrace{\theta^{(i)}(w)w}_{X_i(w)} = \sum_{\Sigma} X_i(\Sigma) \in \mathcal{A}_K,$$

where Σ is a symbol which runs over the set $\mathbf{Symb} = W \cup (\mathbf{SimComp} := \{A_1, \dots, A_q\})$. *As a result of this identification, the spaces* $P_n(\mathcal{A}) := \frac{P_n(A)}{P_n(A)\cap \mathrm{Id}(A)}$ *are viewed throughout this Chapter as subspaces of* $\mathcal{A}_K$*.*

We decompose $P_n(\mathcal{A})$ into subspaces as follows. Consider a monomial $X_{\sigma(1)} \cdots X_{\sigma(n)} \in P_n(\mathcal{A})$, where $\sigma \in S_n$. Clearly, by the identification have we just described, the corresponding monomial in $\mathcal{A}_K$ is equal to the sum

$$\sum_{\Sigma_1,\dots,\Sigma_n \in \mathbf{Symb}} X_{\sigma(1)}(\Sigma_1)\cdots X_{\sigma(n)}(\Sigma_n). \tag{2.1}$$

Note that

1. $X_i(A_k)X_j(A_{k'}) = 0$ if $k \neq k'$.

2. If more than s symbols from $\Sigma_1, \dots, \Sigma_n$ are radical (i.e. from W), then

$$X_{\sigma(1)}(\Sigma_1)\cdots X_{\sigma(n)}(\Sigma_n) = 0.$$

This leads to the following definition.

Definition 2.1.1. A sequence $\overrightarrow{p} = (p_1, \dots, p_n)$ of symbols in $\mathbf{Symb}$ is called a *path* (of length n) if the following two properties are satisfied:

1. If $p_i, p_{i+1} \in \mathbf{SimComp}$, then $p_i = p_{i+1}$.

2. Not more than s symbols (from $p_1, \dots, p_n$) are in W.

Furthermore, suppose

$$\overrightarrow{p} = (A_{k_1}, \dots, A_{k_1}, w_1, A_{k_2}, \dots, A_{k_2}, w_2, \dots, w_{s'}, A_{k_{s'+1}}, \dots, A_{k_{s'+1}}).$$

Then the *path structure* of $\overrightarrow{p}$ is defined to be the sequence

$$\mathbf{struc}(\overrightarrow{p}) := (A_{k_1}, w_1, A_{k_2}, w_2, \dots, w_{s'}, A_{k_{s'+1}})$$

(i.e. we record the simple components with no adjacent repetitions and the radical elements).

Due to the special role in the sequel of the path structure of a path, we take paths of the same length and with the same structure together.

Definition 2.1.2. Two paths $\overrightarrow{p_1}, \overrightarrow{p_2}$ of the same length are called *equivalent*, denoted by $\overrightarrow{p_1} \sim \overrightarrow{p_2}$, if they have the same path structure. (e.g. $(A_{k_1}, A_{k_1}, w_1, A_{k_2})$ and $(A_{k_1}, w_1, A_{k_2}, A_{k_2})$ are equivalent paths whereas $(A_{k_1}, A_{k_1}, w_1, A_{k_2})$ and (A_{k_1}, w_1, A_{k_2}) are not).

The set of all paths of length n is denoted by $\mathbf{Path}_n$ and the set of all equivalence classes of paths of length n is denoted by $\mathbf{Path}_n/\sim$.

Definition 2.1.3. For a given path $\overrightarrow{p} \in \mathbf{Path}_n$ we denote the number of appearances of a symbol Σ from **Symb** by $\overrightarrow{p}(\Sigma)$.

By definition, the expression in (2.1) can now be rewritten as

$$\sum_{\overrightarrow{p}=(p_1,\dots,p_n)\in\mathbf{Path}_n} X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n) = \sum_{[\overrightarrow{p_1}]\in\mathbf{Path}_n/\sim} \left(\sum_{\overrightarrow{p}=(p_1,\dots,p_n)\in[\overrightarrow{p_1}]} X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n) \right)$$

where we ignore some vanishing monomials.

Definition 2.1.4. For a fixed $\overrightarrow{p} \in \mathbf{Path}_n$ denote by $P_{\overrightarrow{p}}(\mathcal{A})$ the F-linear span of all monomials $X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$, where σ varies over S_n. Furthermore, $P_{[\overrightarrow{p_1}]}(\mathcal{A})$ denotes the sum of all $P_{\overrightarrow{p}}(\mathcal{A})$ such that $\overrightarrow{p} \sim \overrightarrow{p_1}$.

Lemma 2.1.5. *The space $P_n(\mathcal{A})$ is embedded in*

$$\bigoplus_{\overrightarrow{p}\in\mathbf{Path}_n} P_{\overrightarrow{p}}(\mathcal{A}) = \bigoplus_{[\overrightarrow{p}]\in\mathbf{Path}_n/\sim} P_{[\overrightarrow{p}]}(\mathcal{A}).$$

As a result,

$$c_n(A) \leq \sum_{[\overrightarrow{p}]\in\mathbf{Path}_n/\sim} \dim_F P_{[\overrightarrow{p}]}(\mathcal{A}).$$

Moreover, the size of the set $\mathbf{Path}_n/\sim$ is bounded by a constant independent of n.

Proof. Only the last part requires an explanation. For this note that the size of $\mathbf{Path}_n/\sim$ is bounded from above by the number of sequences of length at most $2s+1$ whose elements are taken from the finite set **Symb**. So the constant can be taken to be

$$\sum_{t=1}^{2s+1} |\mathbf{Symb}|^t.$$

∎

Remark 2.1.6. *By the previous Lemma, to prove the upper bound it is sufficient to show that* $\dim_F P_{[\overrightarrow{p}]}(\mathcal{A})$*,* $\overrightarrow{p} \in \mathbf{Path}_n$*, is bounded from above by* $Cn^{\frac{q-d}{2}+s}d^n$*, where* C *is some constant independent of* n*.*

We intend to decompose each $P_{[\overrightarrow{p}]}(\mathcal{A})$ into a (direct) sum of some special subspaces.

Definition 2.1.7. Let $\overrightarrow{p} = (p_1, \dots, p_n) \in \mathbf{Path}_n$ and let $Z = X_{\sigma(1)}(p_1)\cdots X_{\sigma(n)}(p_n)$ be a monomial in $P_{\overrightarrow{p}}(\mathcal{A})$ for some $\sigma \in S_n$. For $1 \le l \le q$ we denote by $\mathbf{ind}_l(Z)$ the set of all indices $\sigma(u)$ (here $1 \le u \le n$) for which $p_u = A_l$.

Furthermore, we denote by $\mathbf{seq}_{rad}(Z)$ the sequence of indices $(\sigma(i_1), \dots, \sigma(i_{s'}))$ for which

1. $p_{i_u} \in W$ for every $1 \le u \le s'$,
2. $i_1 < \cdots < i_{s'}$,
3. $\{\sigma(i_1), \dots, \sigma(i_{s'})\} \cup \mathbf{ind}_1(Z) \cup \cdots \cup \mathbf{ind}_q(Z) = \{1, \dots, n\}$ (that is $\mathbf{seq}_{rad}(Z)$ consists of all the indices whose corresponding variables take values in the radical).

Finally, we set $\overrightarrow{\text{ind}}(Z) = (\mathbf{ind}_1(Z), \dots, \mathbf{ind}_q(Z); \mathbf{seq}_{rad}(Z))$. For example for $Z = X_2(A_{k_1})X_5(w_1)X_1(A_{k_2})X_4(A_{k_2})X_3(w_2)X_6(A_{k_1})$, if $k_1 < k_2$, we have that $\overrightarrow{\text{ind}}(Z) = (\{2,6\}, \{1,4\}; 5, 3)$.

Definition 2.1.8. Two monomials Z_1 and Z_2 in $P_{[\overrightarrow{p}]}(\mathcal{A})$ are *equivalent* (or $Z_1 \sim Z_2$) if $\overrightarrow{\text{ind}}(Z_1) = \overrightarrow{\text{ind}}(Z_2)$. The set of all equivalence classes corresponding to this relation is denoted by $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$, where $\mathbf{Mon}_{[\overrightarrow{p}]}$ is the set of monomials in $P_{[\overrightarrow{p}]}(\mathcal{A})$. Furthermore, $P_{[Z]}(\mathcal{A}) (\subseteq P_{[\overrightarrow{p}]}(\mathcal{A}))$ denotes the F-span of all monomials in $P_{[\overrightarrow{p}]}(\mathcal{A})$ which are equivalent to Z.

To illustrate previous equivalence consider

$$Z_1 = X_2(A_{k_1})X_5(w_1)X_1(A_{k_2})X_4(A_{k_2})X_3(w_2)X_6(A_{k_1})$$

and $Z_2 = X_6(A_{k_1})X_5(w_1)X_4(A_{k_2})X_1(A_{k_2})X_3(w_2)X_2(A_{k_1})$, then $Z_1, Z_2 \in P_{[\overrightarrow{p}]}(\mathcal{A})$ with $\overrightarrow{p} = (A_{k_1}, w_1, A_{k_2}, w_2, A_{k_1})$. Furthermore, $\overrightarrow{\text{ind}}(Z_1) = \overrightarrow{\text{ind}}(Z_2) = (\{2,6\},\{1,4\};5,3)$, thus $Z_1 \sim Z_2$. On the other hand for example $Z_1 = X_2(A_{k_1})X_1(w_1)X_3(A_{k_2})$ is not equivalent to $Z_2 = X_3(A_{k_2})X_1(w_1)X_2(A_{k_1})$, even though $\overrightarrow{\text{ind}}(Z_1) = \overrightarrow{\text{ind}}(Z_2) = (\{2\},\{3\};1)$, since they have non-equivalent paths.

Lemma 2.1.9. *The following hold:*

1. *$P_{[\overrightarrow{p}]}(\mathcal{A})$ is equal to*

$$\bigoplus_{[Z]\in \mathbf{Mon}_{[\overrightarrow{p}]}/\sim} P_{[Z]}(\mathcal{A}).$$

2. *Denote by $\mathbf{Mon}_{[\overrightarrow{p}]}(n_1,\dots,n_q)/\sim$ the subset of $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$ consisting of all $[Z]$ for which the corresponding path $\overrightarrow{p}$ satisfies $n_1 = |\mathbf{ind}_1(Z)|,\dots,n_q = |\mathbf{ind}_q(Z)|$. Then, $\mathbf{Mon}_{[\overrightarrow{p}]}/\sim$ is equal to the (disjoint) union*

$$\bigcup_{n_1+\cdots+n_q=n-s'} \mathbf{Mon}_{[\overrightarrow{p}]}(n_1,\dots,n_q)/\sim,$$

where $s' = |\mathbf{seq}_{rad}(Z)|$ is the number of symbols from W in $\overrightarrow{p}$.

3. *The size of $\mathbf{Mon}_{[\overrightarrow{p}]}(n_1,\dots,n_q)/\sim$ is bounded from above by*

$$n^{s'}\cdot\binom{n-s'}{n_1,\dots,n_q}.$$

Proof. Only the third part requires a proof. There are $s'!\cdot\binom{n}{s'}$ options to choose and order s' indices from the set $\{1,\dots,n\}$, i.e. there are $s'!\cdot\binom{n}{s'}$ ways to choose $\mathbf{seq}_{rad}$ for a fixed $1\le s'\le s$. From the remaining $n-s'$ indices there are $\binom{n-s'}{n_1,\dots,n_q}$ options to choose n_1 which will correspond to A_1,..., n_q which will correspond to A_q. Finally, it is clear that

$$s'!\cdot\binom{n}{s'}\binom{n-s'}{n_1,\dots,n_q} \le n^{s'}\binom{n-s'}{n_1,\dots,n_q}.$$

∎

Definition 2.1.10. For $\overrightarrow{i} = (i_1,\dots,i_l)$ we denote the product $X_{i_1}(A_j)\cdots X_{i_l}(A_j)$ by $\mathbf{X}_{\overrightarrow{\mathbf{i}}}(A_j)$. Consider monomials in $P_{[Z]}(\mathcal{A})(\subseteq P_{[\overrightarrow{p}]}(\mathcal{A}))$ of the form

$$\mathbf{X}_{\overrightarrow{\mathbf{i}_1}}(A_{k_1})X_{\nu_1}(w_1)\mathbf{X}_{\overrightarrow{\mathbf{i}_2}}(A_{k_2})X_{\nu_2}(w_2)\cdots\mathbf{X}_{\overrightarrow{\mathbf{i}_{s'+1}}}(A_{k_{s'+1}}),$$

namely monomials which satisfy

1. $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \ldots, A_{k_{s'+1}})$,
2. $\bigcup\limits_{\alpha: k_\alpha = l} Set_{\overrightarrow{\mathbf{i}_\alpha}} = \mathbf{ind}_l(Z)$, where Set_x consists of all indices appearing in the vector x
3. $\mathbf{seq}_{rad}(Z) = (\nu_1, \ldots, \nu_{s'})$.

Remark 2.1.11. It is important to stress that there exist other types of monomials, namely monomials where some radical elements are adjacent or monomials which start or end by radical elements. As it will be clear below, the treatment of these monomials is similar to the monomials of the type considered in Definition 2.1.10.

We now make the last reduction. To this end consider the spaces

$$P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}) := e_{j_0}(A_{k_1}) P_{[Z]}(\mathcal{A}) e_{j_{s'+1}}(A_{k_{s'+1}}) (\subseteq \mathcal{A}_K),$$

where $1 \leq j_0 \leq d_{k_1}$ and $1 \leq j_{s'+1} \leq d_{k_{s'+1}}$ (recall that by $e_j(B)$ we denote the diagonal matrix $e_{j,j}$ in the matrix algebra B). Note that we only consider monomials as in definition 2.1.10, hence the simple components A_{k_1} and $A_{k_{s'+1}}$ are determined by the monomial Z and therefore we do not record the indices k_1 and $k_{s'+1}$ in the definition of $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$. Since any element f of $P_{[Z]}(\mathcal{A})$ can be written as the sum

$$f = 1(A_{k_1}) \cdot f \cdot 1(A_{k_{s'+1}}) = \sum_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} e_{j_0}(A_{k_1}) \cdot f \cdot e_{j_{s'+1}}(A_{k_{s'+1}}),$$

we obtain an injective map

$$P_{[Z]}(\mathcal{A}) \to \bigoplus_{\substack{1 \leq j_0 \leq d_{k_1} \\ 1 \leq j_{s'+1} \leq d_{k_{s'+1}}}} P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A}).$$

So we have proved

Lemma 2.1.12. $\dim_F P_{[Z]}(\mathcal{A}) \leq \sum\limits_{j_0, j_{s'+1}} \dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$.

As a result of this observation we also will fix the indices j_0, $j_{s'+1}$ and work in the space $P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$. In Lemma 2.1.13 we describe the asymptotics of $\dim_F P_{[Z]}^{j_0, j_{s'+1}}(\mathcal{A})$ and then, subsequently, we will sum up all the spaces of that form.

2.1.2 The key lemma and upper bound

In order to be able to carry out manipulations in the vector space $P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$ we introduce the following notation:

$$\theta^{X_1,\ldots,X_l}_{a_1,\ldots,a_{l+1}}(A_k) := \theta^{(X_1)}_{a_1,a_2}(A_k)\theta^{(X_2)}_{a_2,a_3}(A_k)\cdots\theta^{(X_l)}_{a_l,a_{l+1}}(A_k)$$

and

$$\theta^{X_1,\ldots,X_l}_{i|j}(A_k) := \sum_{a_2,\ldots,a_l} \theta^{X_1,\ldots,X_l}_{a_1=i,a_2,\ldots,a_l,a_{l+1}=j}(A_k).$$

Note that we have slightly changed the notation we introduced above by replacing $\theta^{(l)}_{a_k,a_r}$ with $\theta^{(X_l)}_{a_k,a_r}$. Furthermore, if $\overrightarrow{\nu} = (1,\ldots,l)$ we simply write

$$\theta^{X_{\overrightarrow{\nu}}}_{i|j}(A_k) = \theta^{X_1,\ldots,X_l}_{i|j}(A_k).$$

The next lemma is straightforward (proof is omitted).

Lemma 2.1.13. *The following statement hold.*

1. *For $w_1 \in W_{-,i}(A_-, A_k), w_2 \in W_{j,-}(A_k, A_-)$ we have*

$$w_1\mathbf{X}_{\overrightarrow{\nu}}(A_k)w_2 = \theta^{X_{\overrightarrow{\nu}}}_{i|j}(A_k)\cdot w_1 e_{i,j}(A_k)w_2.$$

2. *For $w_1 \in W_{j_1,\tilde{j}_1}(A_{k_1}, A_{k_2}),\ldots, w_{s'} \in W_{j_{s'},\tilde{j}_{s'}}(A_{k_{s'}}, A_{k_{s'+1}})$ we have that*

$$\underbrace{e_{\tilde{j}_0=j_0}(A_{k_1})}_{w_0}\left(\mathbf{X}_{\overrightarrow{\nu_1}}(A_{k_1})X_{i_1}(w_1)\mathbf{X}_{\overrightarrow{\nu_2}}(A_{k_2})X_{i_2}(w_2)\cdots\mathbf{X}_{\overrightarrow{\nu_{s'+1}}}(A_{k_{s'+1}})\right)\underbrace{e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})}_{w_{s'+1}}$$

equals

$$\theta^{(i_1)}(w_1)\cdots\theta^{(i_{s'})}(w_{s'})\left(\prod_{l=1}^{q}\prod_{\alpha:k_\alpha=l}\theta^{X_{\overrightarrow{\nu_\alpha}}}_{\tilde{j}_{\alpha-1}|j_\alpha}(A_l)\right)\cdot\mathbf{w},$$

where

$$\mathbf{w} = w_0 e_{\tilde{j}_0,j_1}(A_{k_1})w_1 e_{\tilde{j}_1,j_2}(A_{k_2})\cdots w_{s'}e_{\tilde{j}_{s'},j_{s'+1}}(A_{k_{s'+1}})w_{s'+1}.$$

Corollary 2.1.14. *There exists an element $w \in A$ such that for any monomial $Z' \in P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$ (and hence for any element of $P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$)*

$$Z' = f\cdot w,$$

where f is a polynomial in $F\left[\theta^{(i)}_{j_1,j_2}(A_l), \theta^{(i)}(w)\,|\, i\in\mathbb{N},\ 1\le l\le q,\ 1\le j_1,j_2\le d_l,\ w\in W\right]$.

We now turn to the construction of the map which enables us to estimate $\dim_F P_{[Z]}^{j_0,j_{s'+1}}$.

For every l, consider the variables $Y_{1,l},\dots,Y_{v_l-1,l}$, where v_l is the number of appearances of A_l in $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \dots, A_{k_{s'+1}})$.

Let $P_{X_{h_1},\dots,X_{h_{n_l}};Y_{1,l},\dots,Y_{v_l-1,l}}$ denote the $(n_l+v_l-1)!$ space of all multilinear polynomials in the prescribed variables, where $\mathbf{ind}_l(Z) = (h_1,\dots,h_{n_l})$ (i.e. the indices of variables which get values from A_l). Let

$$P_{X_{h_1},\dots,X_{h_{n_l}};Y_{1,l},\dots,Y_{v_l-1,l}}(A_l) = P_{X_{h_1},\dots,X_{h_{n_l}};Y_{1,l},\dots,Y_{v_l-1,l}}/(P_{X_{h_1},\dots,X_{h_{n_l}};Y_{1,l},\dots,Y_{v_l-1,l}} \cap Id(A_l)).$$

Denote

$$(U_1,\dots,U_{n_l+v_l-1}) = (X_{h_1},\dots,X_{h_{n_l}},Y_{1,l},\dots,Y_{v_l-1,l})$$

and let $P_{[\widehat{X_{A_l}}]}(A_l) = P_{[\widehat{X_{A_l}}]}/(P_{[\widehat{X_{A_l}}]} \cap Id(A_l))$ be the subspace of $P_{X_{h_1},\dots,X_{h_{n_l}};Y_{1,l},\dots,Y_{v_l-1,l}}(A_l)$ spanned by all monomials $U_{\tau(1)}U_{\tau(2)}\cdots U_{\tau(n_l+v_l-1)}$, where

1. $U_{\tau(1)}, U_{\tau(n_l+v_l-1)} \in \{X_{h_1},\dots,X_{h_{n_l}}\}$.
2. If $i<j$ and $U_{\tau(i)}, U_{\tau(j)} \in \{Y_{1,l},\dots,Y_{v_l-1,l}\}$, then there exists k with $i<k<j$ such that $U_{\tau(k)} \in \{X_{h_1},\dots,X_{h_{n_l}}\}$.
3. The ordering induced by τ on the set $\{Y_{1,l},\dots,Y_{v_l-1,l}\}$ is precisely the ordering $(Y_{1,l},\dots,Y_{v_l-1,l})$. That is, for $n_l+1 \le i,j \le n_l+v_l-1$ we have $i<j \implies \tau(i)<\tau(j)$.

Remark 2.1.15. We abuse notation and terminology here by considering monomials as elements of $P_{[\widehat{X_{A_l}}]}(A_l)$.

Notation 2.1.16. *We denote by* $c_{\widehat{X_{A_l}}}(A_l) = \dim_F P_{[\widehat{X_{A_l}}]}(A_l)$ *and note that* $c_{\widehat{X_{A_l}}}(A_l) \le c_{n_l+v_l-1}(A_l)$.

Now, for $l=1,\dots,q$, let

$$\widehat{X_{A_l}} = X_{\overrightarrow{\mu_{\alpha(1,l)}}} \cdot Y_{1,l} \cdot X_{\overrightarrow{\mu_{\alpha(2,l)}}} \cdot Y_{2,l} \cdots Y_{v_l-1,l} \cdot X_{\overrightarrow{\mu_{\alpha(v_l,l)}}},$$

be a monomial in $P_{[\widehat{X_{A_l}}]}(A_l)$ and let

$$X = (\widehat{X_{A_1}},\dots,\widehat{X_{A_q}}) \in P_{[\widehat{X_{A_1}}]} \times \cdots \times P_{[\widehat{X_{A_q}}]}.$$

Next consider the monomial

$$e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}}) \in P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}),$$

where

$$X(\mathcal{A}) = \mathbf{X}_{\overrightarrow{\nu_{\alpha(t_1,k_1)}}}(A_{k_1})X_{i_1}(w_1)\mathbf{X}_{\overrightarrow{\nu_{\alpha(t_2,k_2)}}}(A_{k_2})X_{i_2}(w_2)\cdots\mathbf{X}_{\overrightarrow{\nu_{\alpha(t_{s'+1},k_{s'+1})}}}(A_{k_{s'+1}})$$

and if $k_{g_1} = k_{g_2} = \ldots = k_{g_{v_l}} = l$ are the indices where the simple component A_l appears in $\mathbf{struc}(\overrightarrow{p}) = (A_{k_1}, w_1, A_{k_2}, w_2, \ldots, A_{k_{s'+1}})$, then

$$(\overrightarrow{\nu_{\alpha(t_{g_1},k_{g_1})}}, \ldots, \overrightarrow{\nu_{\alpha(t_{g_{v_l}},k_{g_{v_l}})}}) = (\overrightarrow{\mu_{\alpha(1,l)}}, \ldots, \overrightarrow{\mu_{\alpha(v_l,l)}}).$$

We now have all ingredients for formulating and proving the key lemma.

Lemma 2.1.17. *The following statements hold.*

1. *The map* $\Psi : P_{\widehat{[X_{A_1}]}} \times \cdots \times P_{\widehat{[X_{A_q}]}} \to P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A})$

$$X \mapsto e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})$$

is well defined, surjective and multilinear. Hence it determines a surjection

$$P_{\widehat{[X_{A_1}]}}(A_1) \otimes \cdots \otimes P_{\widehat{[X_{A_q}]}}(A_q) \to P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A}).$$

2. $\dim_F P^{j_0,j_{s'+1}}_{[Z]}(\mathcal{A}) \leq c_{n_1+s}(A_1)\cdots c_{n_q+s}(A_q) \leq C \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q)$*, where* C *is a constant which is independent of* $n_1, \ldots, n_q$*.*

Proof. Suppose f is a linear combination of monomials in $P_{\widehat{[X_{A_l}]}}(A_l)$ which represents the zero element. Clearly, f represents the zero map in $Hom(A_l^{\otimes(n_l+v_l-1)}, A_l)$ and hence, by evaluating $Y_{1,l}, \ldots, Y_{v_l-1,l}$ on A_l we obtain the zero map in $Hom(A_l^{\otimes(n_l)}, A_l)$. In particular we obtain zero if we evaluate

$$Y_{1,l} = e_{j_{g_1},\tilde{j}_{g_2}-1}(A_l), Y_{2,l} = e_{j_{g_2},\tilde{j}_{g_3}-1}(A_l), \ldots, Y_{v_l-1,l} = e_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}-1}(A_l).$$

But in view of the fact that

$$w_{g_1} \in W_{j_{g_1},\tilde{j}_{g_1}}(A_{k_{g_1}}, A_{k_{g_1+1}}), w_{g_2} \in W_{j_{g_2},\tilde{j}_{g_2}}(A_{k_{g_2}}, A_{k_{g_2+1}}), \ldots,$$

$$w_{g_{v_l}-1} \in W_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}}(A_{k_{g_{v_l}-1}}, A_{k_{g_{v_l}}}), w_{g_{v_l}-1} \in W_{j_{g_{v_l}-1},\tilde{j}_{g_{v_l}}}(A_{k_{g_{v_l}-1}}, A_{k_{g_{v_l}}}),$$

and by Lemma 2.1.13, we see that up to a scalar the bridge between the ith and the $(i+1)$th appearance of A_l in $X(\mathcal{A})$ is given precisely by the matrix $e_{j_{g_i},\tilde{j}_{g_{i+1}}-1}(A_l)$ and hence $e_{\tilde{j}_0=j_0}(A_{k_1})X(\mathcal{A})e_{\tilde{j}_{s'+1}=j_{s'+1}}(A_{k_{s'+1}})=0$. This shows the map Ψ is well defined. It is clear by construction that Ψ is multilinear and onto.

Let us prove the second part. Applying part (1) and the inequalities $c_{\widehat{X_{A_l}}}(A_l) \leq c_{n_l+v_l-1}(A_l)$ we have

$$\dim_F P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}) \leq c_{\widehat{X_{A_1}}}(A_1)\cdots c_{\widehat{X_{A_q}}}(A_q) \leq c_{n_l+v_l-1}(A_l)\cdots c_{n_l+v_l-1}(A_l).$$

Furthermore, since the number of Y's is bounded by s, and the sequence $c_n(A_l)$ is an eventually nondecreasing function in n [GZ14], we obtain

$$\dim_F P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}) \leq c_{n_1+s}(A_1)\cdots c_{n_q+s}(A_q).$$

The last inequality in the lemma follows from the fact that $c_n(B) \simeq \mathcal{O}(n^t d^n)$ for PI algebras B (as proved by Berele and Regev [BR08] for unital algebras and later by Giambruno and Zaicev [GZ14] for arbitrary algebras). Indeed, we have that

$$\lim_{n\to\infty}\frac{c_{n+s}(A_l)}{c_n(A_l)} = K_2$$

for some constant $K_2 \in \mathbb{R}$ and the result follows. This completes the proof of the lemma. ∎

As mentioned in Remark 2.1.11 other type of monomials Z should be considered. The proofs of the statements that correspond to Lemmas 2.1.12 - 2.1.17 are similar and therefore are left to the reader.

Theorem 2.1.18 (Upper bound)**.** *There is a constant C such that*

$$c_n(A) \leq C \cdot n^{\frac{q-d}{2}+s} d^n.$$

Proof. By part (5) of Lemma 2.1.13 it follows that

$$\dim_F P_{[Z]}^{j_0,j_{s'+1}}(\mathcal{A}) \leq C_1 \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q),$$

where $n_1,\ldots,n_q$ are determined by the path corresponding to $[Z]$. Combining this with Lemma 2.1.12 it gives

$$\dim_F P_{[Z]}(\mathcal{A}) \leq C_2 \cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q).$$

By Lemma 2.1.9 it follows that

$$\sum_{[Z]\in \mathbf{Mon}_{[\overrightarrow{p}]}(n_1,\ldots,n_q)/\sim} \dim_F P_{[Z]}(\mathcal{A}) \leq C_3 \cdot n^{s'}\binom{n-s'}{n_1,\ldots,n_q}\cdot c_{n_1}(A_1)\cdots c_{n_q}(A_q),$$

where $s' = n - n_1 - \cdots - n_q$. Thus,

$$\dim_F P_{[\overrightarrow{p}]}(\mathcal{A}) \leq C_4 \cdot n^{s'} \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q).$$

By Lemma 2.1.5 we obtain

$$c_n(A) \leq C_5 \cdot \sum_{s'=0}^{s} \left(n^{s'} \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q) \right).$$

Next, by [Reg84],

$$\sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} c_{n_1}(A_1)\cdots c_{n_q}(A_q) \leq$$
$$C_6 \cdot \sum_{n_1+\cdots+n_q=n-s'} \binom{n-s'}{n_1,\ldots,n_q} n_1^{\frac{1-d_1^2}{2}} d_1^{2n_1} \cdots n_q^{\frac{1-d_q^2}{2}} d_q^{2n_q},$$

for some constant C_6.

By a theorem of Regev and Beckner (see Theorem 2.1.19), this is asymptotically equal to

$$C_7 \cdot (n-s')^{\frac{q-d}{2}} d^n.$$

All in all we have

$$c_n(A) \leq C_8 \cdot \sum_{s'=0}^{s} \left(n^{s'} \cdot (n-s')^{\frac{q-d}{2}} d^n \right) \leq C \cdot n^{\frac{q-d}{2}+s} d^n$$

as desired. ∎

Theorem 2.1.19 (Regev and Beckner [BR98a]). *Let $r_1,\ldots,r_q,k_1,\ldots,k_q \in \mathbb{R}$ be such that $0 < k_1,\ldots,k_q$. Then,*

$$\sum_{n_1+\cdots+n_q=n} \binom{n}{n_1,\ldots,n_q} k_1^{n_1}\cdots k_q^{n_q} n_1^{r_1}\cdots n_q^{r_q} \simeq \left(\left(\frac{k_1}{k}\right)^{r_1} \cdots \left(\frac{k_q}{k}\right)^{r_q} \right) \cdot n^r k^n,$$

where $k = k_1 + \cdots + k_q$ and $r = r_1 + \cdots + r_q$.

2.2 Lower bound

For the lower bound we will work in the basic algebra A itself. As for the upper bound, we write $A \cong (A_{ss} = A_1 \oplus \cdots \oplus A_q) \oplus J$ and denote by d_i^2 the dimension of A_i, $i = 1, \ldots, q$. Furthermore, $d = d_1^2 + \cdots + d_q^2$ and $s = \mathrm{nildeg}(J(A)) - 1$.

Convention 2.2.1. *In the sequel, like we may do by linearity, all evaluations of multilinear polynomials we consider are from $A_1 \cup \cdots \cup A_q \cup J$.*

Since A is basic, by Theorem 1.4.3, it possesses a multilinear polynomial

$$f_0 = f_0\left(z_1, \ldots, z_q; B := B_1 \cup \cdots \cup B_s; E\right),$$

where

(1) $|B_1| = \cdots = |B_s| = d + 1$.

(2) f_0 alternates on each set B_i, $i = 1, \ldots, s$. Therefore, in any nonzero evaluation of the variables of f_0 exactly one variable of every B_i is evaluated by a radical element and the remaining variables (including the z's and the ones from E) by a semisimple element.

(3) There is a nonzero evaluation of the variables of f_0 such that $\tilde{z}_i = 1_{A_i}$ for $i = 1, \ldots, q$.

Throughout this section we fix a nonzero evaluation, denoted by

$$\tilde{f}_0 = f_0\left(\tilde{z}_1, \ldots, \tilde{z}_q; \tilde{B}; \tilde{E}\right),$$

which satisfies (3). Moreover, for $i = 1, \ldots, s$, we denote by $w_i \in B_i$ the variable such that $\tilde{w}_i \in J$.

Remark 2.2.2. In the sequel we will consider partial evaluations of multilinear polynomials. The properties of these evaluations rely on the existence of the evaluation $\tilde{f}_0$ of f_0.

Consider the multilinear polynomial

$$f_1 = f_1(z_1, \ldots, z_q; B; Y; E) = f_0(y_{1,1}y_{1,2}z_1y_{1,3}y_{1,4}, \cdots, y_{q,1}y_{q,2}z_qy_{q,3}y_{q,4}; B; E),$$

where $Y = \{y_{i,j} | \, i = 1, \ldots, q, \; j = 1, \ldots, 4\}$. We will abuse notation by omitting E, Y and B and thus simply writing $f_1(z_1, \ldots, z_q)$. Furthermore, we denote $B \cup Y \cup E$ by BYE. Note that since f_0 is a nonidentity of A, f_1 is a nonidentity as well.

Remark 2.2.3. In what follows, roughly speaking, we shall replace the variables $z_1, \dots, z_q$, with multilinear polynomials $g_1, \dots, g_q$. The polynomials $g_1, \dots, g_q$ will be elements in $F\langle x_1, \dots, x_n\rangle$, $n \in \mathbb{N}$, where different polynomials depend on disjoint sets of variables. This will give rise to a multilinear polynomial in the variables $\{x_1, \dots, x_n\} \cup BYE$.

For any $n \in \mathbb{N}$, let $X_n = \{x_1, \dots, x_n\}$ and let $X_n \cup BYE$ be the corresponding set of variables. Let $P_{X_n;BYE}$ be the F-space of all multilinear polynomials on $X_n \cup BYE$. The symmetric group S_n acts on $P_{X_n;BYE}$ by

$$\sigma \cdot f(x_1, \dots, x_n; BYE) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}; BYE)$$

and, as usual, this action induces an S_n-module structure on the space

$$P_{X_n;BYE}(A) = \frac{P_{X_n;BYE}}{P_{X_n;BYE} \cap id(A)}.$$

Now, consider a partition $\mathbf{p}$ of the set X_n into q subsets, denoted by $X[A_1], \dots, X[A_q]$, where each $X[A_i]$ is nonempty (we are interested in $n \to \infty$, so we may assume that $n \geq q$).

Consider the symmetric groups $S_{X[A_1]}, \dots, S_{X[A_q]}$ and their direct product $S_{\boldsymbol{p}} = S_{X[A_1]} \times \cdots \times S_{X[A_q]} \leq S_n$. Clearly, by restriction, we obtain $S_{\boldsymbol{p}}$-module structures on $P_{X_n;BYE}$ and consequently on $P_{X_n;BYE}(A)$.

Similar to the upper bound, we embed the relatively free algebra of A in

$$U_A = A \otimes_F F(\theta_{i,j} \,:\, i \in \mathbb{N},\, j = 1, \dots, \dim_F A).$$

Due to this we may and will view the space $P_{X_n;BYE}(A)$ as a subspace of U_A. This allows us to consider partial evaluation which is a key idea in this section.

Definition 2.2.4. For $\tilde{\mathbf{j}}, \mathbf{j} \in \{1, \dots, d_1\} \times \cdots \times \{1, \dots, d_q\}$, let $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A) \subseteq U_A$ be the space obtained from $P_{X_{\mathbf{n}};BYE}(A)$ by performing the following evaluations on some of the variables:

1. $y_{i,2} \to \bar{y}_{i,2} = e_{\tilde{j}_i}(A_i)$ and $y_{i,3} \to \bar{y}_{i,3} = e_{j_i}(A_i)$ for $i = 1, \dots, q$.
2. For $i = 1, \dots, s$, and for any $w \in B_i \setminus \{w_i\}$ we replace w by its value $\tilde{w}$. Note that $\tilde{w}$ is a semisimple element.

For $g \in P_{X_n;BYE}(A)$ we denote by $\bar{g}$ its image in $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$.

Notation 2.2.5. *In order to simplify our notation for elements in $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ we shall write $\overline{f(x_1,\dots,x_n)}$ if the variables BYE do not play a role in our variable manipulations. In Lemma 2.2.11 we will need to manipulate the variables of X_n and B and so we will write $\overline{f(x_1,\dots,x_n;w_1,\dots,w_s)}$.*

Observe that

$$\dim_F P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A) \le \dim_F P_{X_n;BYE}(A).$$

Therefore, for the lower bound, it will be sufficient to bound from below one of the spaces $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$.

The elements of $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$, as the elements of $P_{X_n;BYE}(A)$, will be referred to as *multilinear polynomials.*

As for the elements of $P_{X_{\mathbf{n}};BYE}(A)$, we perform the partial evaluations 1 and 2 in Definition 2.2.4 also on the polynomial

$$f_1 = f_1(z_1,\dots,z_q;B;Y;E) = f_0(y_{1,1}y_{1,2}z_1y_{1,3}y_{1.4},\dots,y_{q,1}y_{q,2}z_qy_{q,3}y_{q,4};B;E),$$

and denote the obtained polynomial by $\overline{f_1}$.

Lemma 2.2.6. *For any nonzero evaluation of $\overline{f_1}$ the following hold.*

1. *The variables $\{z_1,\dots,z_q\}\cup Y\cup E$ are all evaluated by semisimple elements.*
2. *Each $w_i\in B_i$, $i=1,\dots,s$, is evaluated by a radical element (note the other elements of B_i have already been replaced by semisimple elements).*
3. *For every $i=1,\dots,q$, the variable z_i is evaluated by an element of A_i.*

Moreover, any nonzero evaluation $\bar{g}_i$ of a polynomial g_i by elements of A_i, $i=1,\dots,q$, such that $e_{j_i}\bar{g}_ie_{\tilde{j}_i}\neq 0$ may be extended to a nonzero evaluation of $\overline{f_{\mathbf{1}}(g_1,\dots,g_q)}$.

Proof. Parts (1) and (2) are clear. Part (3) now follows, since $\bar{y}_{i,2}$ (and $\bar{y}_{i,3}$) is an element from A_i and z_i must be a semisimple element (by (1)).

We turn to the last part of the Lemma. Since $\bar{y}_{i,2}\bar{g}_i\bar{y}_{i,3}\neq 0$, we can find suitable evaluations of $y_{i,1}$ and $y_{i,4}$ so that the expression $y_{i,1}\bar{y}_{i,2}\bar{g}_i\bar{y}_{i,3}y_{i,4}$ is evaluated by any element of A_i. We are done since there is a nonzero evaluation of f_0 were each z_i (as variable of f_0) is evaluated by an element of A_i. ∎

Lemma 2.2.7. *Let V and W be finite dimensional vector spaces and let U be a subspace of $Hom_F(V,W)$. Fix a basis $w_1,\dots,w_{\dim_F W}$ of W. Then there is an element ψ in the dual basis of W such that*

$$\dim_F(\psi\circ U)\geq \frac{\dim_F U}{\dim_F W},$$

where $\psi\circ U$ is the space of all elements $\psi\circ T$ with $T\in U$.

Proof. It is clear that the map $\Psi : Hom_F(V,W)\to V^*\oplus\cdots\oplus V^*$ ($\dim_F W$ times) given by

$$\Psi(T)=(\psi_1\circ T,\dots,\psi_{\dim_F W}\circ T)$$

is injective (in fact, it is an isomorphism), where $\psi_1,\dots,\psi_{\dim_F W}$ is any basis of W^* (in our case it is the dual basis of $(w_1,\dots,w_{\dim_F W})$). As a result,

$$\dim_F U\leq \sum_{i=1}^{\dim_F W}\dim_F \psi_i\circ U.$$

So, there is some i_0 such that

$$\dim_F U\leq \dim_F W\cdot \dim_F \psi_{i_0}\circ U.$$

Hence $\psi=\psi_{i_0}$ is the required dual basis element. ∎

Next, consider the spaces $T_{X[A_i]}(A_i)=Hom_F(A_i^{\otimes n_i},A_i)$ where $i=1,\dots,q$ (n_i is the number of elements of $X[A_i]$ in the partition of n). Furthermore, we may view the space $P_{X[A_i]}(A_i)$ as a subspace of $T_{X[A_i]}(A_i)$ via the embedding $g\to\eta(g)$ which is determined by

$$\eta(g)(a_1\otimes\cdots\otimes a_{n_i})=g(a_1,\dots,a_{n_i}),(a_1,\dots,a_{n_i})\in A_i^{\otimes n_i}.$$

We apply the above lemma in the following setup:

$$V=A_i^{\otimes n_i}, W=A_i \text{ and } U=P_{X[A_i]}(A_i)\subseteq T_{X[A_i]}(A_i).$$

We use the basis of elementary matrices as a basis of W.

Now, for $i=1,\dots,q$, we denote by $\psi_{\tilde{j}_i,j_i}$ the element in the dual basis of $W=A_i$ as given by Lemma 2.2.7. Note that $\psi_{\tilde{j}_i,j_i}:A_i\to F$ is the map assigning to each matrix of A_i its $e_{\tilde{j}_i,j_i}$ coefficient. For the given $\mathbf{j}=(j_1,\dots,j_q)$ and $\tilde{\mathbf{j}}=(\tilde{j}_1,\dots,\tilde{j}_q)$ we simplify our notation and denote the space $P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ by $\bar{P}_{X_n;BYE}(A)$.

Theorem 2.2.8. *The mapping*

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \cdots \otimes P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A)$$

given by

$$\phi_{\mathbf{p}}(g_1 \otimes \cdots \otimes g_q) = \overline{f_1(g_1, \ldots, g_q)},$$

is well defined.

Moreover, if we denote by $M(\mathbf{p})$ the image of $\phi_{\mathbf{p}}$, then

$$\dim_F M(\mathbf{p}) \geq \frac{1}{d_1^2 \cdots d_q^2} \cdot c_{n_1}(A_1) \cdots c_{n_q}(A_q),$$

where $n_1 = |X[A_1]|, \ldots, n_q = |X[A_q]|$.

Proof. It is convenient to introduce the following auxiliary spaces which we denote by

$$\bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A).$$

As for the construction of $\bar{P}_{X_n;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ above, the space

$$\bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A),$$

$i = 1, \ldots, q$, is obtained from the space

$$P_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A)$$

by performing evaluations 1 and 2 of Definition 2.2.4.

Note that

$$\overline{f_1|_{z_i \to g_i}} \in \bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A).$$

Furthermore the map $\phi_i : P_{X[A_i]}(A_i) \to \bar{P}_{z_1,\ldots,z_{i-1},X[A_i],z_{i+1},\ldots,z_q;BYE}(A)$ given by $\phi_i(g_i) = \overline{f_1|_{z_i \to g_i}}$ is well defined, since in any nonzero evaluation of $\overline{f_1}$ all variables of $X[A_i]$ must be evaluated by elements of A_i (see Lemma 2.2.6), so for $g_i \in Id(A_i)$ there is no nonzero evaluation of $\overline{f_1|_{z_i \to g_i}}$. In other words,

$$g_i \in id(A_i) \Longrightarrow \phi_i(g_i) = 0.$$

The same argument shows that the map

$$\phi'_{\mathbf{p}} : P_{X[A_1]}(A_1) \times \cdots \times P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A)$$

given by

$$\phi'_{\mathbf{p}}(g_1,\ldots,g_q) = \overline{f_1(g_1,\ldots,g_q)}$$

is well defined. Since $\phi'_{\mathbf{p}}$ is multilinear, it induces the map

$$\phi_{\mathbf{p}} : P_{X[A_1]}(A_1) \otimes \cdots \otimes P_{X[A_q]}(A_q) \to \bar{P}_{X_n;BYE}(A).$$

Suppose $\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,1)},\ldots,\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,t_i)} \in \left(A_i^{\otimes n_i}\right)^*$ is a basis of $\psi_{\tilde{j}_i,j_i} \circ P_{X[A_i]}(A_i)$, $i = 1,\ldots,q$.

Applying Lemma 2.2.7 it is clear now that, in order to complete the proof, it is enough to prove that $\phi_{\mathbf{p}}$ is injective when restricted to the subspace T spanned by $g_{(A_1,\alpha_1)} \otimes \cdots \otimes g_{(A_q,\alpha_q)}$, where $\alpha_1 = 1,\ldots,t_1;\ldots;\alpha_q = 1,\ldots,t_q$.

Suppose there are scalars $c_{\alpha_1,\ldots,\alpha_q} \in F$ such that

$$0 = \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \phi_{\mathbf{p}}(g_{(A_1,\alpha_1)} \otimes \cdots \otimes g_{(A_q,\alpha_q)}) = \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \overline{f_1\left(g_{(A_1,\alpha_1)},\ldots,g_{(A_q,\alpha_q)}\right)}.$$

For $i = 1,\ldots,q$, let $\mathbf{a}_1^{(A_i)},\ldots,\mathbf{a}_{t_i}^{(A_i)} \in A_i^{\otimes n_i}$ be a dual basis of $\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,1)},\ldots,\psi_{\tilde{j}_i,j_i} \circ g_{(A_i,t_i)} \in \left(A_i^{\otimes n_i}\right)^*$. Write $\mathbf{a}_1^{(A_1)} = \sum_l a_{1,l} \otimes \cdots \otimes a_{n_1,l}$. Recall the action of $g_{(A_1,\alpha_1)}$ on $\mathbf{a}_1^{(A_i)}$ is determined linearly via the action on $a_{1,l} \otimes \cdots \otimes a_{n_1,l}$ and the latter is determined via the substitutions $x_1 \to a_1(l),\ldots,x_{n_1} \to a_{n_1}(l)$, where (without loss of generality) $X[A_1] = \{x_1,\ldots,x_{n_1}\}$.

Recall that $\overline{f_1\left(g_{(A_1,\alpha_1)},\ldots,g_{(A_q,\alpha_q)}\right)} \in \bar{P}_{X_n;BYE}(A) = P^{\tilde{\mathbf{j}},\mathbf{j}}_{X_n;BYE}(A)$ and $\tilde{\mathbf{j}},\mathbf{j}$ were chosen in Lemma 2.2.7. Hence we have,

$$\begin{aligned} 0 &= \sum_{\alpha_1,\ldots,\alpha_q} c_{\alpha_1,\ldots,\alpha_q} \overline{f_1\left(g_{(A_1,\alpha_1)}(\mathbf{a}_1^{(A_1)}),g_{(A_2,\alpha_2)},\ldots,g_{(A_q,\alpha_q)}\right)} \\ &= \sum_{\alpha_1=1,\alpha_2,\ldots,\alpha_q} c_{\alpha_1=1,\alpha_2,\ldots,\alpha_q} \overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1),g_{(A_2,\alpha_2)},\ldots,g_{(A_q,\alpha_q)}\right)}. \end{aligned}$$

By considering $\mathbf{a}_1^{(A_2)},\ldots,\mathbf{a}_1^{(A_q)}$ and applying the same argument on the last expression we conclude that

$$c_{1,\ldots,1} \cdot \overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1),\ldots,e_{\tilde{j}_q,j_q}(A_q)\right)} = 0.$$

By Lemma 2.2.6, it follows that $\overline{f_1\left(e_{\tilde{j}_1,j_1}(A_1),\ldots,e_{\tilde{j}_q,j_q}(A_q)\right)} \neq 0$, hence $c_{1,\ldots,1} = 0$.

It is clear that the same argument will work for every $\alpha_1, \ldots, \alpha_q$, thus every $c_{\alpha_1,\ldots,\alpha_q} = 0$. ∎

Next we study the connection between the different $M(\mathbf{p})$.

Lemma 2.2.9. *Fix some* $\mathbf{p}_0 = (X[A_1], \ldots, X[A_q])$ *and denote* $\overrightarrow{n} = (n_1, \ldots, n_q)$*, where* n_i *is the number of elements in* $X[A_i]$*. Let* $\mathcal{T} = \{e = \tau_1, \tau_2, \ldots, \tau_l\}$ *be a transversal of* $S_{\mathbf{p}_0}$ *in* S_n*. Then, the sum of vector spaces*

$$M_{\mathbf{tot}}(\overrightarrow{n}) := M(\tau_1\mathbf{p}_0) + \cdots + M(\tau_l\mathbf{p}_0),$$

is direct.

Note that $M_{\mathbf{tot}}(\overrightarrow{n}) = \overline{FS_{\mathbf{p}_0} \cdot f_1(x_1, \ldots, x_{n_1}, \ldots, x_{n_1+\cdots+n_{q-1}+1}, \ldots, x_{n_1+\cdots+n_q})}$.

Proof. Suppose

$$0 = \sum_{k=1}^{l} \alpha_k h_k,$$

where $0 \neq h_k \in M(\tau_k\mathbf{p}_0)$ and $\alpha_k \in F$. Here, $l = \frac{n!}{n_1!\cdots n_q!} = \binom{n}{n_1,\ldots,n_q}$.

Since

$$h_1 = \overline{\sum_{\sigma\in S_{p_0}} \beta_\sigma f_1(x_{\sigma(1)} \cdots x_{\sigma(n_1)}, \ldots, x_{\sigma(n_1+\cdots+n_{q-1}+1)} \cdots x_{\sigma(n)})} \neq 0$$

we obtain by Lemma 2.2.6 a nonzero evaluation $\xi : F\{X_n, BYE\} \to A$ which maps the variables of $X[A_i]$ to elements of A_i (for $i = 1, \ldots, q$). We claim this evaluation maps each h_k ($k \neq 1$) to zero. Indeed, at least one of the sets $\tau_k X[A_1], \ldots, \tau_k X[A_q]$ must have an element $x \in \tau_{i_0} X[A_{i_0}]$ which is mapped to some A_i, $i \neq i_0$ and hence

$$h_k = \overline{\sum_{\sigma\in S_{p_0}} \beta_{\tau_k\sigma} f_1(x_{\tau_k\sigma(1)} \cdots x_{\tau_k\sigma(n_1)}, \ldots, x_{\tau_k\sigma(n_1+\cdots+n_{q-1}+1)} \cdots x_{\tau_k\sigma(n)})}$$

is mapped to zero. It follows that $0 = \alpha_1 \cdot \xi(h_1)$ and hence $\alpha_1 = 0$. Repeating this argument for $h_2, \ldots, h_l$ yields $\alpha_2 = \alpha_3 = \cdots = \alpha_l = 0$. ∎

Write

$$M_{\mathbf{tot}}(n) := \sum_{n_1+\cdots+n_q=n} M_{\mathbf{tot}}\left(\overrightarrow{n} = (n_1, \ldots, n_q)\right).$$

Note that, in view of our notation above for $M_{\mathbf{tot}}(\overrightarrow{n})$, we have

$$M_{\mathbf{tot}}(n) = \overline{FS_n \cdot f_1(x_1, \ldots, x_{n_1}, \ldots, x_{n_1+\cdots+n_{q-1}+1}, \ldots, x_{n_1+\cdots+n_q})}.$$

Using a similar argument as in the previous lemma we obtain the following (a proof is omitted).

Lemma 2.2.10. *For every n, the above sum in $M_{\mathbf{tot}}(n)$ is direct.*

Consider now the group $S_{X_n \cup \{w_1,\dots,w_s\}}$ (see the beginning of the section for the definition of $w_1,\dots,w_s$) and let $\{e = \sigma_1, \sigma_2, \dots, \sigma_u\}$ be a transversal of S_{X_n} inside the aforementioned group. Note that $u = \frac{(n+s)!}{n!} = \mathcal{O}(n^s)$.

Lemma 2.2.11. *The sum*

$$M_{\mathbf{total}}(n) = \sum_{k=1}^{u} \sigma_k M_{\mathbf{tot}}(n)$$

is direct, where

$$\sigma_k M_{\mathbf{tot}}(n) = \overline{\sigma_k F S_n \cdot f_1(x_1,\dots,x_{n_1},\dots,x_{n_1+\cdots+n_{q-1}+1},\dots,x_{n_1+\cdots+n_q}; w_1,\dots,w_s)}.$$

Proof. The idea used in the proof of Lemma 2.2.9 works also here. Suppose

$$0 = \sum_{k=1}^{u} \alpha_k h_k,$$

where $0 \neq h_k \in \sigma_k M_{\mathbf{tot}}(n)$ and $\alpha_k \in F$. Since $h_1 \neq 0$, we obtain by Lemma 2.2.6, a nonzero evaluation $\xi : F\{X_n, BYE\} \to A$ which maps the variables in X_n to elements of A_{ss} and $w_1,\dots,w_q$ to elements of J. This evaluation maps each h_k $(k \neq 1)$ to zero. Indeed, $\sigma_k B_1,\dots,\sigma_k B_s$ are alternating sets of cardinality $d+1$ in h_k. However, there is some i_0 for which $\sigma_k B_{i_0}$ does not contain any of the variables $w_1,\dots,w_s$. Hence $\xi(\sigma_k B_{i_0}) \subseteq A_{ss}$. As a result, $\xi(h_k) = 0$. Thus, $0 = \alpha_1 \xi(h_1)$ and so $\alpha_1 = 0$.

Repeating this argument to $h_2,\dots,h_l$ yields $\alpha_2 = \cdots = \alpha_l = 0$. ∎

Now we show that asymptotically $\dim_F M_{\mathbf{tot}}(n)$ has the desired lower bound. Recall that $c_{n_i}(A_i) \simeq K_i n^{\frac{1-d_i^2}{2}} d_i^{2n}$ for some constant $K_i \in \mathbb{R}$ [Reg84]. As a result of Lemma 2.2.9, Lemma 2.2.10 and Lemma 2.2.11 we get asymptotically the inequality

$$\sum_{n_1+\ldots+n_q=n} \frac{K_1\cdots K_q}{d_1^2\cdots d_q^2} \cdot \binom{n}{n_1,\dots,n_q} \cdot n_1^{\frac{1-d_1^2}{2}} \cdots n_q^{\frac{1-d_q^2}{2}} \cdot d_1^{2n_1}\cdots d_q^{2n_q} \cdot n^s \lesssim \dim_F M_{\mathbf{total}}(n).$$

Finally we apply Theorem 2.1.19 and obtain

$$\dim_F M_{\mathbf{total}}(n) \geq C n^{\frac{q-d}{2}+s} d^n$$

for some nonnegative constant C.

Corollary 2.2.12 (Lower bound)**.** *Let $A = A_1 \oplus \cdots \oplus A_q \oplus J(A)$ be a basic algebra of Kemer index $\kappa_A = (d, s)$. Then*

$$c_n(A) \geq Cn^{\frac{q-d}{2}+s}d^n.$$

for some constant $C \in \mathbb{R}$.

Proof. It is enough to show that

$$\dim_F c_{n+\gamma}(A) \geq \dim_F M_{\mathbf{total}}(n),$$

for some γ independent of n. Indeed,

$$M_{\mathbf{total}}(n) \subseteq \overline{P_{X_n;BYE}(A)}$$

and $\overline{P_{X_n;BYE}(A)}$ is a projection of $P_{X_n;BYE}(A)$. Hence, the statement stands for $\gamma = |BYE|$. ∎

So combined with Theorem 2.1.18 we finally get a positive answer for Giambruno's conjecture.

Theorem 2.2.13. *Let $A = A_1 \times \cdots \times A_q \oplus J(A)$ be a basic algebra of Kemer index $\kappa_A = (d, s)$. Then*

$$c_n(A) = \mathcal{O}(n^{\frac{q-d}{2}+s}d^n).$$

In the special case where the algebra A has a unit we have

$$c_n(A) \simeq Cn^{\frac{q-d}{2}+s}d^n.$$

for some constant $0 < C \in \mathbb{R}$.

3

Codimension theory for rings

The Noblest pleasure is the joy of
understanding.
Leonardo da Vinci

As shown in the previous chapters, the asymptotics of the codimension sequence for an algebra over a field of characteristic zero has been intensively and successfully investigated with the proof of Amitsur's conjecture as one of the main milestones. However, the theory is based on the one hand on the fact that $P_n(F)/P_n(F) \cap \mathrm{Id}(A)$ is an FS_n-module over a field of characteristic zero, and thus can be decomposed into simple Specht modules, and on the other hand on the decomposition of Wedderburn-Malcev, both facts that completely disappear if one is not working over a 'nice field'. In particular, over $\mathbb{Z}$ or $\mathbb{F}_p$ the theory does not work. In this chapter we address the former case, or more precisely we investigate what happens if merely the ring structure of A and the corresponding polynomials which have coefficients in $\mathbb{Z}$ are taken into account. The investigation of polynomial identities with integer coefficients was initiated by W. Specht [Spe50] who proved, among others, that over $\mathbb{Z}$ all polynomial identities are still consequences of multilinear polynomial identities with integral coefficients. Thus, from now on we will only consider $P_n(\mathbb{Z}) \cap \mathrm{Id}(R)$ or rather $P_n(\mathbb{Z})/P_n(\mathbb{Z}) \cap \mathrm{Id}(R)$ for a ring R. As explained in Section 1.1.2, the latter is a finitely generated $\mathbb{Z}S_n$-module and in particular a finitely generated abelian group. Therefore

$$\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R)} \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{c_n(R,\mathbb{Z},0)} \oplus \bigoplus_{\substack{p \text{ is a prime} \\ \text{number}}} \bigoplus_{k \in \mathbb{N}} \Big(\underbrace{\mathbb{Z}_{p^k} \oplus \cdots \oplus \mathbb{Z}_{p^k}}_{c_n(R,\mathbb{Z},p^k)} \Big).$$

Throughout the chapter we work with rings, so from now on we write $c_n(R, p^k)$ and $c_n(R, 0)$ instead of $c_n(R, \mathbb{Z}, p^k)$ and $c_n(R, \mathbb{Z}, 0)$. This chapter is based on our joint work with Alexey Gordienko [GJ13] where we introduced these numbers for rings.

The first natural question which arises is how the usual F-codimensions $c_n(A, F)$ of an F-algebra A relate to its $\mathbb{Z}$-codimensions $c_n(A, q)$ and, vice versa, how the $\mathbb{Z}$-codimensions of a ring R relate to the F-codimensions of the F-algebra $R \otimes_{\mathbb{Z}} F$. The former question has been solved in Propositions 1.1.11, 1.1.12 and 1.1.15. In Theorem 3.1.1 we answer the latter question. The answer to this will moreover enable us to deduce a positive answer to an analogue of Amitsur and Regev's conjecture for $\mathbb{Z}$-codimensions for (additive) torsion-free rings. Recall that an element $r \in R$ is called torsion if there exists a positive integer m such that $mr = 0$ and we denote by $\operatorname{Tor}(R) := \{r \in R \mid mr = 0 \text{ for some } m \in \mathbb{N}\}$ the ideal of R consisting of the additive torsion of R.

It is not hard to see that in case R is torsion-free, i.e. $\operatorname{Tor}(R) = \{0\}$, only the codimension $c_n(R, 0)$ is non-zero and moreover in Theorem 3.1.1 we prove that $c_n(R, 0) = c_n(R/\operatorname{Tor}(R), 0) = c_n(R \otimes_{\mathbb{Z}} F, F)$ for a field of characteristic zero.

Consequently, for general rings R we will not be able to pass to some algebra A and use its rich theory on codimensions. One will rather have to understand in a precise way the impact of the torsion in R on $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$. Therefore, in the next sections we investigate the $\mathbb{Z}S_n$-module structure of it. Unfortunately, as mentioned in the previous chapter, it is no longer decomposable into simple $\mathbb{Z}S_n$-modules. However, it might still have a 'sufficiently nice' filtration of $\mathbb{Z}S_n$-submodules which would find its origins in the Specht Series of $\mathbb{Z}S_n$ as discussed in Section 1.2.5. In general we put and investigate the following question.

Question 3.0.1. *Let R be a ring. Does $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ have a chain of submodules with factors that are isomorphic to $S(\lambda)/mS(\lambda)$, where λ is a partition of n and m is an integer connected to the torsion of R?*

We call a series such as in Question 3.0.1 a generalized Specht Series. We do not know yet if this could be expected in general since not all submodules of $S(\lambda)$ are of the form $mS(\lambda)$ for some $m \in \mathbb{Z}$ (as is already shown for $S((2, 1))$).

In a first step towards Question 3.0.1, in the case R is unital, we relate in Section 3.2 the ordinary and proper codimensions and moreover prove the existence of a chain of $\mathbb{Z}S_n$-modules

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R, \mathbb{Z})} \supsetneq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(R, \mathbb{Z})},$$

where the quotient M_t/M_{t+1} is the induction of $\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ as $\mathbb{Z}(S_t \times S_{n-t})$-module to $\mathbb{Z}S_n$, where S_{n-t} is acting trivially.

Therefore, we are left with the problems to describe $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ and how to compute these consecutive inductions afterwards. Unfortunately, we are not able to answer the former question. However, in case $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})} \cong S(\lambda)/mS(\lambda)$ we prove in Theorem 3.3.1 a generalized version of Young's rule 1.2.31 which, in this case, solves the latter problem.

Finally, in the case R is a generalized upper-triangular matrix ring or an infinitely generated Grassmann algebra, we prove in Sections 3.4 and 3.5 that indeed $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ is isomorphic to $S(\lambda)/mS(\lambda)$ for some partition λ of n and $m \in \mathbb{Z}$ related to the torsion in R. In particular, in combination with all the previous results we confirm in Theorem 3.4.8 and 3.5.4 Question 3.0.1 for these cases.

In this chapter all Specht modules will be over $\mathbb{Z}$ and therefore we will no longer emphasize it and write $S(\lambda)$ instead of $S^{\mathbb{Z}}(\lambda)$.

3.1 Codimensions of Torsion-free rings

In this section we will prove a $\mathbb{Z}$-version of Regev's Conjecture, as formulated in [GJ13], in case R is a unital torsion-free ring. If R is non-unital, a weakened version of the $\mathbb{Z}$-version of Regev's Conjecture is obtained.

Conjecture (Regev's conjecture for rings)**.**
Let R be a ring. Then, for all prime powers $q = p^k$ there exist constants $C_q > 0, t_q \in \frac{\mathbb{Z}}{2}$ and $d_q \in \mathbb{N}$ such that

$$c_n(R, \mathbb{Z}, q) \simeq C_q n^{t_q} (d_q)^n.$$

In case R is torsion-free we will be able to do an extension of scalars of R to get some algebra and keep the codimensions invariant. Using the known results for algebras over fields of characteristic zero will then yield the result. Therefore we first investigate for a general ring what is the exact influence of extension of scalars. Recall from Section 1.1.2 that in case of an F-algebra A there is only one codimension sequence, namely $c_n(A, F) := c_n(A, F, 0)$, which may be non-zero.

Theorem 3.1.1. *Let R be a ring and let F be a field. Then*

$$c_n(R \otimes_{\mathbb{Z}} F, F) = \begin{cases} c_n(R/\mathrm{Tor}(R), \mathbb{Z}, 0) & \text{if } \operatorname{char} F = 0, \\ c_n(R/pR, \mathbb{Z}, p) & \text{if } \operatorname{char} F = p. \end{cases}$$

The following lemma is well known, but for completeness sake we add a proof.

Lemma 3.1.2. *Let R be a ring and let F be a field. Then,*

$$R \otimes_{\mathbb{Z}} 1_F \cong \begin{cases} R/\mathrm{Tor}(R) & \text{if } \operatorname{char} F = 0, \\ R/pR & \text{if } \operatorname{char} F = p \end{cases}$$

as rings, where $R \otimes_{\mathbb{Z}} 1_F \subseteq R \otimes_{\mathbb{Z}} F$ is a subring.

Proof. Consider the natural homomorphism $\varphi\colon R \to R \otimes_{\mathbb{Z}} 1_F$ where $\varphi(a) = a \otimes 1_F$, $a \in R$. Since $1_F = 1_K$ with K the prime field of F, it is enough to prove the lemma in the case $F = \mathbb{Q}$ or $F = \mathbb{F}_p$.

Suppose $F = \mathbb{Q}$. If $ma = 0$ for some non-zero $m \in \mathbb{N}$ and $a \in R$, then $\varphi(a) = a \otimes 1_{\mathbb{Q}} = ma \otimes \frac{1_{\mathbb{Q}}}{m} = 0$. Hence $\mathrm{Tor}(R) \subseteq \ker \varphi$. We claim that $\ker \varphi = \mathrm{Tor}(R)$.

Let $a \in \ker \varphi$, i.e., $a \otimes 1_{\mathbb{Q}} = 0$. By construction of the tensor product,

$$\begin{aligned} (a, 1_{\mathbb{Q}}) = & \sum_i \ell_i((a_i + b_i, q_i) - (a_i, q_i) - (b_i, q_i)) \\ & + \sum_i m_i((c_i, s_i + t_i) - (c_i, s_i) - (c_i, t_i)) \\ & + \sum_i n_i((k_i d_i, u_i) - (d_i, k_i u_i)) \end{aligned} \tag{3.1}$$

holds for some $a_i, b_i, c_i, d_i \in R$, $k_i, \ell_i, m_i, n_i \in \mathbb{Z}$, and $q_i, s_i, t_i, u_i \in \mathbb{Q}$ in the free $\mathbb{Z}$-module $H_{R \times \mathbb{Q}}$ with the basis $R \times \mathbb{Q}$. We can find such $m \in \mathbb{N}$ that all $mq_i, ms_i, mt_i, mu_i \in \mathbb{Z}$. Then by multiplying equation (3.1) from the right with $(1, m)$ we get that

$$\begin{aligned} (a, m) = & \sum_i \ell_i((a_i + b_i, mq_i) - (a_i, mq_i) - (b_i, mq_i)) \\ & + \sum_i m_i((c_i, ms_i + mt_i) - (c_i, ms_i) - (c_i, mt_i)) \\ & + \sum_i n_i((k_i d_i, mu_i) - (d_i, k_i mu_i)) \end{aligned}$$

holds in the free $\mathbb{Z}$-module $H_{R \times \mathbb{Z}}$ with the basis $R \times \mathbb{Z}$. Note that on the right hand side of the latter equality we have a relation in $R \otimes_{\mathbb{Z}} \mathbb{Z}$. Hence $a \otimes m = 0$ in $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ and $ma = 0$. Thus $a \in \mathrm{Tor}\, R$. Therefore, $\ker \varphi = \mathrm{Tor}\, R$ and $R \otimes 1_{\mathbb{Q}} \cong R/\mathrm{Tor}\, R$.

Suppose $F = \mathbb{F}_p$. Then $\varphi(pR) = R \otimes p1_{\mathbb{F}_p} = 0$ and $pR \subseteq \ker\varphi$. Let $a \in \ker\varphi$, i.e., $a \otimes 1_{\mathbb{F}_p} = 0$. Then

$$\begin{aligned}(a, 1_{\mathbb{F}_p}) = &\sum_i q_i((a_i + b_i, \bar{\ell}_i) - (a_i, \bar{\ell}_i) - (b_i, \bar{\ell}_i)) \\ &+ \sum_i s_i((c_i, \bar{m}_i + \bar{n}_i) - (c_i, \bar{m}_i) - (c_i, \bar{n}_i)) \\ &+ \sum_i t_i((k_i d_i, \bar{u}_i) - (d_i, k_i \bar{u}_i))\end{aligned}$$

holds for some $a_i, b_i, c_i, d_i \in R$ and $k_i, \ell_i, m_i, n_i, q_i, s_i, t_i, u_i \in \mathbb{Z}$ in the free $\mathbb{Z}$-module $H_{R\times\mathbb{F}_p}$ with the basis $R \times \mathbb{F}_p$. Note that $H_{R\times\mathbb{F}_p}$ is the factor module of $H_{R\times\mathbb{Z}}$ by the subgroup $\langle (a, m) - (a, m + p) \mid a \in R,\ m \in \mathbb{Z}\rangle_{\mathbb{Z}}$. Hence

$$\begin{aligned}(a, 1_{\mathbb{Z}}) = &\sum_i q_i((a_i + b_i, \ell_i) - (a_i, \ell_i) - (b_i, \ell_i)) \\ &+ \sum_i s_i((c_i, m_i + n_i) - (c_i, m_i) - (c_i, n_i)) \\ &+ \sum_i t_i((k_i d_i, u_i) - (d_i, k_i u_i)) + \sum_i \alpha_i((r_i, \beta_i) - (r_i, \beta_i + p))\end{aligned}$$

holds in $H_{R\times\mathbb{Z}}$ for some $r_i \in R$ and $\alpha_i, \beta_i \in \mathbb{Z}$. Thus $a \otimes 1_{\mathbb{Z}} = \sum_i \alpha_i r_i \otimes p$. Now we use the isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$ and get $a = \sum_i \alpha_i r_i p \in pR$. Therefore, $\ker\varphi = pR$ and $R \otimes 1_{\mathbb{F}_p} \cong R/pR$. ∎

Proof.[Proof of Theorem 3.1.1] Recall that $R \otimes_{\mathbb{Z}} 1_F$ is a subring of $R \otimes_{\mathbb{Z}} F$. Hence $P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} F, \mathbb{Z}) \subseteq P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} 1_F, \mathbb{Z})$. Conversely, $P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} F, \mathbb{Z}) \supseteq P_n(\mathbb{Z}) \cap \mathrm{Id}(R \otimes_{\mathbb{Z}} 1_F, \mathbb{Z})$ since $R \otimes_{\mathbb{Z}} 1_F$ generates $R \otimes_{\mathbb{Z}} F$ as an F-vector space. Thus $c_n(R \otimes_{\mathbb{Z}} 1_F, \mathbb{Z}, q) = c_n(R \otimes_{\mathbb{Z}} F, \mathbb{Z}, q)$ for all prime powers $q = p^k$. On the other hand by Propositions 1.1.11, 1.1.12, 1.1.15 we also have that $c_n(R \otimes_{\mathbb{Z}} F, \mathbb{Z}, q) = 0$ for $q \neq \operatorname{char} F$ and $c_n(R \otimes_{\mathbb{Z}} F, \mathbb{Z}, \operatorname{char} F) = c_n(R \otimes_{\mathbb{Z}} F, K)$ where K is the prime field of F.

Therefore we get Theorem 3.1.1 for $F = \mathbb{Q}$ and $F = \mathbb{Z}_p$ from Lemma 3.1.2. The general case follows from the fact $(R \otimes_{\mathbb{Z}} F) \otimes_F K \cong R \otimes_{\mathbb{Z}} K$ (as a K-algebra) for any field extension $F \subseteq K$ and by Theorem 1.1.14,

$$c_n(R \otimes_{\mathbb{Z}} K, K) = c_n((R \otimes_{\mathbb{Z}} F) \otimes_F K, K) = c_n(R \otimes_{\mathbb{Z}} F, F).$$

∎

Now we have all ingredients for a direct proof of an analog of Amitsur's Conjecture and Regev's Conjecture for torsion-free rings. Recall that for torsion-free rings $c_n(R, \mathbb{Z}, q) = 0$ if $q \neq 0$, so it is enough to consider $c_n(R, \mathbb{Z}, 0)$.

Theorem 3.1.3. *Let R be a torsion-free ring satisfying a non-trivial polynomial identity. Then,*

1. *either $c_n(R,\mathbb{Z},0) = 0$ for all $n \geqslant n_0$, $n_0 \in \mathbb{N}$, or there exist $d \in \mathbb{N}, t \in \frac{\mathbb{Z}}{2}$ and $C_1, C_2 > 0$, such that $C_1 n^t d^n \leqslant c_n(R,\mathbb{Z},0) \leqslant C_2 n^t d^n$ for all $n \in \mathbb{N}$; in particular there exists $\lim_{n\to\infty} \sqrt[n]{c_n(R,\mathbb{Z},0)} \in \mathbb{N}$;*

2. *if R contains 1, then there exist $C > 0$ and $t \in \frac{\mathbb{Z}}{2}$ such that $c_n(R,\mathbb{Z},0) \simeq Cn^t d^n$ as $n \to \infty$.*

Proof. By Theorem 3.1.1, $c_n(R,\mathbb{Z},0) = c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q})$. Now we apply respectively Theorem 1.3.3 and Theorem 1.3.5. ∎

Remark. One can also deduce an interpretation for d and t in terms of the algebraic structure of $R \otimes_{\mathbb{Z}} \mathbb{C}$. An interpretation internal to the ring structure of R would however also be interesting.

We conclude the section with an example.

Example. Let $R = \bigoplus_{k=1}^{\infty} \mathbb{Z}_{2^k}$. Then $c_n(R,\mathbb{Z},0) = 1$ and $c_n(R,\mathbb{Z},q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. Although $mR \neq 0$ for all positive integers m, $R \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$.

Proof. The ring R is commutative. Hence all monomials from $P_n(\mathbb{Z})$ are proportional to $x_1 x_2 \dots x_n$ modulo $\mathrm{Id}(R,\mathbb{Z})$. However, $m x_1 x_2 \dots x_n \notin \mathrm{Id}(R,\mathbb{Z})$ for all $m \in \mathbb{N}$. (It is sufficient to substitute $x_1 = x_2 = \dots = x_n = \bar{1}_{\mathbb{Z}_{2^k}}$ for $2^k > m$.) Thus $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})} \cong \mathbb{Z}$ and $c_n(R,\mathbb{Z},0) = 1$ and $c_n(R,\mathbb{Z},q) = 0$ for all $q \neq 0$ and $n \in \mathbb{N}$. However $a \otimes q = 2^k a \otimes \frac{q}{2^k}$ for all $a \in R$, $q \in \mathbb{Q}$, and $k \in \mathbb{N}$. Choosing k sufficiently large, we get $a \otimes q = 2^k a \otimes \frac{q}{2^k} = 0$. Thus $R \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $c_n(R \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) = 0$ for all $n \in \mathbb{N}$. ∎

3.2 A chain from Ordinary to Proper Polynomial Functions

Now we prove an analogue of Drensky's theorem [Dre00, Theorem 12.5.4] that enables us to reduce the problem of describing concretely a generalized Specht series of $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ to the module of proper multilinear polynomial functions.

Theorem 3.2.1. *Let R be a unitary ring and* $\operatorname{char} R = \ell$ *a positive integer. Consider for every $n \in \mathbb{N}$ the series of $\mathbb{Z}S_n$-submodules*

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \supsetneqq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$$

where each M_k is the image of $\bigoplus_{t=k}^n \mathbb{Z}S_n(x_{t+1}\dots x_n\Gamma_t(\mathbb{Z}))$ and $M_{n+1} := 0$. Then $M_0/M_2 \cong \mathbb{Z}/\ell\mathbb{Z}$ (trivial S_n-action) and

$$\begin{array}{rcl} M_t/M_{t+1} & \cong & \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \uparrow S_n \\ & := & \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t\times S_{n-t})} \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right), \end{array}$$

for all $2 \leqslant t \leqslant n$ where S_{n-t} is permuting $x_{t+1},\dots,x_n$ and $\mathbb{Z}$ is a trivial $\mathbb{Z}S_{n-t}-$module. If $\ell = 0$ we set by definition $\mathbb{Z}/\ell\mathbb{Z} = 0$.

Proof. It is easy to see that M_0/M_2 is generated by the image of $x_1x_2\dots x_n$ and moreover, isomorphic as $\mathbb{Z}S_n$-module to $\mathbb{Z}/\ell\mathbb{Z}$ which has trivial S_n-action. This is also a direct consequence of expressions (1.1) and (1.2).

Note that $\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})}$ as $\mathbb{Z}(S_t \times S_{n-t})$-module, where S_{n-t} acts trivially on $\mathbb{Z}$. Consider the bilinear map

$$\varphi\colon \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \to M_t/M_{t+1},$$

defined by $\varphi(\sigma, f) = x_{\sigma(t+1)}x_{\sigma(t+2)}\dots x_{\sigma(n)}(\sigma f) \mod M_{t+1}$ for $\sigma \in S_n$, $f \in \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})}$. We will now show that M_t/M_{t+1}, together with this bilinear map, satisfies the universal property of the tensor product.

Note that $\varphi(\sigma\pi, f) = \varphi(\sigma, \pi f)$ for all $\pi \in S_t \times S_{n-t}$ and M_t/M_{t+1} is generated by all $\varphi(\sigma, f)$ with $\sigma \in S_n$ and $f \in \Gamma_t(\mathbb{Z})$.

Suppose L is an abelian group and $\psi\colon \mathbb{Z}S_n \times \frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap\operatorname{Id}(R,\mathbb{Z})} \to L$ is a $\mathbb{Z}$-bilinear map and $\psi(\sigma\pi, f) = \psi(\sigma, \pi f)$ for all $\pi \in S_t \times S_{n-t}$. First we define $\bar{\psi}\colon M_t \to L$ on the elements that generate M_t modulo M_{t+1}:

$$\bar{\psi}(x_{i_1}x_{i_2}\dots x_{i_{n-t}}f) = \psi(\sigma, \sigma^{-1}f),$$

where $\sigma(k) = i_k$ for $1 \leqslant k \leqslant n-t$. Suppose the image $\bar{f}_0$ of a polynomial

$$f_0 = \sum_{i_1<\dots<i_{n-t}} x_{i_1}x_{i_2}\dots x_{i_{n-t}} f_{i_1,\dots,i_{n-t}}$$

belongs to M_{t+1} for some $f_{i_1,\dots,i_{n-t}} \in \Gamma_t(\mathbb{Z})$. Substituting

$$x_{i_1} = x_{i_2} = \cdots = x_{i_{n-t}} = 1_R$$

and arbitrary values for the other x_j, we get that in any long commutator 1_R is substituted and thus we get zero for every $i_1 < \cdots < i_{n-t}$. Hence $f_{i_1,\dots,i_{n-t}} \in \mathrm{Id}(R,\mathbb{Z})$ and $\bar{\psi}(\bar{f}_0) = 0$. Thus we can define $\bar{\psi}$ to be zero on M_{t+1} and we may assume that $\bar{\psi}\colon M_t/M_{t+1} \to L$. By definition $\bar{\psi}\varphi = \psi$ which proves that $(M_t/M_{t+1}, \varphi)$ indeed satisfy the universal property. Hence $M_t/M_{t+1} \cong \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t\times S_{n-t})} \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right)$ as $\mathbb{Z}S_n$-modules with an isomorphism given by $\varphi(\sigma, f) \mapsto \sigma \otimes f$. ∎

Remark. Due to the nature of the result and its proof we can deduce immediately the same statement over any Principal ideal domain.

3.3 A particular case of the Littlewood-Richardson rule

In order to fully describe the $\mathbb{Z}S_n$-structure of $P_n(\mathbb{Z})/(P_n(\mathbb{Z}) \cap \mathrm{Id}(R))$ for the upper-triangular matrix algebra and the Grassman algebra we will need a slight generalization of Young's rule (cf. Corollary 1.2.31).

Theorem 3.3.1. *Let $t, n \in \mathbb{N}$, $t < n$, m a non-zero positive integer and $\lambda \vdash t$ and let $\mathbb{Z}$ be the trivial $\mathbb{Z}S_{n-t}$-module. Then*

$$(S(\lambda)/mS(\lambda)) \uparrow S_n := \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t\times S_{n-t})} ((S(\lambda)/mS(\lambda)) \otimes_{\mathbb{Z}} S((n-t)))$$

has a series of submodules with factors $S(\nu)/mS(\nu)$, where ν runs over the set of all partitions $\nu \vdash n$ such that

$$\lambda_n \leqslant \nu_n \leqslant \lambda_{n-1} \leqslant \nu_{n-1} \leqslant \cdots \leqslant \lambda_2 \leqslant \nu_2 \leqslant \lambda_1 \leqslant \nu_1.$$

Moreover, each factor occurs exactly once.

Proof. Suppose $\lambda = (\lambda_1, \dots, \lambda_s)$, $\lambda_s > 0$. Then as we saw in Section 1.2.5 $S(\lambda) \uparrow S_n \cong S(\lambda;\mu)$, where $\mu = (\lambda_1, \dots, \lambda_s, n-t)$. Now, Theorem 1.2.30 implies the theorem for $m = 0$ since the conditions $\lambda_i \le \nu_i$ and $\nu'_i \le \lambda'_i + 1$ can clearly be rephrased into the condition on ν from the Theorem.

Suppose $m > 0$. Then $(S(\lambda)/mS(\lambda)) \uparrow S_n \cong (S(\lambda) \uparrow S_n)/(m(S(\lambda) \uparrow S_n))$. Let

$$S(\lambda) \uparrow S_n = M_0 \supsetneqq M_1 \supsetneqq M_2 \supsetneqq \cdots \supsetneqq M_t = 0,$$

where $M_{i-1}/M_i \cong S(\lambda^{(i)})$, $\lambda^{(i)} \vdash n$, $1 \leqslant i \leqslant t$.

Hence

$$(S(\lambda) \uparrow S_n)/(m(S(\lambda) \uparrow S_n)) = \overline{M_0} \supsetneqq \overline{M_1} \supsetneqq \overline{M_2} \supsetneqq \cdots \supsetneqq \overline{M_t} = 0,$$

where $\overline{M_i} \cong (M_i + m(S(\lambda) \uparrow S_n))/m(S(\lambda) \uparrow S_n)$ and

$$\begin{aligned} \overline{M_{i-1}}/\overline{M_i} &\cong (M_{i-1} + m(S(\lambda) \uparrow S_n))/(M_i + m(S(\lambda) \uparrow S_n)) \\ &\cong M_{i-1}/M_{i-1} \cap (M_i + m(S(\lambda) \uparrow S_n)) \\ &= M_{i-1}/(M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n)) \\ &\cong (M_{i-1}/M_i)/((M_i + M_{i-1} \cap m(S(\lambda) \uparrow S_n))/M_i). \end{aligned}$$

By Theorem 1.2.30, $(S(\lambda) \uparrow S_n)/M_{i-1}$ is torsion-free. Hence $M_{i-1} \cap m(S(\lambda) \uparrow S_n) = mM_{i-1}$ and

$$\begin{aligned} \overline{M_{i-1}}/\overline{M_i} &\cong (M_{i-1}/M_i)/((M_i + mM_{i-1})/M_i) \\ &= (M_{i-1}/M_i)/(m(M_{i-1}/M_i)) \\ &\cong S(\lambda^{(i)})/mS(\lambda^{(i)}). \end{aligned}$$

∎

3.4 Algebras of Upper Triangular Matrices

Let M be an (R_1, R_2)-bimodule for commutative rings R_1, R_2 with 1 and let

$$R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}.$$

In this section, we calculate $c_n(R, q)$ for all prime powers $q = p^k$ and $q = 0$, describe the structure of the $\mathbb{Z}S_n$-module $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \mathrm{Id}(R,\mathbb{Z})}$ and find such multilinear polynomials so that elements of $\mathrm{Id}(R,\mathbb{Z}) \cap \bigcup_{n\in\mathbb{N}} P_n(\mathbb{Z})$ are all consequences of them. We say that a polynomial f *is a consequence of* a set of polynomials $\{g_1, \dots, g_t\}$ if f is in the T-ideal generated by $\{g_1, \dots, g_t\}$.

Remark 3.4.1. If F is a field of characteristic 0 and $A = \mathrm{UT}_2(F) := \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, then $c_n(A, F)$ and generators of $\mathrm{Id}(A, F)$ as a T-ideal can be found, e.g., in [GZ05, Theorem 4.1.5]. The structure of the FS_n-module $\frac{P_n(F)}{P_n(F) \cap \mathrm{Id}(A,F)}$ can be determined using proper cocharacters [Dre00, Theorem 12.5.4].

3.4.1 Codimensions and multilinear identities

Theorem 3.4.2. *All polynomials from $P_n(\mathbb{Z})\cap \operatorname{Id}(R,\mathbb{Z})$, $n\in\mathbb{N}$, are consequences of the left hand sides of the following polynomial identities in R:*

$$[x,y][z,t]\equiv 0, \tag{3.2}$$

$$\ell x\equiv 0, \tag{3.3}$$

$$m[x,y]=0 \tag{3.4}$$

where $[x,y]:=xy-yx$,

$$\ell:=\min\{n\in\mathbb{N}\mid n\neq 0 \text{ and } na=0 \text{ for all } a\in R_1\cup R_2\},$$

$$m:=\min\{n\in\mathbb{N}\mid n\neq 0 \text{ and } na=0 \text{ for all } a\in M\}.$$

(Again we agree that if one of the corresponding sets is empty, we define $\ell=0$ or $m=0$, respectively and $\mathbb{Z}_0=\mathbb{Z}$. Note that $m\mid \ell$.)

Moreover, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \operatorname{Id}(R,\mathbb{Z})}\cong \mathbb{Z}_\ell\oplus(\mathbb{Z}_m)^{(n-2)2^{n-1}+1}$, *where* $\mathbb{Z}_0:=\mathbb{Z}$.

Remark 3.4.3. Now $c_n(R,q)$ can simply be read off the decomposition. If $R_1=R_2=M$ and $R_1=R_2$ is a field, we obtain the same numbers as in [GZ05, Theorem 4.1.5].

Proof.[Proof of Theorem 3.4.2.] Denote by e_{ij} the matrix units. Then $R=R_1e_{11}\oplus R_2e_{22}\oplus Me_{12}$ (direct sum of abelian groups), $[R,R]\subseteq Me_{12}$, and (3.2)–(3.4) are indeed polynomial identities of R.

Now we consider an arbitrary mononomial from $P_n(\mathbb{Z})$ and find the first inversion among the indexes of its variables. Using $x_jx_i=x_ix_j+[x_j,x_i]$, we replace the corresponding product of variables with the sum of their commutator and their product in the right order. Note that $[x,y]u[z,t]=[x,y][z,t]u+[x,y][u,[z,t]]\equiv 0$ is a consequence of (3.2). Therefore, we may assume that all the variables to the right of the commutator have increasing indexes. For example:

$$\begin{aligned}
x_3x_1x_4x_2 &= x_1x_3x_4x_2+[x_3,x_1]x_4x_2\\
&\overset{(3.2)}{\equiv} x_1x_3x_2x_4+x_1x_3[x_4,x_2]+[x_3,x_1]x_2x_4\\
&= x_1x_2x_3x_4+x_1[x_3,x_2]x_4+x_1x_3[x_4,x_2]+[x_3,x_1]x_2x_4.
\end{aligned}$$

Continuing this procedure, we present any element of $P_n(\mathbb{Z})$ modulo the consequences of (3.2) as a linear combination of polynomials $f_0:=x_1x_2\dots x_n$ and

$$x_{i_1}\dots x_{i_k}[x_s,x_r]x_{j_1}\dots x_{j_{n-k-2}} \text{ for } i_1<\dots<i_k<s,\ r<s,\ j_1<\dots<j_{n-k-2}. \tag{3.5}$$

Denote the set consisting of the polynomials (3.5) by Ξ.

Consider the free abelian group $\mathbb{Z}(\Xi \cup \{f_0\})$ with the basis $\Xi \cup \{f_0\}$. Now we have the surjective homomorphism $\varphi\colon \mathbb{Z}(\Xi \cup \{f_0\}) \to \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$, where $\varphi(f)$ is the image of $f \in \Xi \cup \{f_0\}$ in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$. We claim that $\ker\varphi$ is generated by ℓf_0 and all mf, where $f \in \Xi$.

Suppose that a linear combination f_1 of f_0 and elements from Ξ is a polynomial identity, but that f_1 is not a linear combination of ℓf_0 and mf, $f \in \Xi$. If we substitute

$$x_1 = x_2 = \dots = x_n = 1_{R_i} e_{ii} \text{ where } i \in \{1, 2\},$$

then all $f \in \Xi$ vanish. Therefore, the coefficient of f_0 is a multiple of ℓ. Now we find $f_2 := x_{i_1} \dots x_{i_k}[x_s, x_r]x_{j_1} \dots x_{j_{n-k-2}} \in \Xi$ with the largest k such that the coefficient β of f_2 in f_1 is not a multiple of m. Then we substitute $x_{i_1} = \dots = x_{i_k} = x_s = 1_{R_1}e_{11}$, $x_r = ae_{12}$, $x_{j_1} = \dots = x_{j_{n-k-2}} = 1_{R_1}e_{11} + 1_{R_2}e_{22} = 1_R$, where $a \in M$ and $\beta a \neq 0$. Our choice of f_2 implies that f_2 is the only summand in f_1 that could be nonzero under this substitution. Hence f_1 does not vanish and we get a contradiction. Therefore, $\ker\varphi$ is generated by ℓf_0 and mf, $f \in \Xi$. In particular, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \cong \mathbb{Z}_\ell \oplus (\mathbb{Z}_m)^{|\Xi|}$ and every multilinear polynomial identity of R is a consequence of (3.2)–(3.4).

Note that

$$\begin{aligned} |\Xi| &= \sum_{k=2}^{n} (k-1)\binom{n}{k} = \sum_{k=2}^{n} \frac{n!}{(k-1)!(n-k)!} - \sum_{k=2}^{n} \binom{n}{k} \\ &= n \sum_{k=1}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} - (2^n - n - 1) \\ &= n(2^{n-1} - 1) - (2^n - n - 1) = (n-2)2^{n-1} + 1 \end{aligned}$$

and the theorem follows. ∎

Corollary 3.4.4. *Multilinear polynomial identities of* $\mathrm{UT}_2(\mathbb{Q})$ *as a ring are generated by (3.2).*

3.4.2 $\mathbb{Z}S_n$-modules

Note that the Jacobi identity $[a, b, c] + [c, a, b] = [c, b, a]$ and (3.2) imply that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is generated as a $\mathbb{Z}$-module by $[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$, where $1 \leqslant i \leqslant n-1$. By $[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$ we denote the commutator where only x_i is omitted.

Lemma 3.4.5. *With notations as before and* $T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$. *Then,*

$$b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv n(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \pmod{P_n(\mathbb{Z}) \cap \operatorname{Id}(R)}. \tag{3.6}$$

Proof. Indeed,

$$\begin{aligned} b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] &\equiv b_{T_\lambda}(n-2)! \sum_{i=1}^{n-1} [x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}] \\ &= (n-2)! \sum_{i=2}^{n-1} ([x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}] \\ &\qquad - [x_i, x_1, x_n, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]) \\ &\qquad + 2(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \\ &\equiv n(n-2)![x_1, x_n, x_2, x_3, \dots, x_{n-1}] \end{aligned}$$

since, by the Jacobi identity, $[x_i, x_1, x_n] = [x_i, x_n, x_1] + [x_n, x_1, x_i]$. ∎

First, we determine the structure of $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R, \mathbb{Z})}$ for $R = \mathrm{UT}_2(\mathbb{Q})$.

Lemma 3.4.6. *Let* $T_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & \dots & n-1 \\ \hline n \\ \cline{1-1}\end{array}$. *Then* $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z})} \cong (\mathbb{Z}S_n) b_{T_\lambda} a_{T_\lambda}$.

Proof. We claim that if $u b_{T_\lambda} a_{T_\lambda} = 0$ for some $u \in \mathbb{Z}S_n$, then

$$u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \in \Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z}).$$

Indeed, by (3.6),

$$n(n-2)!\, u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv u b_{T_\lambda} a_{T_\lambda}[x_1, x_n, x_2, x_3, \dots, x_{n-1}] = 0.$$

Since $\mathrm{UT}_2(\mathbb{Q})$ has no torsion, $u[x_1, x_n, x_2, x_3, \dots, x_{n-1}] \equiv 0$ is a polynomial identity of $\mathrm{UT}_2(\mathbb{Q})$.

Thus, we can define the surjective homomorphism $\varphi \colon (\mathbb{Z}S_n) b_{T_\lambda} a_{T_\lambda} \to \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Z})}$ by $\varphi(\sigma b_{T_\lambda} a_{T_\lambda}) = \sigma[x_1, x_n, x_2, x_3, \dots, x_{n-1}]$ for $\sigma \in S_n$.

Analogously, we can define the surjective homomorphism

$$\varphi_0 \colon (\mathbb{Q}S_n) b_{T_\lambda} a_{T_\lambda} \to \frac{\Gamma_n(\mathbb{Q})}{\Gamma_n(\mathbb{Q}) \cap \operatorname{Id}(\mathrm{UT}_2(\mathbb{Q}), \mathbb{Q})},$$

by $\varphi(\sigma b_{T_\lambda} a_{T_\lambda}) = \sigma[x_1, x_n, x_2, x_3, \dots, x_{n-1}]$ for $\sigma \in S_n$. Since $(\mathbb{Q}S_n)b_{T_\lambda}a_{T_\lambda}$ is an irreducible $\mathbb{Q}S_n$-module by Theorem 1.2.8, φ_0 is an isomorphism of $\mathbb{Q}S_n$-modules. Further note that
$\varphi_0(\mathbb{Z}S_n b_{T_\lambda} a_{T_\lambda}) \subseteq \overline{\Gamma_n(\mathbb{Z})}$. Thus φ equals φ_0 on $\mathbb{Z}S_n b_{T_\lambda} a_{T_\lambda}$ and therefore it is also a monomorphism, hence an isomorphism and the lemma is proven. ∎

Theorem 3.4.7. *Let R and m be, respectively, the ring and the number defined in the statement of Theorem 3.4.2. Then $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})} \cong S(\lambda)/mS(\lambda)$, where $\lambda = (n-1, 1)$, for all $n \geq 2$.*

Proof. Recall that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is generated as a $\mathbb{Z}$-module by

$$[x_i, x_n, x_1, x_2, \dots, \hat{x}_i, \dots, x_{n-1}]$$

where $1 \leqslant i \leqslant n-1$. These elements are moreover $\mathbb{Z}$-linear independent. This can be seen by using the substitution $x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_n = 1_{R_1} e_{11}$, $x_i = a e_{12}$ where $a \in M$. So we obtain that $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ is the direct sum of $n-1$ cyclic groups isomorphic to $\mathbb{Z}_m$.

By Theorem 3.4.2 and Corollary 3.4.4, we have the natural surjective homomorphism $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})} \to \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$. The remarks above imply that the kernel equals $m\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z})\cap \mathrm{Id}(\mathrm{UT}_2(\mathbb{Q}),\mathbb{Z})}$. Now the theorem follows from Lemma 3.4.6. ∎

Applying Theorems 3.2.1, 3.3.1 and 3.4.7 we immediately get the following result.

Theorem 3.4.8. *Let R, ℓ, and m be, respectively, the ring and the numbers as in Theorem 3.4.2. Then there exists a chain of $\mathbb{Z}S_n$-submodules in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ with the set of factors that consists of one copy of $\mathbb{Z}_\ell$ and $(\lambda_1-\lambda_2+1)$ copies of $S(\lambda_1,\lambda_2,\lambda_3)/mS(\lambda_1,\lambda_2,\lambda_3)$ where $(\lambda_1,\lambda_2,\lambda_3) \vdash n$, $\lambda_2 \geqslant 1$, $\lambda_3 \in \{0,1\}$.*

3.5 Grassmann Algebras

Let R be a commutative ring with a unit element 1_R, $\operatorname{char} R = \ell$, where either ℓ is an odd natural number or $\ell = 0$. We define the Grassman algebra G_R over a ring R as the R-algebra with a unit, generated by the countable set of generators e_i, $i \in \mathbb{N}$, and the anti-commutative relations $e_i e_j = -e_j e_i$, $i, j \in \mathbb{N}$. We now consider the same questions as for the upper triangular matrices.

3.5.1 Codimensions and polynomial identities

This lemma is known but we provide its proof for the reader's convenience.

Lemma 3.5.1. *The polynomial identity* $[y,x][z,t]+[y,z][x,t]\equiv 0$ *is a consequence of* $[x_1,x_2,x_3]\equiv 0$. *In particular,* $[x,y]u[z,t]+[x,t]u[z,y]\equiv 0$, *for all* $u\in\mathbb{Z}\langle X\rangle$.

Proof. Note that

$$\begin{aligned}[x,yt,z] &= [[x,y]t,z]+[y[x,t],z]\\ &= [x,y,z]t+[x,y][t,z]+[y,z][x,t]+y[x,t,z]\\ &\equiv [x,y][t,z]+[y,z][x,t]=[y,x][z,t]+[y,z][x,t]\end{aligned}$$

modulo $[x_1,x_2,x_3]\equiv 0$. (Here we have used Jacobi's identity too.) Hence

$$\begin{aligned}u[z,t]+[x,t]u[z,y] &= [x,y][u,[z,t]]+[x,y][z,t]u\\ &\quad +[x,t][u,[z,y]]+[x,t][z,y]u\\ &\equiv [x,y][z,t]u+[x,t][z,y]u\\ &\equiv 0.\end{aligned} \tag{3.7}$$

∎

Theorem 3.5.2. *All polynomials from* $P_n(\mathbb{Z})\cap\operatorname{Id}(G_R,\mathbb{Z})$, *for any* n, *are consequences of the left hand sides of the following polynomial identities of* G_R:

$$[x,y,z]\equiv 0, \tag{3.8}$$

$$\ell x\equiv 0. \tag{3.9}$$

Moreover, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap\operatorname{Id}(G_R,\mathbb{Z})}\cong(\mathbb{Z}_\ell)^{2^{n-1}}$.

Proof. Define $G_R^{(0)}=\langle e_{i_1}e_{i_2}\dots e_{i_{2k}}\mid 0\neq k\in\mathbb{N}\rangle_R$ and $G_R^{(1)}=\langle e_{i_1}e_{i_2}\dots e_{i_{2k+1}}\mid 0\neq k\in\mathbb{N}\rangle_R$. Clearly, $G_R=G_R^{(0)}\oplus G_R^{(1)}$ (direct sum of R-submodules), $[G_R,G_R]\subseteq G_R^{(0)}$, $G_R^{(0)}=Z(G_R)$. Hence $[x_1,x_2,x_3]\equiv 0$ is a polynomial identity for G_R. Obviously, (3.9) is a polynomial identity too. Let

$$\begin{aligned}\Xi=\{x_{i_1}\dots x_{i_k}[x_{j_1},x_{j_2}]\dots[x_{j_{2m-1}},x_{j_{2m}}]\mid i_1<\dots<i_k,\\ j_1<\dots<j_{2m},\ k+2m=n,\ k,m\in\mathbb{Z}_+\}\subset P_n(\mathbb{Z}).\end{aligned}$$

By Lemma 3.5.1, every polynomial from $P_n(\mathbb{Z})$ can be presented modulo (3.8) as a linear combination of polynomials from Ξ. For example,

$$\begin{aligned}
x_3x_2x_4x_1 =& -[x_2,x_3]x_4x_1 + x_2x_3x_4x_1 \\
=& ([x_2,x_3][x_1,x_4] - [x_2,x_3]x_1x_4) + (x_2x_3x_1x_4 - x_2x_3[x_1,x_4]) \\
\equiv& -[x_2,x_1][x_3,x_4] - x_1x_4[x_2,x_3] + x_2x_1x_3x_4 \\
&- x_2[x_1,x_3]x_4 - x_2x_3[x_1,x_4] \\
\equiv& [x_1,x_2][x_3,x_4] - x_1x_4[x_2,x_3] + x_1x_2x_3x_4 \\
&- [x_1,x_2]x_3x_4 - x_2x_4[x_1,x_3] - x_2x_3[x_1,x_4] \\
\equiv& [x_1,x_2][x_3,x_4] - x_1x_4[x_2,x_3] + x_1x_2x_3x_4 \\
&- x_3x_4[x_1,x_2] - x_2x_4[x_1,x_3] - x_2x_3[x_1,x_4].
\end{aligned}$$

Consider the free abelian group $\mathbb{Z}\Xi$ with basis Ξ. Now we have the surjective homomorphism $\varphi\colon \mathbb{Z}\Xi \to \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$, where $\varphi(f)$ is the image of $f \in \Xi$ in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$. We claim that $\ker\varphi$ is generated by ℓf where $f \in \Xi$.

Suppose that a linear combination f_1 of elements from Ξ is a polynomial identity, but f_1 is not a linear combination of ℓf, $f \in \Xi$. Now we find

$$f_2 := x_{i_1}\dots x_{i_k}[x_{j_1},x_{j_2}]\dots[x_{j_{2m-1}},x_{j_{2m}}] \in \Xi,$$

with the largest k such that the coefficient β of f_2 in f_1 is not a multiple of ℓ. Then we substitute $x_{i_1} = \dots = x_{i_k} = 1_{G_R}$, $x_{j_i} = e_i$, $1 \leqslant i \leqslant 2m$. Our choice of f_2 implies that f_2 is the only summand in f_1 that could be nonzero under this substitution. Hence the value of f_1 equals $(2^m\beta\, 1_R)e_1e_2\dots e_m = 0$. However, G_R is a free R-module and $e_1e_2\dots e_m$ is one of its basis elements. Therefore $2^m\beta\, 1_R = 0$, $\ell \mid (2^m\beta)$ and $\ell \mid \beta$ since $2 \nmid \ell$. We get a contradiction.

Thus $\ker\varphi$ is generated by ℓf, $f \in \Xi$. In particular, $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong (\mathbb{Z}_\ell)^{|\Xi|}$ and every multilinear polynomial identity of G_R is a consequence of (3.8) and (3.9).

We now calculate $|\Xi|$. The number of these polynomials equals the number of choices of $x_{i_1},\dots,x_{i_k}$. If n is odd, this number equals $\binom{n}{1}+\binom{n}{3}+\dots+\binom{n}{n}$. If n is even, the number equals $\binom{n}{0}+\binom{n}{2}+\dots+\binom{n}{n}$. But both are equal to 2^{n-1}. Indeed, denote $s_0 = \sum\limits_{i \text{ even}} \binom{n}{i}$ and $s_1 = \sum\limits_{i \text{ odd}} \binom{n}{i}$. Then $2^n = (1+1)^n = s_0 + s_1$ and $0 = (1-1)^n = s_0 - s_1$. So $|\Xi| = s_0 = s_1 = 2^{n-1}$. ∎

3.5.2 $\mathbb{Z}S_n$-modules

First we determine the structure of $\mathbb{Z}S_n$-modules of proper polynomial functions.

Theorem 3.5.3. *Let* $\lambda = (1^{2m})$ *and* $T_\lambda = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline 2m \\ \hline \end{array}$. *Then* $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong S(\lambda)/\ell S(\lambda)$, *as* $\mathbb{Z}S_n$*-modules for all positive integers* m , *where* $\ell = \operatorname{char} R$, *and* $\frac{\Gamma_{2m+1}(\mathbb{Z})}{\Gamma_{2m+1}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} = 0$ *for all* m.

Proof. Note that $S(\lambda)$ is isomorphic to the trivial $\mathbb{Z}S_n$-module $\mathbb{Z}$, is generated by $b_{T_\lambda}[T_\lambda]$ and $\sigma b_{T_\lambda}[T_\lambda] = (\operatorname{sign}\sigma) b_{T_\lambda}[T_\lambda]$ for all $\sigma \in S_n$.

The proof of Theorem 3.5.2 implies that $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong \mathbb{Z}_\ell$ is a cyclic group generated by $[x_1, x_2] \ldots [x_{2m-1}, x_{2m}]$. By Lemma 3.5.1,

$$\sigma[x_1, x_2] \ldots [x_{2m-1}, x_{2m}] \equiv (\operatorname{sign}\sigma)[x_1, x_2] \ldots [x_{2m-1}, x_{2m}] \text{ for all } \sigma \in S_n.$$

Hence $\frac{\Gamma_{2m}(\mathbb{Z})}{\Gamma_{2m}(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})} \cong S(\lambda)/\ell S(\lambda)$. The first assertion of the Theorem is proven. The second assertion is evident since every long commutator of a length greater than 2 is a polynomial identity of G_R. ∎

Theorem 3.5.4. *Let* G_R *be the Grassmann algebra over the* R. *Then there exists a chain of* $\mathbb{Z}S_n$*-submodules in* $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(G_R,\mathbb{Z})}$ *with factors* $S(n-k, 1^k)/\ell S(n-k, 1^k)$ *for each* $0 \leqslant k \leqslant n-1$ *(each factor occurs exactly once) where* $\ell = \operatorname{char} R$.

Proof. Now we apply Theorems 3.2.1, 3.3.1, and 3.5.3. By Theorem 3.3.1, a diagram consisting of a single column can generate only diagrams $D_{(n-k,1^k)}$. Since we have diagrams of an even length only, each factor occurs only once. ∎

4

Semigroup Graded-Simple Algebras

One cannot guess the real
difficulties of a problem before
having solved it.
Carl Ludwig Siegel

In this chapter we consider finite dimensional algebras A, not necessarily unital, over a field F of characteristic 0 and endowed with a gradation by a semigroup, say S. The main goal is to classify those algebras which are simple with respect to this gradation, i.e. their only graded ideals are $\{0\}$ and A.

For group gradations a classification was obtained by Bahturin, Zaicev and Sehgal [BZ02, BZS08]. Unfortunately, we are not able to solve the classification problem for a general semigroup, but we achieve this in an important case which is somehow the opposite case of the group-graded context. Suppose now that A is an S-graded-simple algebra. In Section 4.1 we start by explaining how the S-graded-simplicity can be used to reduce the grading semigroup S to three types of semigroups. Moreover, in case one assumes that S is finite only one type occurs, namely so called 'completely 0-simple semigroups'. The latter have been heavily investigated in semigroup-theory and are isomorphic to 'generalized matrix groups', denoted $\mathcal{M}(G^0; I; I; P)$. Section 4.1 also serves to recall the necessary background.

If $|I| = |J| = 1$, then $\mathcal{M}(G^0; I; I; P) = G$ is a group and so this case is described by [BZS08]. In Section 4.3 and Section 4.4 we classify the S-graded-simple algebras where $S = \mathcal{M}(\{e\}^0; I; I; P)$, i.e. S has trivial maximal subgroups. The main results are

Theorem 4.3.2, Theorem 4.3.7 and Theorem 4.4.2.

More precisely, in the ungraded setting, in order to understand the structure of a finite dimensional algebra A it boils down to three problems:

1. the determination of the structure of $J(A)$.
2. describing the division algebras and the degrees appearing in the Wedderburn-Artin decomposition $A/J(A) \cong \bigoplus_{i=1}^{q} M_{n_i}(D_i)$.
3. understanding the interaction of the Wedderburn-Artin components of $A/J(A)$ with $J(A)$.

In case the ground field F is algebraically closed the second problem is reduced to finding the numbers n_i. At a vector space level the third problem is solved by means of Wedderburn-Malcev's Theorem 1.3.2 whereas the PI-exponent by Giambruno-Zaicev's Theorem 1.3.3 delivers partial information on how they multiply. The first problem is usually a hopeless endeavour. In order to classify finite dimensional S-graded-simple algebras it is natural to also work along the same lines, but in a graded context.

First we show in Theorem 4.3.2, without any restriction on the base field F, that there exists an S-graded subalgebra B such that B is simple and $A = B \oplus J(A)$, a direct sum of vector spaces (which in particular provides a constructive proof of a graded Wedderburn-Malcev type theorem). However $J(A)$ is not a graded ideal and does not even contain homogeneous elements, cf. Lemma 4.3.1. Furthermore, we show that problem 1 above also can be solved in our setting. Namely in Theorem 4.3.7 we provide a specific decomposition of $J(A)$ as a direct sum of simple B-modules. Although the homogeneous part of $J(A)$ is trivial, these summands are strongly related to the description of the homogeneous components of A.

This chapter is based on joint work with Gordienko and Jespers [GJJ17].

Finally, note that characterizing semigroup-graded rings that are simple has received a lot of attention, see for example [BZ02, BZ03, BZ06, JW89, NO14, NO15] and the references therein. For an introduction to graded ring theory, respectively semigroup theory, we refer to [Kel02, NVO04] and [CP61, Okn98].

4.1 General results and Reducing the Grading

Reduction of S to (completely) 0-simple semigroups

Let S be an arbitrary semigroup and let R be an S*-graded ring*, that is

$$R = \bigoplus_{s \in S} R^{(s)},$$

a direct sum of additive subgroups $R^{(s)}$ of R such that $R^{(s)}R^{(t)} \subseteq R^{(st)}$, for all $s, t \in S$.

Definition 4.1.1. Let R be an S-graded ring for a semigroup S and P an additive subgroup of R. Then

- P is said to be *homogeneous* or *graded* if $P = \bigoplus_{s \in S}(R^{(s)} \cap P)$.
- R is called *(S-)graded-simple* if $R^2 \neq 0$ and $\{0\}$ and R are the only homogeneous ideals of R.

From now on we assume R is S-graded-simple.

We want to reduce S to a list of three types of semigroups. As a first step, remark that we may, without loss of generality, replace S by the semigroup generated by $\operatorname{supp}(R) = \{s \in S \mid R^{(s)} \neq 0\}$.

Next note that if S has a zero element θ, then $R^{(\theta)}$ is a homogeneous ideal of R. So if $R^{(\theta)} \neq 0$, then $R^{(\theta)} = R$ by the graded-simplicity and $\operatorname{supp}(R) = \{\theta\}$, a trivial graded ring. Thus in order to investigate graded-simple rings, without loss of generality, we may assume that if $\theta \in S$, then $R^{(\theta)} = 0$. Altogether, replacing S by $S^0 := S \cup \{\theta\}$, if necessary, we assume that S has a zero element, $R^{(\theta)} = 0$ and $S = \langle \operatorname{supp}(R) \rangle \cup \{\theta\}$. *These assumptions will, from now on, be used implicitly throughout the chapter without mentioning.*

Note that if I is an ideal of S, then $R_I = \bigoplus_{s \in I} R^{(s)}$ is a graded ideal of R. Hence, by the graded-simplicity, either $R_I = 0$ or $R_I = R$. The latter implies that $I = S$. Thus, if I is a proper ideal of S, then by the former $I \cap \operatorname{supp}(R) = \emptyset$. Therefore, we may replace S by the Rees factor semigroup S/I and consequently assume that S itself does not have proper ideals and $S^2 \neq \{\theta\}$. Such a semigroup S is called a 0*-simple semigroup*. If $T = S \setminus \{\theta\}$ is a semigroup, then we may replace S by the simple semigroup T.

At this stage we know that, if an S-graded ring R is S-graded-simple, then without loss of generality we may assume that S is a 0-simple semigroup. Now we can subdivide 0-simple semigroups in three types:

(i) S is *completely* 0-*simple*, i.e. S is 0-simple and contains a non-zero primitive idempotent;

(ii) S is 0-simple and contains a non-zero idempotent, but does not contain primitive idempotents (recall that in this case each idempotent is the identity element of a subsemigroup isomorphic to the bicylic semigroup [CP61, Theorem 2.54] and, in particular, S is an infinite semigroup);

(iii) S does not have non-zero idempotents. It is not hard to see that in this case S must be infinite.

Definition 4.1.2. Let G be a group, I and J sets, and $P = (p_{ji})$ a $J \times I$ matrix with entries in $G^0 := G \cup \{\theta\}$. Then

$$\mathcal{M}(G^0; I, J; P) := \{(g, i, j) \mid i \in I,\ j \in J,\ g \in G^0\},$$

where we identify all elements (θ, i, j) with the zero element θ, and with the following associative multiplication

$$(g, i, j)(h, k, \ell) = (gp_{jk}h, i, \ell)$$

is called the *Rees $I \times J$ matrix semigroup over the group with zero G^0 and sandwich matrix P.*

Now, by the Rees theorem [CP61, Theorem 3.5], every completely 0-simple semigroup S is isomorphic to $\mathcal{M}(G^0; I, J; P)$ for a maximal subgroup G of S, sets I, J and a matrix P such that every row and every column of P has at least one nonzero element.

If we assume that S is finite, then the S-graded simplicity of R implies that S is non-nilpotent and contains a nonzero primitive idempotent. In other words, for finite semigroups we may restrict our consideration to gradings by completely 0-simple semigroups

$$S = \mathcal{M}(G^0, n, m; P) = \{(g, i, j) \mid 1 \le i \le n,\ 1 \le j \le m,\ g \in G^0\},$$

where G is a group, n and m are positive integers, and P is an $n \times m$ matrix with entries in G^0. Note that R is then also graded by the semigroup $S' = \mathcal{M}(\{e\}^0, n, m; P')$, where the (i, j) component of P' is e if $p_{ij} \neq 0$ and θ otherwise.

One useful feature of a $\mathcal{M}(\{e\}^0; I, J; P)$-gradation is that if L is a nonzero ideal of R, then RLR is an homogeneous ideal. As a first consequence of this feature, we can put the finger on the difference between being simple and $\mathcal{M}(\{e\}^0; I, J; P)$-graded-simple. Namely, a $\mathcal{M}(\{e\}^0; I, J; P)$-graded-simple ring R can still have non-zero annihilators. Recall that a ring is *faithful* if its left and right annihilators are trivial, i.e. if $aR = 0$ or $Ra = 0$ for some $a \in R$, then $a = 0$.

Proposition 4.1.3. *Let R be a ring graded by a finite 0-simple semigroup S with trivial maximal subgroups. Then, R is simple if and only if R is S-graded-simple and R is faithful.*

Proof. The necessity of the conditions is obvious. Suppose R is faithful and S-graded-simple. Since we assume S is a finite 0-simple semigroup with trivial maximal subgroups, $S \cong \mathcal{M}(\{e\}^0; n, m; P)$ by [CP61, Theorem 3.5]. Now, as mentioned before, if L is a nonzero ideal of R, then RLR is an S-homogeneous ideal of R. Since R is faithful, RLR is nonzero and thus $R = RLR \subseteq L$. Hence R is simple. ∎

The faithfulness condition can not be removed. For example, consider the semigroup algebra FT of a right zero band $T = \{e, f\}$ consisting of two elements over a field F. Recall that a semigroup T is a *right zero band* if $st = t$ for any $s, t \in T$. Now clearly T^0 is a 0-simple semigroup and FT is graded-simple. However, this algebra is not simple as it contains the proper two-sided ideal $F(e - f)$. Note that $(e - f)FT = 0$ and thus FT is not faithful.

Some general results

To finish this section we show that if $R \neq J(R)$ and R is S-graded-simple, for a semigroup S, then $J(R)$ does not contain any specific information concerning the S-grading and, in some sense, even the structure of R. First we show that a non-trivial ideal cannot contain homogeneous elements.

Lemma 4.1.4. *Let $I \neq R$ be a two-sided ideal of an S-graded-simple ring $R = \bigoplus_{s\in S} R^{(s)}$ for some semigroup S. Then $R^{(s)} \cap I = 0$ for all $s \in S$.*

Proof. Suppose $r \in R^{(s)} \cap I$ for some $s \in T$. Then the smallest two-sided ideal I_0 containing r is homogeneous. Since $I_0 \subseteq I \subsetneq R$, we get $I_0 = 0$ and $r = 0$. ∎

Recall that a homomorphism $\varphi\colon R_1 \to R_2$ of S-graded rings R_1 and R_2 is *graded* if $\varphi\left(R_1^{(s)}\right) \subseteq R_2^{(s)}$ for all $s \in S$. Two S-graded rings R_1 and R_2 are *isomorphic as* *graded rings* if there exists a graded isomorphism $R_1 \to R_2$. In this case we say that the gradings on R_1 and R_2 are *isomorphic*.

Theorem 4.1.5. *Let S be a semigroup and let $R_i = \bigoplus_{s\in S} R_i^{(s)}$, $i = 1, 2$, be two S-graded-simple rings, $R_i \neq J(R_i)$ for both $i = 1, 2$. If there exists a ring isomorphism $\bar\varphi\colon R_1/J(R_1) \to R_2/J(R_2)$ such that $\bar\varphi\left(\pi_1\left(R_1^{(s)}\right)\right) = \pi_2\left(R_2^{(s)}\right)$ for every $s \in S$ where $\pi_i\colon R_i \to R_i/J(R_i)$, $i = 1, 2$, are the natural epimorphisms, then there exists an isomorphism $\varphi\colon R_1 \to R_2$ of graded rings such that $\pi_2\varphi = \bar\varphi\pi_1$.*

Conversely, if $\varphi\colon R_1 \to R_2$ is an isomorphism of graded rings, we can define the ring isomorphism $\bar\varphi\colon R_1/J(R_1) \to R_2/J(R_2)$ by $\bar\varphi(\pi_1(a)) = \pi_2\varphi(a)$ for all $a \in R_1$ and get $\bar\varphi\left(\pi_1\left(R_1^{(s)}\right)\right) = \pi_2\left(R_2^{(s)}\right)$ for every $s \in S$.

Proof. Suppose that there exists such an isomorphism $\bar\varphi :: R_1/J(R_1) \to R_2/J(R_2)$. Lemma 4.1.4 implies that

$$\pi_i\big|_{R_i^{(s)}}\colon R_i^{(s)} \to \pi_i\left(R_i^{(s)}\right)$$

is an isomorphism of additive groups for every $s \in S$ and $i = 1, 2$. Define $\varphi\colon R_1 \to R_2$ by

$$\varphi(r) := \left(\pi_2\big|_{R_2^{(s)}}\right)^{-1} \bar\varphi\pi_1(r) \text{ for } r \in R_1^{(s)} \text{ and } s \in S$$

and extend it additively. Clearly, $\varphi\left(R_1^{(s)}\right) = R_2^{(s)}$ and φ is a graded surjective homomorphism of additive groups. Moreover $\pi_2\varphi = \bar\varphi\pi_1$ holds.

Suppose $\varphi\left(\sum_{s\in S} r^{(s)}\right) = 0$ for some $r^{(s)} \in R_1^{(s)}$ and $s \in S$. Since φ is graded, we have $\varphi\left(r^{(s)}\right) = 0$ for every $s \in S$. Hence $\pi_1\left(r^{(s)}\right) = 0$ and thus $r^{(s)} = 0$, since by Lemma 4.1.4 we have $R_1^{(s)} \cap J(R_1) = 0$ for every $s \in S$. Therefore, φ is a bijection.

Now we prove that φ is an isomorphism of rings. Indeed, suppose $r^{(s)} \in R_1^{(s)}$ and $r^{(t)} \in R_1^{(t)}$. Then

$$\pi_2\varphi\left(r^{(s)}r^{(t)}\right) = \bar\varphi\pi_1\left(r^{(s)}r^{(t)}\right) = \bar\varphi\pi_1\left(r^{(s)}\right)\bar\varphi\pi_1\left(r^{(t)}\right) = \pi_2\left(\varphi\left(r^{(s)}\right)\varphi\left(r^{(t)}\right)\right).$$

Since both $\varphi\left(r^{(s)}r^{(t)}\right)$ and $\varphi\left(r^{(s)}\right)\varphi\left(r^{(t)}\right)$ belong to $R_2^{(st)}$ and $\pi_2\big|_{R_2^{(st)}}$ is an isomorphism, we get

$$\varphi\left(r^{(s)}r^{(t)}\right) = \varphi\left(r^{(s)}\right)\varphi\left(r^{(t)}\right) \text{ for all } r^{(s)} \in R_1^{(s)} \text{ and } r^{(t)} \in R_1^{(t)}$$

and the first assertion is proved.

The second assertion is obvious since the Jacobson radical is stable under isomorphisms. ∎

4.2 Left ideals of matrix algebras

Here we state some propositions which turn out to be very useful in order to classify all possible finite dimensional T-graded-simple algebras for some (right) zero band T.

These results are known, however, for the reader's convenience, we include their proofs.

Lemma 4.2.1. *Let F be a field and let $k \in \mathbb{N}$. Consider the natural $M_k(F)$-action on the coordinate space F^k by linear operators. Then there exists a one-to-one correspondence between left ideals I in $M_k(F)$ and subspaces $W \subseteq F^k$ such that*

$$I = \operatorname{Ann} W := \{a \in M_k(F) \mid aW = 0\}, \qquad W = \bigcap_{a \in I} \ker a, \tag{4.1}$$

and $\dim I = k(k - \dim W)$. Moreover, if $I_1 = \operatorname{Ann} W_1$ and $I_2 = \operatorname{Ann} W_2$, then $I_1 + I_2 = \operatorname{Ann}(W_1 \cap W_2)$ and $I_1 \cap I_2 = \operatorname{Ann}(W_1 + W_2)$.

Proof. Let W be a subspace of F^k. Let $w_{k+1-\dim W}, \ldots, w_k$ be a basis in W. Choose $w_1, \ldots, w_{k-\dim W} \in F^k$ such that $w_1, \ldots, w_k$ is a basis in F^k. Then $\operatorname{Ann} W$ consists of all $a \in M_k(F)$ that have columns in the basis $w_1, \ldots, w_k$ with zeros in the last $\dim W$ columns. Note that $\bigcap_{a \in \operatorname{Ann} W} \ker a = W$ and $\dim_F \operatorname{Ann} W = k(k - \dim_F W)$.

Let $I \subseteq M_k(F)$ be a left ideal. Since I is a left ideal in the semisimple artinian algebra $M_k(F)$, by [Her68, Theorem 1.4.2], there exists an idempotent $e \in I$ such that $I = M_k(F)e$. Thus $I(\ker e) = 0$. Note that e is acting on F^k as a projection. Hence $F^k = \ker e \oplus \operatorname{im} e$. We choose a basis in F^k that is the union of bases in $\operatorname{im} e$ and $\ker e$. Then the matrix of e in this basis is $\left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and I contains all operators with zeros in the last $\dim \ker e$ columns. Thus $\bigcap_{a \in I} \ker a = \ker e$ and $\operatorname{Ann} \ker e = I$. Together with the first paragraph this implies that (4.1) is indeed a one-to-one correspondence.

Suppose $I_1 = \operatorname{Ann} W_1$ and $I_2 = \operatorname{Ann} W_2$. Then

$$I_1 \cap I_2 = \operatorname{Ann} W_1 \cap \operatorname{Ann} W_2 = \operatorname{Ann}(W_1 + W_2).$$

Moreover, $(I_1 + I_2)(W_1 \cap W_2) = 0$ and $I_1 + I_2 \subseteq \operatorname{Ann}(W_1 \cap W_2)$. Now

$$\begin{aligned} \dim_F(I_1 + I_2) &= \dim_F I_1 + \dim_F I_2 - \dim_F(I_1 \cap I_2) \\ &= k(2k - \dim_F W_1 - \dim_F W_2) - k(k - \dim_F(W_1 + W_2)) \\ &= k(k - (\dim_F W_1 + \dim_F W_2 - \dim_F(W_1 + W_2))) \\ &= \dim_F \operatorname{Ann}(W_1 \cap W_2) \end{aligned}$$

implies the lemma. ∎

Theorem 4.2.2. *Let $k, s \in \mathbb{N}$ and let F be a field. Assume I_i are left ideals of $M_k(F)$ such that $M_k(F) = \bigoplus_{i=1}^s I_i$. Suppose $\dim I_i = n_i k$, $n_i \in \mathbb{Z}_+$. Then there exists $P \in \mathrm{GL}_k(F)$ such that $P^{-1} I_i P$ consists of all matrices with zeros in all columns except those that have numbers*

$$1 + \sum_{j=1}^{i-1} n_j,\ 2 + \sum_{j=1}^{i-1} n_j,\ \dots,\ n_i + \sum_{j=1}^{i-1} n_j.$$

Proof. Consider the standard action of $M_k(F)$ on the coordinate space F^k. By Lemma 4.2.1, $I_i = \operatorname{Ann} V_i$ for some $V_i \subseteq F^k$. Applying the duality from Lemma 4.2.1 to $M_k(F) = \bigoplus_{i=1}^s I_i$, we get $\bigcap_{i=1}^s V_i = 0$ and

$$V_i + \bigcap_{\substack{j=1,\\ j\ne i}}^{s} V_j = F^k \text{ for all } 1 \le i \le s.$$

Denote $W_i = \bigcap\limits_{\substack{j=1,\\ j\ne i}}^{s} V_j$. Then $F^k = V_i \oplus W_i$. Note that

$$\operatorname{Ann} W_i = \bigoplus_{\substack{j=1,\\ j\ne i}}^{s} I_j.$$

Since $\bigcap\limits_{\substack{j=1,\\ j\ne i}}^{s} \operatorname{Ann} W_j = I_i$, we have $V_i = \bigoplus\limits_{\substack{j=1,\\ j\ne i}}^{s} W_j$.

Now, choose a basis in F^k that is a union of bases in W_i. Denote the transition matrix from the standard basis to this basis by $P \in \mathrm{GL}_k(F)$. Then each $P^{-1} I_i P$ consists of all matrices with zeros in all columns except those that correspond to W_i. ∎

Lemma 4.2.3. *Let I be a minimal left ideal of $M_k(F)$ where $k \in \mathbb{N}$ and F is a field. Then there exist $\mu_j \in F$, $1 \le j \le k$, such that $I = span_F\{\sum\limits_{j=1}^{k} \mu_j e_{ij} \;\Big|\; 1 \le i \le k\}$.*

Proof. Let $a = \sum\limits_{i,j=1}^{k} \mu_{ij} e_{ij} \in I \backslash \{0\}$. Since $\sum_{\ell=1}^k e_{\ell\ell} a = a$, we have $e_{\ell\ell} a \ne 0$ for some $1 \le \ell \le k$. Define $\mu_j := \mu_{\ell j}$ for all $1 \le j \le k$. Then

$$\operatorname{span}_F\{\sum_{j=1}^{k} \mu_j e_{ij} \;\Big|\; 1 \le i \le k\} = \operatorname{span}_F\{e_{i\ell} a \mid 1 \le i \le k\}$$

is a left ideal contained in I. Since I is a minimal left ideal, we get the lemma. ∎

Lemma 4.2.4. *Let D be a finite dimensional division algebra over a field F and let $k \in \mathbb{N}$. Let I and V be, respectively, a left and a right ideal of $M_k(D)$. Then* $\dim_F(VI) = \frac{\dim_F V \dim_F I}{k^2 \dim_F D}$.

Proof. Note that

$$I \cong \underbrace{M_k(D)e_{11} \oplus \ldots \oplus M_k(D)e_{11}}_{\dim_F I/(k \dim_F D)} \text{ and } V \cong \underbrace{e_{11}M_k(D) \oplus \ldots \oplus e_{11}M_k(D)}_{\dim_F V/(k \dim_F D)}$$

as respectively, left and right $M_k(D)$-modules. Hence

$$\begin{aligned} \dim_F(VI) &= \frac{\dim_F I}{k \dim_F D} \dim_F(VM_k(D)e_{11}) \\ &= \frac{\dim_F V \dim_F I}{k^2(\dim_F D)^2} \dim_F(e_{11}M_k(D)e_{11}) = \frac{\dim_F V \dim_F I}{k^2 \dim D}. \end{aligned}$$

∎

4.3 Graded-simple algebras

Throughout this section A is a finite dimensional S-graded F-algebra, where F is a field and

$$S = \mathcal{M}(\{e\}^0, n, m; P) = \langle \operatorname{supp}(A) \rangle \cup \{\theta\}$$

is a finite completely 0-simple semigroup having trivial maximal subgroups. Denote the homogeneous component corresponding to (e, i, j) by A_{ij}. Then

$$A = \bigoplus_{\substack{1 \le i \le n, \\ 1 \le j \le m}} A_{ij}$$

and

$$A_{ij}A_{k\ell} \subseteq A_{i\ell}.$$

In particular, each homogeneous component A_{ij} is a subalgebra of A. If $p_{jk} = 0$, where $P := (p_{jk})_{j,k}$, then $A_{ij}A_{k\ell} = 0$.

Note that A is $\mathcal{M}(\{e\}^0, n, m; P)$-graded-simple for some matrix P if and only if A is $\mathcal{M}(\{e\}^0, n, m; P')$-graded-simple with P' the matrix with all the entries being equal to e.

We begin with some basic observations.

Lemma 4.3.1. *The following properties hold for an S-graded-simple algebra A:*

1. $A_{ij} \cap J(A) = 0$ *for all* i, j;

2. *if* $I \subseteq A$ *is a set, then* AIA *is a homogeneous ideal (and thus* AIA *equals either* 0 *or* A*);*

3. $AJ(A)A = 0$.

Proof. Part (1) is a direct consequence of Lemma 4.1.4. Part (2) is obvious, Part (3) is a direct consequence of (2). ∎

If $n = 1$, i.e S has only one row, then all the graded components are left ideals and thus $J(A)A$ is homogeneous. So, if A is graded-simple, then $J(A)A = 0$. Due to this, one can reformulate Theorem 4.3.7 in a simpler form in this case. In the lower bound part of Chapter 5 we will be working in this setting.

Define now the left, respectively right ideals

$$L_i := \bigoplus_{k=1}^{n} A_{ki} \text{ and } R_i := \bigoplus_{k=1}^{m} A_{ik}. \tag{4.2}$$

Note that $L_j \cap R_i = A_{ij}$.

The following theorem is the best one can expect for a graded version of the Wedderburn-Malcev Theorem 1.3.2 for S-graded-simple algebras, since $J(A)$ is not graded by the previous Lemma. It is namely proven that there exist orthogonal "column" (respectively, "row") homogeneous idempotents that define a semisimple complement of the radical. This result is a first step towards the classification of S-graded-simple algebras.

Theorem 4.3.2. *Let* $A = \bigoplus_{i,j} A_{ij}$ *be a finite dimensional* S*-graded* F*-algebra over a field* F *such that* $AJ(A)A = 0$*. Then, there exist orthogonal idempotents* $f_1, \ldots, f_m$ *and orthogonal idempotents* $f'_1, \ldots, f'_n$ *(some of them could be zero) such that*

$$B = \bigoplus_{i,j} f'_i A f_j = \bigoplus_{i,j} (B \cap A_{ij})$$

is an S*-graded maximal semisimple subalgebra of* A*,* $f'_i \in B \cap R_i$ *for* $1 \leq i \leq n$*,* $f_j \in B \cap L_j$ *for* $1 \leq j \leq m$*,* $\sum_{i=1}^{n} f'_i = \sum_{j=1}^{m} f_j = 1_B$*, and* $A = B \oplus J(A)$ *(direct sum of subspaces).*

Proof. We write $\bar{X}$ for the image of a subset X of A in the algebra $A/J(A)$ under the natural epimorphism $A \to A/J(A)$.

Note that $\bar{A} = \sum_{j=1}^{m} \bar{L}_j$. Since $\bar{A} = A/J(A)$ is semisimple and completely reducible as a left $A/J(A)$-module, there exist left ideals $\tilde{L}_i \subseteq \bar{L}_i$ complementary to $\bar{L}_i \bigcap \sum\limits_{j=1}^{i-1} \bar{L}_j$ in $\bar{L}_i$. Clearly, $\bar{A} = \bigoplus\limits_{i=1}^{m} \tilde{L}_i$. The decomposition $1_{\bar{A}} = \sum\limits_{i=1}^{m} \bar{\omega}_i$ of the identity element of $\bar{A}$ yields orthogonal idempotents $\bar{\omega}_i \in \tilde{L}_i$. The idempotents $\bar{\omega}_i$ can be lifted to homogeneous idempotents $\omega_i \in L_i$ of A using the natural epimorphisms $\pi\big|_{L_i} \colon L_i \to L_i/L_i \cap J(A)$ since $J(A)$ is nilpotent. The idempotents $\omega_1, \ldots, \omega_m$ are orthogonal too since $\omega_i\omega_j = \omega_i(\omega_i\omega_j)\omega_j \in AJ(A)A = 0$.

Analogously, one gets orthogonal idempotents $\omega'_1, \ldots, \omega'_n \in A$, $\omega'_i \in R_i$ for $1 \leq i \leq n$, such that $\sum_{i=1}^{n} \bar{f}'_i = 1_{\bar{A}}$. Define now $B = \bigoplus\limits_{\substack{1\leq i\leq n,\\ 1\leq j\leq m}} \omega'_i A \omega_j$. Note that $\omega'_i A\omega_j \subseteq A_{ij}$ and B is an S-graded subalgebra of A. Suppose $a = \sum\limits_{\substack{1\leq i\leq n,\\ 1\leq j\leq m}} \omega'_i a_{ij}\omega_j \in J(A)$ for some $a_{ij} \in A$. Then $AJ(A)A = 0$ implies $\omega'_i a_{ij}\omega_j = \omega'_i a\omega_j = 0$ for all i, j. Hence $a = 0$ and $J(A) \cap B = 0$. Moreover, $\bar{B} = 1_{\bar{A}}\bar{A}1_{\bar{A}} = \bar{A}$. Hence B is an S-graded maximal semisimple subalgebra of A and $A = B \oplus J(A)$ (direct sum of subspaces).

Decomposing 1_B with respect to the left ideals $\bigoplus_{i=1}^{n} \omega'_i A\omega_j$, $1 \leq j \leq m$, and with respect to the right ideals $\bigoplus_{j=1}^{m} \omega'_i A\omega_j$, $1 \leq i \leq n$, we get orthogonal idempotents $f_i \in B \cap L_i$ for $1 \leq i \leq n$, and orthogonal idempotents $f'_j \in B \cap R_j$ for $1 \leq j \leq m$ such that $\sum_{i=1}^{n} f'_i = \sum_{j=1}^{m} f_j = 1_B$. Then

$$B = 1_B B 1_B = \bigoplus_{\substack{1\leq i\leq n,\\ 1\leq j\leq m}} f'_i B f_j = \bigoplus_{\substack{1\leq i\leq n,\\ 1\leq j\leq m}} f'_i A f_j$$

since $A = B \oplus J(A)$ and $AJ(A)A = 0$. ∎

Example 4.3.3 below shows that the S-gradings on different B in Theorem 4.3.7 can be non-isomorphic. Interestingly, for connected $\mathbb{N}$-gradings generated in degree 1, such phenomena can not happen by [BZ17].

Example 4.3.3. Let F be a field, let I be the left $M_2(F)$-module isomorphic to $\operatorname{span}_F\{e_{12}, e_{22}\}$, and let $\varphi \colon I \xrightarrow{\sim} \operatorname{span}_F\{e_{12}, e_{22}\}$ be the corresponding isomorphism. Let $A = M_2(F) \oplus I$, direct sum of $M_2(F)$-modules, where $IM_2(F) = I^2 = 0$. Define on A the following T_3-grading:

$$A^{(e_1)} = (M_2(F), 0) \text{ and } A^{(e_2)} = \{(\varphi(a), a) \mid a \in I\}.$$

Then the algebra A is T_3-graded-simple and both

$$B_1 = A^{(e_1)} \text{ and } B_2 = \operatorname{span}_F\{(e_{11},0),(e_{21},0)\} \oplus A^{(e_2)}$$

are graded maximal semisimple subalgebras of A. However $B_1 \ncong B_2$ as graded algebras.

Now we present a finite dimensional S-graded non-graded-simple algebra that does not have an S-graded maximal semisimple subalgebra complementary to the radical. So, in Theorem 4.3.2, the assumption $AJ(A)A = 0$ is essential.

Example 4.3.4. Let $R = F[X]/(X^2)$ and let $A = M_2(R)$. Put $v_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), v_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), w_1 = \left(\begin{smallmatrix} 1 & X \end{smallmatrix}\right), w_2 = \left(\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right)$. Consider the following F-subspaces of A:

$$A_{11} = Rv_1w_1 = R\left(\begin{smallmatrix} 1 & X \\ 0 & 0 \end{smallmatrix}\right), \quad A_{12} = Rv_1w_2 = R\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),$$

$$A_{21} = Rv_2w_1 = R\left(\begin{smallmatrix} 0 & 0 \\ 1 & X \end{smallmatrix}\right), \quad A_{22} = Rv_2w_2 = R\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Then $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ is an S-grading for $S = \mathcal{M}\left(\{e\}^0, 2, 2, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$. However, there does not exist an S-graded maximal semisimple subalgebra B of A such that $A = B \oplus J(A)$.

Proof. First we notice that $A = A_{11} \oplus A_{12} \oplus A_{21} \oplus A_{22}$ is indeed an S-grading since $(a\, v_i w_j)(b\, v_\ell w_k) = ab(w_j v_\ell)\, v_i w_k$ for all $a, b \in R$ and $1 \leq i, j, k, \ell \leq 2$ since $w_j v_\ell$ is a 1×1 matrix which can be identified with the corresponding element of the field F. Clearly, $J(A) = \left(\begin{smallmatrix} (X) & (X) \\ (X) & (X) \end{smallmatrix}\right)$ and $A/J(A) \cong M_2(F)$.

Fix the following bases in the homogeneous components:

$$A_{11} = \operatorname{span}_F\{\left(\begin{smallmatrix} 1 & X \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right)\}, \quad A_{12} = \operatorname{span}_F\{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)\},$$

$$A_{21} = \operatorname{span}_F\{\left(\begin{smallmatrix} 0 & 0 \\ 1 & X \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ X & 0 \end{smallmatrix}\right)\}, \quad A_{22} = \operatorname{span}_F\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & X \end{smallmatrix}\right)\}.$$

Now it is clear that $J(A)$ is a homogeneous ideal and $A/J(A) \cong M_2(F)$ is an S-graded algebra too.

Suppose that there exists an S-graded maximal semisimple subalgebra $B = \bigoplus\limits_{1 \leq i,j \leq 2} B_{ij}$ such that $A = B \oplus J(A)$ and $B_{ij} \subseteq A_{ij}$. Then there exists a graded isomorphism $\varphi\colon B \to M_2(F)$.

In particular, $B_{ii} = \operatorname{span}_F\{b_{ii}\}$ where $\varphi(b_{ii}) = e_{ii}$ and $b_{ii}^2 = b_{ii}$ for $i = 1, 2$. Hence $b_{11} = (1 + \alpha X)\left(\begin{smallmatrix} 1 & X \\ 0 & 0 \end{smallmatrix}\right)$ for some $\alpha \in F$ and $b_{22} = (1 + \beta X)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $\beta \in F$. (In fact, if $\operatorname{char} F \neq 2$, then $\alpha = \beta = 0$.) Then

$$b_{11}b_{22} = (1 + (\alpha+\beta)X)\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right) \in J(A)$$

and we get a contradiction. ∎

Theorem 4.3.2 describes the semisimple part of an S-graded-simple algebra. We proceed with the description of the radical and hence we obtain a characterization of the finite dimensional S-graded-simple algebras. In Section 4.4 we will show that this description delivers a complete classification.

For $r \in A$, we denote $x - xr$ (respectively $x - rx$) by $x(1-r)$ (respectively, $(1-r)x$), even if A does not contain a unit element.

Lemma 4.3.5. *Suppose A is S-graded-simple and let $A = B \oplus J(A)$ be the direct vector space decomposition from Theorem 4.3.2. Then the following properties hold:*

1. $J(A)^2 A = AJ(A)^2 = 0$;
2. *B is a simple subalgebra;*
3. $A = A1_B A$;
4. $J(A) = (1-1_B)A1_B \oplus 1_B A(1-1_B) \oplus J(A)^2$ *(direct sum of subspaces);*
5. $J(A)^2 = (1-1_B)A1_B A(1-1_B) = (1-1_B)A(1-1_B)$.

Proof. Part (3) is a direct consequence of Part (2) of Lemma 4.3.1. Let f be a primitive central idempotent of B. Then $A = AfA$ and, by Part (3) of Lemma 4.3.1,

$$B = 1_B(B \oplus J(A))1_B = 1_B A 1_B = 1_B A f A 1_B = 1_B A 1_B f 1_B A 1_B = BfB = Bf = fB.$$

Hence f is a left identity of B and thus $f = 1_B$. Therefore, B is simple and we get Part (2).

Using $B = 1_B A 1_B$ and the Pierce decomposition with respect to the idempotent 1_B, we get

$$J(A) = (1-1_B)A1_B \oplus 1_B A(1-1_B) \oplus (1-1_B)A(1-1_B) \text{ (direct sum of subspaces).}$$

Part (3) implies

$$(1-1_B)A(1-1_B) = (1-1_B)A1_B A(1-1_B) \subseteq J(A)^2.$$

Hence

$$(1-1_B)A(1-1_B)J(A) \subseteq J(A)^3 = 0 \text{ and } J(A)(1-1_B)A(1-1_B) \subseteq J(A)^3 = 0.$$

Since

$$1_B A(1-1_B)A1_B \subseteq 1_B J(A)1_B \subseteq AJ(A)A = 0,$$

we get $J(A)^2 \subseteq (1-1_B)A(1-1_B)$ and statements (1), (4) and (5) follow. ∎

Remark that if S has only one row then $a - 1_B a \subseteq J(A) \cap A_{1i} = 0$ for every $a \in A_{1i}$ and $1 \le i \le m$. Thus in this case 1_B acts as a left identity on $J(A)$.

It is also interesting to note that condition (1) of Lemma 4.3.1 and condition (2) of Lemma 4.3.5, together with $A^2 = A$, are equivalent to the graded S-simplicity.

Proposition 4.3.6. *Suppose that the base field F is perfect, $A/J(A)$ is a simple algebra, $A^2 = A$, and $A_{ij} \cap J(A) = 0$ for all $1 \le i \le n$ and $1 \le j \le m$. Then A is S-graded-simple.*

Proof. Let I be a nonzero two-sided homogeneous ideal of A. Denote by $\pi\colon A \twoheadrightarrow A/J(A)$ the natural epimorphism. Then $\pi(I) \ne 0$. Since $A/J(A)$ is simple, we get $\pi(I) = A/J(A)$ and $A = I + J(A)$. By the Wedderburn-Malcev Theorem 1.3.2, there exists a maximal semisimple subalgebra $B \subseteq I$ such that $I = B \oplus J(I)$ (direct sum of subspaces). Recall that $J(I) = J(A) \cap I$. Thus $A = B \oplus J(A)$. Note that $\pi(A(1-1_B)A) = 0$. Hence $A(1-1_B)A \subseteq J(A)$. Since $A(1-1_B)A$ is a graded ideal, we get $A(1-1_B)A = 0$ and $ab = a1_B b \in I$ for all $a, b \in A$. Thus $A = A^2 \subseteq I$ and $I = A$. ∎

From Lemma 4.3.5 we have that

$$J(A) = 1_B A(1-1_B) \oplus (1-1_B)A1_B \oplus J(A)^2 = \sum_{j=1}^{m} 1_B L_j (1-1_B) \oplus \sum_{i=1}^{n} (1-1_B) R_i 1_B \oplus J(A)^2.$$

Therefore, consider for $1 \le i \le n$ and $1 \le j \le m$ the subspaces

$$J_{ij}^{10} := f_i' L_j (1-1_B) \quad \text{and} \quad J_{ij}^{01} := (1-1_B) R_i f_j.$$

Also, put

$$J_{*j}^{10} := \sum_{1 \le i \le n} J_{ij}^{10} = 1_B L_j (1-1_B) \quad \text{and} \quad J_{i*}^{01} := \sum_{1 \le j \le m} J_{ij}^{01} = (1-1_B) R_i 1_B.$$

We will show that these subspaces form the building blocks of $J(A)$.

Theorem 4.3.7. *Let A be a finite dimensional S-graded-simple F-algebra. Let B and let $f_1, \dots, f_m,\ f'_1, \dots, f'_n$ be, respectively, a graded subalgebra and orthogonal idempotents from Theorem 4.3.2.*

Then each J^{10}_{*j} is a left B-submodule of $J(A)$ and $J^{10}_{*j} = \bigoplus_{i=1}^n J^{10}_{ij}$. Also each J^{01}_{i*} is a right B-submodule of $J(A)$ and $J^{01}_{i*} = \bigoplus_{j=1}^m J^{01}_{ij}$. Moreover,

$$J(A) = \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \oplus J(A)^2 \quad \text{and} \quad J(A)^2 = \bigoplus_{i=1}^n \bigoplus_{j=1}^m J^{01}_{i*} J^{10}_{*j},$$

direct sums of subspaces.

In addition, there exists an F-linear map

$$\varphi\colon \bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j} \to B$$

defined by

$$\varphi(a - 1_B a 1_B) = 1_B a 1_B - f'_i a f_j, \quad \text{for } a \in f'_i A_{ij} + A_{ij} f_j, \tag{4.3}$$

and such that $\varphi\big|_{\bigoplus_{j=1}^m J^{10}_{*j}}$ is a homomorphism of left B-modules,

$$J^{10}_{*j} \cap \ker\varphi = 0, \qquad \varphi(J^{10}_{*j}) \cap B f_j = \varphi(J^{10}_{*j}) f_j = 0, \qquad \text{for every } 1 \le j \le m, \tag{4.4}$$

$\varphi\big|_{\bigoplus_{i=1}^n J^{01}_{i*}}$ is a homomorphism of right B-modules,

$$J^{01}_{i*} \cap \ker\varphi = 0, \qquad \varphi(J^{01}_{i*}) \cap f'_i B = f'_i \varphi(J^{01}_{i*}) = 0, \qquad \text{for every } 1 \le i \le n. \tag{4.5}$$

Moreover,

$$\begin{aligned} A_{ij} = f'_i B f_j &\oplus \left\{ \varphi(v) + v \mid v \in J^{10}_{ij} \oplus J^{01}_{ij} \right\} \\ &\oplus span_F \left\{ \varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J^{01}_{i*},\ w \in J^{10}_{*j} \right\} \end{aligned} \tag{4.6}$$

is a direct sum of subspaces, for all $1 \le i \le n$, $1 \le j \le m$.

If $s \in \mathbb{N}$, $v_\ell \in J^{01}_{i*}$ and $w_\ell \in J^{10}_{*j}$ for $1 \le \ell \le s$, then $\sum_{\ell=1}^s v_\ell w_\ell = 0$ if and only if $\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = 0$.

Furthermore, $B \cong M_k(D)$ for some $k \in \mathbb{N}$ and a division algebra D satisfying

$$\begin{aligned} \dim_F \bigoplus_{i=1}^n J^{01}_{i*} &\le (n-1)\dim_F B = (n-1)k^2 \dim_F D, &(4.7)\\ \dim_F \bigoplus_{j=1}^m J^{10}_{*j} &\le (m-1)\dim_F B = (m-1)k^2 \dim_F D, &(4.8)\\ \dim_F J(A) &\le (nm-1)\dim_F B = (|S|-1)\dim_F B = (|S|-1)k^2 \dim_F D. &(4.9) \end{aligned}$$

Proof. By Lemma 4.3.5,

$$J(A) = 1_B A(1-1_B)\oplus(1-1_B)A1_B\oplus J(A)^2 = \sum_{j=1}^m 1_B L_j(1-1_B)\oplus\sum_{i=1}^n(1-1_B)R_i 1_B\oplus J(A)^2.$$

Note that if $\sum_{j=1}^m 1_B a_j(1-1_B) = 0$ for some $a_j \in L_j$, then $\sum_{j=1}^m 1_B a_j 1_B = \sum_{j=1}^m 1_B a_j$. Since $1_B A 1_B = B$ is a graded subalgebra, we get $1_B a_j \in B \cap L_j$, $1_B a_j 1_B = 1_B a_j$ and all $1_B a_j(1-1_B) = 0$. Hence the sum $\bigoplus_{j=1}^m 1_B L_j(1-1_B)$ is direct. Analogously, the sum $\bigoplus_{i=1}^n (1-1_B)R_i 1_B$ is direct too and

$$J(A) = \bigoplus_{j=1}^m 1_B L_j(1-1_B) \oplus \bigoplus_{i=1}^n (1-1_B)R_i 1_B \oplus J(A)^2 = \bigoplus_{j=1}^m J^{10}_{*j} \oplus \bigoplus_{i=1}^n J^{01}_{i*} \oplus J(A)^2.$$

Using Part (5) of Lemma 4.3.5, we get

$$J(A)^2 = \sum_{i=1}^n\sum_{j=1}^m (1-1_B)R_i 1_B 1_B L_j(1-1_B) = \sum_{i=1}^n\sum_{j=1}^m J^{01}_{i*}J^{10}_{*j}. \tag{4.10}$$

Now we show that φ can be defined by (4.3). First, $f'_i L_j \subseteq A_{ij}$ and thus $f'_i L_j = f'_i A_{ij}$, $R_i f_j \subseteq A_{ij}$ and thus $R_i f_j = A_{ij} f_j$, $J^{10}_{ij} = f'_i L_j(1-1_B) = \{a - 1_B a 1_B \mid a \in f'_i A_{ij}\}$, $J^{01}_{ij} = (1-1_B)R_i f_j = \{a - 1_B a 1_B \mid a \in A_{ij} f_j\}$. If $a - 1_B a 1_B = 0$ for some $a \in A_{ij}$, then $a \in B \cap A_{ij} = f'_i B f_j$ and $1_B a 1_B - f'_i a f_j = 0$. Hence (4.3) can indeed be used to define φ on J^{10}_{ij} and J^{01}_{ij} and the definition is consistent. We extend φ on $\bigoplus_{i=1}^n J^{01}_{i*} \oplus \bigoplus_{j=1}^m J^{10}_{*j}$ by linearity.

Note that if $a \in f'_i A_{ij} = f'_i L_j$, then we have $a - 1_B a 1_B = a(1-1_B)$ and $\varphi(a(1-1_B)) = a(1_B - f_j)$. By the linearity, this formula holds for every $a \in 1_B L_j$. Hence $\varphi\big|_{J^{10}_{*j}}$ is a homomorphism of left B-modules. By the linearity, $\varphi\big|_{\bigoplus_{j=1}^m J^{10}_{*j}}$ is a homomorphism of left B-modules too. Analogously, $\varphi((1-1_B)a) = (1_B - f'_i)a$ for all $a \in R_i 1_B$ and $\varphi\big|_{\bigoplus_{i=1}^n J^{01}_{i*}}$ is a homomorphism of right B-modules.

Suppose $\varphi(1_B a(1-1_B)) = 0$ for some $a \in L_j$. Then $1_B a(1_B - f_j) = 0$, $1_B a f_j = 1_B a 1_B$ and $f'_i a f_j = f'_i a 1_B$ for all $1 \leq i \leq n$. Hence $f'_i a(1-1_B) = f'_i a - f'_i a f_j \in A_{ij} \cap J(A) = 0$. Thus $1_B a(1-1_B) = 0$ and $1_B L_j(1-1_B) \cap \ker\varphi = J^{10}_{*j} \cap \ker\varphi = 0$. Analogously,

$$(1-1_B)R_i 1_B \cap \ker\varphi = J^{01}_{i*} \cap \ker\varphi = 0.$$

Moreover

$$\varphi(J^{10}_{*j}) \cap Bf_j = \varphi(1_B L_j(1-1_B)) \cap Bf_j \subseteq \varphi(1_B L_j(1-1_B))f_j = 0,$$

for every $1 \leq j \leq m$, and (4.4) is proven. Analogously,

$$\varphi(J_{i*}^{01}) \cap f_i'B = \varphi((1-1_B)R_i 1_B) \cap f_i'B \subseteq f_i'\varphi((1-1_B)R_i 1_B) = 0,$$

for every $1 \leq i \leq n$, and (4.5) is proven.

By Part (2) of Lemma 4.3.5, we have $B \cong M_k(D)$ for some $k \in \mathbb{N}$ and a division algebra D. Now

$$\begin{aligned}\dim_F \bigoplus_{i=1}^n J_{i*}^{01} = \sum_{i=1}^n \dim_F \varphi(J_{i*}^{01}) &\leq \sum_{i=1}^n (\dim_F B - \dim_F f_i'B) \\ &= (n-1)\dim_F B = (n-1)k^2 \dim_F D\end{aligned}$$

and we get (4.7). Analogously we obtain (4.8).

Now we prove (4.6). Let

$$\begin{aligned}\tilde{A}_{ij} = &\left(f_i'Bf_j \oplus \left\{\varphi(v) + v \mid v \in J_{ij}^{10} \oplus J_{ij}^{01}\right\}\right) \\ &+\operatorname{span}_F \left\{\varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10}\right\}.\end{aligned}$$

We first show that $\tilde{A}_{ij} = A_{ij}$. To do so, note that if $a \in f_i'L_j$, then $\varphi(a(1-1_B)) + a(1-1_B) = a(1_B - f_j) + a(1-1_B) = a - af_j \in A_{ij}$. Therefore $\varphi(w) + w \in A_{ij}$ for every $w \in J_{ij}^{10}$. Analogously, $\varphi(v) + v \in A_{ij}$ for every $v \in J_{ij}^{01}$. Hence $\tilde{A}_{ij} \subseteq A_{ij}$.

Obviously, $B \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$ and therefore $1_B \in \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$. By Lemma 4.3.5,

$$1_B \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}(1-1_B) = \bigoplus_{j=1}^m J_{*j}^{10}$$

and

$$(1-1_B) \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij} 1_B = \bigoplus_{i=1}^n J_{i*}^{01}.$$

Hence

$$\bigoplus_{i=1}^n J_{i*}^{01} \oplus \bigoplus_{j=1}^m J_{*j}^{10} \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}.$$

In addition, (4.10) implies

$$(1-1_B) \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}(1-1_B) = J^2(A)$$

and $J(A)^2 \subseteq \bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij}$. Hence $\bigoplus_{i=1}^n \bigoplus_{j=1}^m \tilde{A}_{ij} = A$ and $\tilde{A}_{ij} = A_{ij}$. Equality (4.6) will follow from the fact that the sum in the definition of $\tilde{A}_{ij}$ is direct. We prove the last fact below.

Suppose $s \in \mathbb{N}$ and $v_\ell \in J^{01}_{i*}$ and $w_\ell \in J^{10}_{*j}$ for $1 \le \ell \le s$. Assume $\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = 0$. Since $\varphi\big|_{\bigoplus_{j=1}^m J^{01}_{i*}}$ is a homomorphism of right B-modules, we get that $\varphi\left(\sum_{\ell=1}^s v_\ell\varphi(w_\ell)\right) = 0$ and thus, by (3), $\sum_{\ell=1}^s v_\ell\varphi(w_\ell) = 0$. Analogously, $\sum_{\ell=1}^s \varphi(v_\ell)w_\ell = 0$. Hence

$$\sum_{\ell=1}^s v_\ell w_\ell = \sum_{\ell=1}^s (\varphi(v_\ell)+v_\ell)(\varphi(w_\ell)+w_\ell) \in A_{ij} \cap J(A) = 0.$$

Conversely, suppose $\sum_{\ell=1}^s v_\ell w_\ell = 0$ for some $v_\ell \in J^{01}_{i*}$ and $w_\ell \in J^{10}_{*j}$, $1 \le \ell \le s$, $s \in \mathbb{N}$. Let $a = \sum_{\ell=1}^s (\varphi(v_\ell)+v_\ell)(\varphi(w_\ell)+w_\ell)$,

$$b = \sum_{\ell=1}^s (\varphi(\varphi(v_\ell)w_\ell) + \varphi(v_\ell)w_\ell) + \sum_{\ell=1}^s (\varphi(v_\ell\varphi(w_\ell)) + v_\ell\varphi(w_\ell)) - \sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell).$$

Then $a - b = \sum_{\ell=1}^s v_\ell w_\ell = 0$. Thus $b = a \in A_{ij}$. However

$$\begin{aligned} b = &\sum_{q=1}^n \sum_{\ell=1}^s (f'_q\varphi(\varphi(v_\ell)w_\ell) + f'_q\varphi(v_\ell)w_\ell) \\ &+ \sum_{r=1}^m \sum_{\ell=1}^s (\varphi(v_\ell\varphi(w_\ell))f_r + v_\ell\varphi(w_\ell)f_r) - \sum_{q=1}^n \sum_{r=1}^m \sum_{\ell=1}^s f'_q\varphi(v_\ell)\varphi(w_\ell)f_r. \end{aligned}$$

Taking the homogeneous component of b, corresponding to A_{ij}, i.e. the summand with $q = i$ and $r = j$, we obtain $a = b = 0$ since by (4.4) and (4.5) we have

$$f'_i\varphi(v_\ell) = \varphi(w_\ell)f_j = 0.$$

The projection of a on B with the kernel $J(A)$ yields $\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = 0$. Hence

$$\sum_{\ell=1}^s \varphi(v_\ell)\varphi(w_\ell) = \sum_{\ell=1}^s v_\ell\varphi(w_\ell) = \sum_{\ell=1}^s \varphi(v_\ell)w_\ell = 0.$$

Now we are ready to prove that the sum in the definition of $\tilde{A}_{ij}$ is direct. Suppose

$$\begin{aligned} \sum_{\ell=1}^s \left(\varphi(v_\ell)\varphi(w_\ell) + v_\ell\varphi(w_\ell) + \varphi(v_\ell)w_\ell + v_\ell w_\ell\right) &\in f'_iBf_j \oplus \left\{\varphi(v)+v \mid v \in J^{10}_{ij} \oplus J^{01}_{ij}\right\} \\ &\subseteq B \oplus \bigoplus_{j=1}^m J^{10}_{*j} \oplus \bigoplus_{i=1}^n J^{01}_{i*} \end{aligned}$$

for some $v_\ell \in J^{01}_{i*}$ and $w_\ell \in J^{10}_{*j}$. Since $\sum_{\ell=1}^s v_\ell w_\ell \in J(A)^2$ and

$$\sum_{\ell=1}^s \left(\varphi(v_\ell)\varphi(w_\ell) + v_\ell\varphi(w_\ell) + \varphi(v_\ell)w_\ell\right) \in B \oplus \bigoplus_{j=1}^m J^{10}_{*j} \oplus \bigoplus_{i=1}^n J^{01}_{i*},$$

we have $\sum_{\ell=1}^{s} v_\ell w_\ell = 0$ and, by the previous remarks,

$$\sum_{\ell=1}^{s} \varphi(v_\ell)\varphi(w_\ell) = \sum_{\ell=1}^{s} v_\ell \varphi(w_\ell) = \sum_{\ell=1}^{s} \varphi(v_\ell) w_\ell = 0$$

and $\sum_{\ell=1}^{s} \left(\varphi(v_\ell)\varphi(w_\ell) + v_\ell\varphi(w_\ell) + \varphi(v_\ell)w_\ell + v_\ell w_\ell\right) = 0$.

In particular, the sum in the definition of $\tilde{A}_{ij}$ is direct and the proof of (4.6) is complete.

Now we prove that the sum $J(A)^2 = \sum_{i=1}^{n}\sum_{j=1}^{m} J_{i*}^{01} J_{*j}^{10}$ is direct. Indeed, suppose $\sum_{i=1}^{n}\sum_{j=1}^{m} u_{ij} = 0$ for some $u_{ij} \in J_{i*}^{01} J_{*j}^{10}$. By (4.6), $u_{ij} = a_{ij} - v_{ij}$ where $a_{ij} \in A_{ij}$ and v_{ij} is a linear combination of homogeneous elements from B and homogeneous elements from $\left\{\varphi(v) + v \mid v \in J_{i*}^{01} \oplus J_{*j}^{10}\right\}$. Now $\sum_{i=1}^{n}\sum_{j=1}^{m}(a_{ij} - v_{ij}) = 0$ implies that each a_{ij} is a linear combination of elements from $f_i' B f_j$ and elements from $\left\{\varphi(v) + v \mid v \in J_{ij}^{01} \oplus J_{ij}^{10}\right\}$. Thus $u_{ij} = (1 - 1_B)u_{ij}(1 - 1_B) = (1 - 1_B)(a_{ij} - v_{ij})(1 - 1_B) = 0$ and the sum $J(A)^2 = \bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{i*}^{01} J_{*j}^{10}$ is indeed direct.

Only (4.9) still has to be proven. Note that Lemma 4.2.4 implies

$$\begin{aligned}
\dim_F J(A)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{m} \dim_F J_{i*}^{01} J_{*j}^{10} = \sum_{i=1}^{n}\sum_{j=1}^{m} \dim_F \varphi(J_{i*}^{01})\varphi(J_{*j}^{10}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\dim_F \varphi(J_{i*}^{01}) \dim_F \varphi(J_{*j}^{10})}{k^2 \dim_F D} \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{(\dim_F B - \dim_F f_i' B)(\dim_F B - \dim_F B f_j)}{k^2 \dim_F D} \\
&= \frac{(\dim_F B)^2 (n-1)(m-1)}{k^2 \dim_F D} \\
&= (n-1)(m-1)\dim_F B.
\end{aligned}$$

Finally, (4.9) follows at once from (4.7) and (4.8). ∎

As an example, we conclude this section with a specific class of algebras for which we give an explicit description of the graded Wedderburn-Malcev decomposition. Let R be a finite dimensional F-algebra with identity element and let P be an $m \times n$ matrix with entries in R such that each row and each column contains at least one invertible element in R. The Munn algebra $A := \mathcal{M}(R, n, m, P)$ is, by definition, the F-vector space of all $n \times m$-matrices over R with multiplication defined by $DE := D \circ P \circ E$, for $D, E \in \mathcal{M}(R, n, m, P)$, and where $D \circ P$ is the usual matrix multiplication. Clearly,

$A = \oplus_{i,j} Re_{ij}$ and it is an S-graded-simple algebra, where S is the completely 0-simple semigroup $S = \mathcal{M}(\{e\}^0, n, m, P')$, with P' the $n \times m$-matrix with e in every entry of P'.

By Lemma 4.3.1 and Theorem 4.3.2, there exists an S-graded maximal semisimple subalgebra B such that $A = B \oplus J(A)$. In case $R = F$, one can give an explicit description of B. Indeed, let k denote the rank of P. Reindexing if needed, we may assume that the first k rows and the first k columns are F-linearly independent. Let $B = \{(a_{ij}) \in A \mid a_{ij} = 0 \text{ for } i > k \text{ or } j > k\}$. Note that B can be identified with $\mathcal{M}(F, k, k, Q)$ where the sandwich matrix Q consists of the first k rows and columns of P. Clearly, B is a graded subalgebra and the mapping $N \mapsto N \circ Q$ is an algebra isomorphism $B \to M_k(F)$. Thus $B \cong M_k(F)$ is a simple algebra. Furthermore, $A/J(A) \cong M_k(F) \cong B$ (Proposition 23 in Chapter 5 of [Okn91]) and thus $A = B \oplus J(A)$ and A is Wedderburn-Malcev graded. Also recall that $J(A) = \{N \in A \mid P \circ N \circ P = 0\}$ (see e.g. Corollary 15 in Chapter 5 of [Okn91]).

4.4 Existence theorems for graded-simple algebras

In Theorem 4.3.2 and Theorem 4.3.7 we obtained a description of an arbitrary S-graded-simple algebra A. More precisely we have shown that A has a graded Malcev-Wedderburn decomposition $B \oplus J(A)$ and that $J(A)$ is roughly the direct sum of left and right B-modules that are isomorphic to left and right ideals of B and that also satisfy some other restrictions. To complete the description, we now show that any such collection of left and right ideals of a finite dimensional simple algebra B yields an S-graded-simple algebra A with B as a maximal semsimple graded subalgebra. We now formulate this in precise detail.

Let $k, m, n \in \mathbb{N}$ and let D be a division algebra. Put $B \cong M_k(D)$ and assume $f_1, \ldots, f_n \in B$ and $f'_1, \ldots, f'_n \in B$ are two sets of idempotents (some of them could be zero) such that $\sum_{i=1}^n f'_i = \sum_{j=1}^m f_j = 1_B$ and so that the idempotents in each set are pairwise orthogonal.

Let $J^{10}_{*1}, \ldots, J^{10}_{*m}$ and $J^{01}_{1*}, \ldots, J^{01}_{n*}$ be, respectively, left and right B-modules such that there exist embeddings

$$\varphi\colon J^{10}_{*j} \hookrightarrow B \quad \text{and} \quad \varphi\colon J^{01}_{i*} \hookrightarrow B$$

which are, respectively, left and right B-module homomorphisms and we have

$$\varphi(J^{10}_{*j}) f_j = 0 \quad \text{and} \quad f'_i \varphi(J^{01}_{i*}) = 0.$$

For sake of convenience, we denote both maps by the same letter φ and we also assume that the linear map φ is defined on the additive group $\bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10}$.

Further define additive groups J_{ij} as being isomorphic copies of $\varphi(J_{i*}^{01})\varphi(J_{*j}^{10}) \subseteq B$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Let

$$\Theta_{ij} \colon \varphi(J_{i*}^{01})\varphi(J_{*j}^{10}) \to J_{ij}$$

be the corresponding linear isomorphisms and let

$$\mu \colon J_{i*}^{01} \times J_{*j}^{10} \to J_{ij}$$

be the bilinear map defined by

$$\mu(v, w) := \Theta_{ij}(\varphi(v)\varphi(w)),$$

for $v \in J_{i*}^{01}$ and $w \in J_{*j}^{10}$. We extend μ by linearity to the map

$$\mu \colon \bigoplus_{i=1}^{n} J_{i*}^{01} \times \bigoplus_{j=1}^{m} J_{*j}^{10} \to \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{ij}.$$

Let Q denote the $n \times m$ matrix with each entry being equal to e.

Theorem 4.4.1. *The additive group*

$$A = B \oplus \bigoplus_{i=1}^{n} J_{i*}^{01} \oplus \bigoplus_{j=1}^{m} J_{*j}^{10} \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{ij}$$

is a $\mathcal{M}(\{e\}^0, n, m; Q)$-graded-simple ring for the multiplication defined by

$$(b_1, v_1, w_1, u_1)(b_2, v_2, w_2, u_2) = (b_1 b_2, v_1 b_2, b_1 w_2, \mu(v_1, w_2)),$$

for $b_1, b_2 \in B$, $v_1, v_2 \in \bigoplus_{i=1}^{n} J_{i*}^{01}$, $w_1, w_2 \in \bigoplus_{j=1}^{m} J_{*j}^{10}$, $u_1, u_2 \in \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{ij}$.
The homogeneous components are

$$A_{ij} = (f_i' B f_j, 0, 0, 0) \oplus \left\{(\varphi(v), v, 0, 0) \mid v \in J_{i*}^{01} f_j\right\} \oplus \left\{(\varphi(w), 0, w, 0) \mid w \in f_i' J_{*j}^{10}\right\}$$
$$\oplus \mathit{span}_{\mathbb{Z}} \left\{(\varphi(v)\varphi(w), v\varphi(w), \varphi(v)w, \mu(v, w)) \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10}\right\}.$$

Proof. Making use of all the assumptions, direct computations show that the multiplication defines a graded ring structure and also that $A = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} A_{ij}$.

Clearly, $J(A) = (0, \bigoplus_{i=1}^{n} J_{i*}^{01}, \bigoplus_{j=1}^{m} J_{*j}^{10}, \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{ij})$, since the third power of the right-hand side is zero.

Note that for every $1 \le i \le n$ and $1 \le j \le m$ we have

$$1_B(f_i' B f_j, 0,0,0)1_B = (f_i' B f_j, 0,0,0)\,,$$

$$1_B \left\{ (\varphi(v), v, 0, 0) \mid v \in J_{i*}^{01} f_j \right\} 1_B \subseteq \left(\bigoplus_{\ell \ne i} f_\ell' B f_j, 0,0,0 \right),$$

$$1_B \left\{ (\varphi(w), 0, w, 0) \mid w \in f_i' J_{*j}^{10} \right\} 1_B \subseteq \left(\bigoplus_{r \ne j} f_i' B f_r, 0,0,0 \right),$$

$$1_B \left\langle (\varphi(v)\varphi(w), v\varphi(w), \varphi(v)w, \mu(v,w)) \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10} \right\rangle_{\mathbb{Z}} 1_B \subseteq \left(\bigoplus_{\substack{\ell \ne i, \\ r \ne j}} f_\ell' B f_r, 0,0,0 \right),$$

where $\langle \cdot \rangle_{\mathbb{Z}}$ denotes the $\mathbb{Z}$-span. Thus, if $1_B a 1_B = 0$ for some $a \in A_{ij}$, then $a = 0$. Hence $A_{ij} \cap J(A) = 0$.

Suppose I is a graded two-sided ideal of A. Let $a \in I$, $a \ne 0$, be a homogeneous element. By the previous, $a = (b, u, v, w)$ with $b \ne 0$. Hence $(1_B, 0,0,0)a(1_B,0,0,0) = (b,0,0,0) \in I$. Since B is a simple ring, $(B,0,0,0) \subseteq I$. Thus $(1_B,0,0,0)A \subseteq I$ and $A(1_B,0,0,0) \subseteq I$. Since

$$A = (B,0,0,0) + (1_B,0,0,0)A + A(1_B,0,0,0) + (1_B,0,0,0)A^2(1_B,0,0,0),$$

we get $I = R$, and A is graded-simple. ∎

In case all modules involved, for example J_{i*}^{01} and J_{ij}, are left and right F-vector spaces on which the left and right F-structure is compatible and all maps involved are (left and right) F-linear, then the construction in Theorem 4.4.1 yields an S-graded F-algebra. If moreover B is a finite dimensional algebra over F, then a left B-module embedding $\varphi\colon J_{*j}^{10} \to \bigoplus_{\substack{r=1,\\ r \ne j}}^{m} B f_r$ exists if and only if $\dim_F J_{*j}^{10} \le \dim_F B - \dim_F (B f_j)$. A right B-module embedding

$$\varphi\colon J_{*i}^{01} \to \bigoplus_{\ell \ne i} f_\ell' B$$

exists if and only if

$$\dim_F J_{i*}^{01} \le \dim_F B - \dim_F (f_i' B).$$

Theorem 4.1.5 shows that the grading on an algebra A is completely defined by the images of the graded components in $A/J(A)$. We show that every such decomposition determines some S-grading.

Theorem 4.4.2. *Let D be a division algebra over a field F and let $B \cong M_k(D)$. Suppose*

$$B = \sum_{i=1}^{n} \sum_{j=1}^{m} B_{ij},$$

a sum of subspaces B_{ij} of B, with $B_{ij}B_{\ell r} \subseteq B_{ir}$, for all $1 \le i, \ell \le n$, $1 \le j, r \le m$. Let $P = (p_{ij})_{i,j}$ be an $n \times m$ matrix where $p_{ij} \in \{0, e\}$ such that $B_{ij}B_{\ell r} = 0$ for every (j, ℓ) with $p_{j\ell} = 0$. Then there exists an

$$\mathcal{M}(\{e\}^0, n, m; P)\text{-graded algebra} \quad A = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} A_{ij}$$

and a surjective algebra homomorphism $\psi \colon A \to B$ such that $\ker \psi = J(A)$ *and $\psi(A_{ij}) = B_{ij}$, or all $1 \le i, \ell \le n$, $1 \le j, r \le m$.*

Proof. Let $\bar{L}_j := \bigoplus_{i=1}^{n} B_{ij}$ for $1 \le j \le m$ and $\bar{R}_i := \bigoplus_{j=1}^{m} B_{ij}$ for $1 \le i \le n$. Note that $\bar{L}_1, \dots, \bar{L}_m$ are left ideals and $\bar{R}_1, \dots, \bar{R}_n$ are right ideals. Moreover, since $B \cong M_k(D)$ is semisimple, B is completely reducible as a left and a right B-module. Define $\tilde{L}_j$ as a complementary left B-submodule to $\bar{L}_j \cap \sum_{\ell=1}^{j-1} \bar{L}_\ell$ in $\bar{L}_j$, $1 \le j \le m$. Analogously, define $\tilde{R}_i$ as a complementary right B-submodule to $\bar{R}_i \cap \sum_{\ell=1}^{i-1} \bar{R}_\ell$ in $\bar{R}_i$, $1 \le i \le n$. Then $\bigoplus_{\ell=1}^{i} \tilde{R}_\ell = \sum_{\ell=1}^{i} \bar{R}_\ell$ for $1 \le i \le n$ and $\bigoplus_{\ell=1}^{j} \tilde{L}_\ell = \sum_{\ell=1}^{j} \bar{L}_\ell$ for $1 \le j \le m$. In particular, $B = \bigoplus_{i=1}^{n} \tilde{R}_i = \bigoplus_{j=1}^{m} \tilde{L}_j$. Decomposing 1_B into the sum of elements of, respectively, $\tilde{L}_1, \dots, \tilde{L}_m$ and $\tilde{R}_1, \dots, \tilde{R}_n$, we find two sets of orthogonal idempotents $f_1, \dots, f_m$ and $f'_1, \dots, f'_n$ such that $\tilde{L}_j = Bf_j$, $\tilde{R}_i = Bf'_i$, $\sum_{i=1}^{n} f'_i = \sum_{j=1}^{m} f_j = 1_B$.

Let $W^{10}_{*j} = \bar{L}_j(1 - f_j) \subseteq \bar{L}_j$ for $1 \le j \le m$ and $W^{01}_{i*} = (1 - f'_i)\bar{R}_i \subseteq \bar{R}_i$ for $1 \le i \le n$. Then $\bar{L}_j = \bar{L}_j f_j \oplus W^{10}_{*j}$ and $\bar{R}_i = f'_i \bar{R}_i \oplus W^{01}_{i*}$ (direct sums of, respectively, left and right ideals). Denote by J^{10}_{*j} the isomorphic copy of W^{10}_{*j} and by J^{01}_{i*} the isomorphic copy of W^{01}_{i*}. Denote the corresponding left B-module isomorphisms $\varphi \colon J^{10}_{*j} \to W^{10}_{*j}$ and right B-module isomorphisms $\varphi \colon J^{01}_{i*} \to W^{01}_{i*}$ by the same letter φ. Now extend φ to an F-linear map

$$\varphi \colon \bigoplus_{i=1}^{n} J^{01}_{i*} \oplus \bigoplus_{j=1}^{m} J^{10}_{*j} \to \sum_{i=1}^{n} W^{01}_{i*} + \sum_{j=1}^{m} W^{10}_{*j} \subseteq B.$$

Let $A = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} A_{ij}$ be the ring constructed in Theorem 4.4.1. Note that this will then be an algebra by the remark given after the proof of the theorem. We claim that A satisfies all the conditions of Theorem 4.4.2. Indeed, define ψ as the projection of

$$A = B \oplus \bigoplus_{i=1}^{n} J^{01}_{i*} \oplus \bigoplus_{j=1}^{m} J^{10}_{*j} \oplus \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} J_{ij}$$

on B with the kernel

$$J(A)=\bigoplus_{i=1}^{n} J_{i*}^{01}\oplus\bigoplus_{j=1}^{m} J_{*j}^{10}\oplus\bigoplus_{i=1}^{n}\bigoplus_{j=1}^{m} J_{ij}.$$

Since B is semisimple, there exist idempotents e_i' and e_j such that $\bar{R}_i=e_i'B$ and $\bar{L}_j=Be_j'$. Then

$$\begin{aligned}
\psi(A_{ij}) &= f_i'Bf_j'+\varphi(J_{i*}^{01}f_j)+\varphi(f_i'J_{*j}^{10})+\varphi(J_{i*}^{01})\varphi(J_{*j}^{10})\\
&= f_i'Bf_j+(1-f_i')\bar{R}_if_j+f_i'\bar{L}_j(1-f_j)+(1-f_i')\bar{R}_i\bar{L}_j(1-f_j)\\
&= f_i'\bar{R}_if_j+(1-f_i')\bar{R}_if_j+f_i'\bar{R}_i\bar{L}_j(1-f_j)+(1-f_i')\bar{R}_i\bar{L}_j(1-f_j)\\
&= \bar{R}_if_j+\bar{R}_i\bar{L}_j(1-f_j)=\bar{R}_i\bar{L}_j=e_i'Be_j=\bar{R}_i\cap\bar{L}_j\\
&= B_{ij}.
\end{aligned}$$

Moreover, $p_{j\ell}=0$ always implies $B_{ij}B_{\ell r}=0$, $\psi(A_{ij}A_{\ell r})=0$ and $A_{ij}A_{\ell r}=0$. The last equality follows from $(\ker\psi)\cap A_{ir}=J(A)\cap A_{ir}=0$. ∎

Note that any simple algebra $B=M_k(D)$ can be decomposed into the sum of B_{ij}'s (as needed in Theorem 4.4.2) by taking a collection $\bar{L}_1,\ldots,\bar{L}_m$ and a collection $\bar{R}_1,\ldots,\bar{R}_n$ of, respectively, left and right ideals of B with $B=\sum_{i=1}^{n}\bar{R}_i=\sum_{j=1}^{m}\bar{L}_j$ and define $B_{ij}=\overline{R}_i\cap\overline{L}_j$.

5 Exponential Growth of Semigroup Graded Algebras

Je serais reconnaissant a toute
personne ayant compris cette
demonstration de me l'expliquer.

Pierre Deligne

An F-algebra A may be endowed with additional structures such as a gradation or an action of some algebra. The fact that a certain gradation or action on A exists, limits the possible algebraic structure of A. As such it may be interesting to take this information into account. In this chapter we do so for semigroup gradings by considering graded versions of polynomial identities, codimensions and its exponential growth rate. For group gradings the story remains as nice as in the ungraded case, in the sense that a graded version of Amitsur's conjecture holds and the graded PI-exponent contains information on a graded Wedderburn-Malcev decomposition of A, see [GLM10, AGLM11, AG13, Gor16a]. Actually a G-grading by a finite group G can alternatively be viewed as the action of the Hopf algebra $H = (FG)^*$ on A and graded codimensions then coincide with so-called H-codimensions. In case of Hopf algebra actions again a version of Amitsur's conjecture holds, see [Gor13a] and it even holds for 'sufficiently nice generalized actions' [Gor13b]. Now in the case of a S-grading for some semigroup S, or alternatively an action of the bialgebra $(FS)^*$, interestingly no longer versions of the classical results hold. The first examples of non-integer semigroup-graded PI-exponent were discovered by Gordienko in [Gor15b]. The main goal of this chapter is to construct an infinite family of semigroup-graded simple PI-algebras with arbitrary large graded PI-exponent. The algebras will be a subset of the algebras classified in Chapter 4.

In Section 5.1 we recall all necessary background on graded PI-theory for associative algebras and their asymptotic methods, such as the relevant definitions and how one can reduce the investigations to the simple objects. Next in Sections 5.3, 5.4 and 5.5 we compute the exponential growth rate of the graded codimensions of the family of semigroup-graded algebras under consideration where the final result is Theorem 5.5.5. As usual, we first deal with the upper-bound and then the lower-bound. The former actually works for a more general class of algebras than those considered in the lower-bound. These sections are based on joint work with Gordienko and Jespers [GJJ17].

Whereas in the associative case one can reduce all investigations to the simple objects, this is not the case for Lie-algebras. In Section 5.6 we will namely explain that semigroup-graded simple Lie algebras are actually group-graded and therefore have an integer graded PI-exponent, but on the other hand we construct (the first example of) a finite dimensional semigroup-graded Lie algebra with an irrational PI-exponent. Finally, we end by emphasizing the connection between integer PI-exponent and graded versions of the classical structure theorems in Lie theory.

In Section 5.6 we start again by introducing the necessary definitions. Next we discuss graded-simple Lie algebras and subsequently we construct a Lie algebra with non-integer graded PI-exponent which is obtained in Theorem 5.7.12. Finally, in Section 5.8 we prove some positive results. The Lie algebra part has been presented during the conference "Groups and Rings Theory and Applications 2015, Sofia" by the author but has not appeared in written form before.

5.1 Introduction to Graded and Generalized Codimensions

Let T be a finite semigroup and let F be a field. Denote by $F\langle X^{T\text{-gr}}\rangle$ the free T-graded associative algebra over F on the countable set

$$X^{T\text{-gr}} := \bigcup_{t\in T} X^{(t)},$$

$X^{(t)} = \{x_1^{(t)}, x_2^{(t)}, \ldots\}$, i.e. the algebra of polynomials in non-commuting variables from $X^{T\text{-gr}}$. If we define the indeterminates from $X^{(t)}$ to be of (homogeneous) degree t, and

$x_{i_1}^{(t_1)} \dots x_{i_s}^{(t_s)}$ of T-degree $t_1 t_2 \dots t_s$, we get the natural T-grading

$$F\langle X^{T\text{-gr}}\rangle = \bigoplus_{t\in T} F\langle X^{T\text{-gr}}\rangle^{(t)}.$$

Analogously to the classical case we now define (mulitlinear) graded polynomial identities and graded codimensions.

Definition 5.1.1. Let $f = f(x_{i_1}^{(t_1)}, \dots, x_{i_s}^{(t_s)}) \in F\langle X^{T\text{-gr}}\rangle$ and $A = \bigoplus_{t\in T} A^{(t)}$ a T-graded algebra. We say that f is a *graded polynomial identity* of A if $f(a_{i_1}^{(t_1)}, \dots, a_{i_s}^{(t_s)}) = 0$ for all $a_{i_j}^{(t_j)} \in A^{(t_j)}$. By $\mathrm{Id}^{T\text{-gr}}(A)$ we denote the set of graded polynomial identities of A. Further let $P_n^{T\text{-gr}} := \operatorname{span}_F\{x_{\sigma(1)}^{(t_1)} x_{\sigma(2)}^{(t_2)} \dots x_{\sigma(n)}^{(t_n)} \mid t_i \in T, \sigma \in S_n\} \subset F\langle X^{T\text{-gr}}\rangle$. The number

$$c_n^{T\text{-gr}}(A) := \dim\left(\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)}\right)$$

is called the nth *graded codimension* of A.

Note that $\mathrm{Id}^{T\text{-gr}}(A)$ is graded ideal of $F\langle X^{T\text{-gr}}\rangle$. Also if $T = \{e\}$ is the trivial group, then we simply obtain the classical notions.

Example 5.1.2. Consider the multiplicative semigroup $T = \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and the T-grading $\mathrm{UT}_2(F) = \mathrm{UT}_2(F)^{(\bar{0})} \oplus \mathrm{UT}_2(F)^{(\bar{1})}$ on the algebra $\mathrm{UT}_2(F)$ of upper triangular 2×2 matrices over a field F defined by $\mathrm{UT}_2(F)^{(\bar{0})} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $\mathrm{UT}_2(F)^{(\bar{1})} = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. We have

$$[x^{(\bar{0})}, y^{(\bar{0})}] := x^{(\bar{0})} y^{(\bar{0})} - y^{(\bar{0})} x^{(\bar{0})} \in \mathrm{Id}^{T\text{-gr}}(\mathrm{UT}_2(F))$$

and $x^{(\bar{1})} y^{(\bar{1})} \in \mathrm{Id}^{T\text{-gr}}(\mathrm{UT}_2(F))$.

We can now formulate an analogue of Amitsur's conjecture for graded codimensions, which we expect to be true.

Conjecture 3. *The limit* $\mathrm{PIexp}^{T\text{-gr}}(A) := \lim_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)}$ *exists.*

In contrast to the ungraded context the graded PI-exponent can be a non-integer. The first examples were constructed in [Gor15b, Theorems 3–5] and in this chapter we even construct an infinite family. Nevertheless, in many instances the graded PI-exponent is still an integer. Integrality has been proven for finite group gradations in subsequent papers by Aljadeff, Giambruno and La Mattina [GLM10, AGLM11, AG13] and for arbitrary groups by Gordienko in [Gor16a] and for cancellative semigroups in [Gor15b]. If, moreover, A is unital then existence and integrality also holds, by [Gor15b], if T is a left or right zero band. Surprisingly, the condition on the existence of a unit element in aforementioned positive results is really essential. Indeed, the class of algebras with non-integer graded PI-exponent that we construct, consists of finite dimensional algebras graded by a left zero band which are moreover simple as graded algebras (but non-semisimple as ungraded algebra).

In our case, instead of working with graded codimensions directly, it is more convenient to replace the grading with the corresponding dual structure and study the asymptotic behaviour of polynomial H-identities for a suitable associative algebra H. We recall now the appropriate definitions.

Definition 5.1.3. Let H be an arbitrary unital associative F-algebra. We say that A is an algebra with a *generalized H-action* if A is endowed with a homomorphism $H \to \mathrm{End}_F(A)$ and for every $h \in H$ there exist $h'_i, h''_i, h'''_i, h''''_i \in H$ with $1 \le i \le s$ and $s \in \mathbb{N}$ such that

$$h(ab) = \sum_{i=1}^{s} \big((h'_i a)(h''_i b) + (h'''_i b)(h''''_i a)\big) \text{ for all } a, b \in A. \tag{5.1}$$

We use the term "generalized H-action" in order to distinguish this from the case when an algebra is an H-module algebra for some Hopf algebra H which is a particular case of the generalized H-action. Consider now the algebra

$$F\langle X|H\rangle := \bigoplus_{n=1}^{\infty} H^{\otimes n} \otimes F\langle X\rangle^{(n)},$$

with the usual multiplication $(u_1 \otimes w_1)(u_2 \otimes w_2) := (u_1 \otimes u_2) \otimes w_1 w_2$ for all $u_1 \in H^{\otimes j}$, $u_2 \in H^{\otimes k}$, $w_1 \in F\langle X\rangle^{(j)}$, $w_2 \in F\langle X\rangle^{(k)}$. We use the notation

$$x_{i_1}^{h_1} x_{i_2}^{h_2} \dots x_{i_n}^{h_n} := (h_1 \otimes h_2 \otimes \dots \otimes h_n) \otimes x_{i_1} x_{i_2} \dots x_{i_n}.$$

Note that if $(\gamma_\beta)_{\beta \in \Lambda}$ is a basis in H, then $F\langle X|H\rangle$ is isomorphic to the free associative algebra over F with free formal generators $x_i^{\gamma_\beta}$, $\beta \in \Lambda$, $i \in \mathbb{N}$. The elements of $F\langle X|H\rangle$ are called *H-polynomials*.

Remark that any map $\psi \colon X \to A$ has the unique homomorphic extension $\bar{\psi} \colon F\langle X|H\rangle \to A$ such that $\bar{\psi}(x_i^h) = h\psi(x_i)$ for all $i \in \mathbb{N}$ and $h \in H$. An H-polynomial $f \in F\langle X|H\rangle$ is called an *H-identity* of A if $\bar{\psi}(f) = 0$ for all maps $\psi\colon X \to A$. The set of all H-identities is denoted by $\mathrm{Id}^H(A)$ and is an ideal of $F\langle X|H\rangle$. Analogously to before we denote by P_n^H the space of all multilinear H-polynomials in $x_1, \dots, x_n$, i.e.

$$P_n^H = \mathrm{span}_F\{x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \dots x^{h_n}_{\sigma(n)} \mid h_i \in H, \sigma \in S_n\} \subset F\langle X|H\rangle.$$

Then the number $c_n^H(A) := \dim\left(\frac{P_n^H}{P_n^H \cap \mathrm{Id}^H(A)}\right)$ is called the nth *H-codimension* of A.

Consider now the vector space $(FT)^*$ dual to FT. Then $(FT)^*$ is an algebra with the multiplication defined by $(hw)(t) = h(t)w(t)$ and identity $1_{(FT)^*}(t) = 1$ for $h, w \in (FT)^*$ and $t \in T$. If $A = \oplus_{t\in T} A^{(t)}$ is T-graded, then it has also the following natural $(FT)^*$-action : $ha^{(t)} = h(t)a^{(t)}$ for all $h \in (FT)^*$, $a^{(t)} \in A^{(t)}$ and $t \in T$. If T were a finite group, this would turn A into a $(FT)^*$-module algebra. However, when T is simply a semigroup, this still endows A with a generalized $(FT)^*$-action. Indeed

$$h_t(ab) = \sum_{\substack{g,w \in \mathrm{supp}A,\\ gw=t}} h_g(a) h_w(b) \text{ for all } a, b \in A,$$

where $\{h_t \mid t \in T\}$ is the dual basis and $\mathrm{supp}A = \{t \in T \mid A^{(t)} \neq 0\}$.

The following lemma is intuitively clear and enables us the pass from graded polynomial identities to polynomial H-identities.

Lemma 5.1.4 ([Gor15b, Lemma 1])**.** *Let A be a finite dimensional algebra, over a field F, and graded by a semigroup T. Then $c_n^{T\text{-gr}}(A) = c_n^{(FT)^*}(A)$ for all $n \in \mathbb{N}$.*

In order to compute graded codimensions we can still use S_n-representation theory by letting, as before, S_n act on $P_n^H(F)$ by permuting the indices of each monomial. So, if $\mathrm{char}(F) = 0$ then one would like to use similar techniques as those explained in Section 1.3. Some of the differences are of course that now we have to produce graded polynomials, or equivalently $(FT)^*$-polynomials, and that an algebra possesses more graded polynomial identities than ordinary ones. The attentive reader will also point out the strong dependence on Wedderburn-Malcev's Theorem 1.3.2 in Giambruno-Zaicev's integrality result. Actually in all positive results by Giambruno, Aljadeff and La Mattina, the authors first needed to prove a graded analogue of Wedderburn-Malcev's Theorem (which for example does not always exists for infinite groups).

In case of group gradings, [GLM10, AGLM11, AG13], the authors even needed the classification of finite dimensional graded-simple algebras in the proof of the lower-bound. However, as remarked by Gordienko in [Gor13a, Theorem 7], based on a trick by Razmyslov [Raz94, Chapter III], this can be avoided by a clever use of the density theorem for graded-simple algebras and the non-degeneracy of the trace form for a semisimple algebra.

In [Gor13b, Theorem 1] it is proven that it is sufficient that $J(A)$ is H-invariant and $A/J(A)$ is a direct sum of H-simple algebras (i.e. a H-version of Wedderburn-Artin's Theorem holds), instead of an H-invariant version of Wedderburn-Malcev's Theorem. Actually the H-invariance of the Jacobson radical can even further be weakened to a condition on the simple objects. We denote by $J^H(A)$ the maximal nilpotent H-invariant two-sided ideal of A. Remark that saying that an ideal I is $(FT)^*$-invariant is equivalent to say it is graded.

Theorem 5.1.5 (Gordienko, [Gor16b])**.** *Let A be a finite dimensional non-nilpotent algebra with a generalized H-action for some associative algebra H with 1 over an algebraically closed field of characteristic 0. Suppose $A/J^H(A) = B_1 \oplus \ldots \oplus B_q$ is a direct sum of H-invariant ideals, with each B_i an H-simple algebra. Suppose $\exp^H(B_i) = \dim_F B_i$ for all i, then there exists $C_1, C_2 > 0$ and $r_1, r_2 \in \mathbb{R}$ such that $C_1 n^{r_1} d^n \leq c_n^H(A) \leq C_2 n^{r_2} d^n$ for all n. Moreover d can explicitly be computed.*

The class of algebras we construct in this chapter are a subset of the finite dimensional $\mathcal{M}(\{e\}^0, n, m; P)$-graded-simple algebras classified in Chapter 4. As shown there, these algebras satisfy a graded-version of Wedderburn-Malcev's theorem; however, $J(A)$ is not a graded ideal and even does not contain homogeneous elements. As will be proven in Theorems 5.4.5 and 5.5.5, the graded PI-exponent $\lim \sqrt[n]{c_n^{T\text{-gr}}(A)}$ of the aforementioned algebras exist but are not necessarily equal to $\dim_F A$, showing that no condition in Theorem 5.1.5 is redundant.

We should also point out that in [Gor16b] the author actually assumes the B_i to satisfy some 'Property (*)', which is a combinatorial condition about the existence of multilinear non-polynomial identities with sufficiently numerous alternations (such as in Section 1.3.2). Anyhow, this condition boils down to $\exp^H(B_i) = \lim \sqrt[n]{c_n^H(A)} = \dim_F B_i$.

To end this section we provide a relation between the ordinary and the graded codimensions and an exponential upper bound of the latter.

Proposition 5.1.6. *Let A be a T-graded algebra over a field F for some semigroup T (not necessarily finite). Then $c_n(A) \leq c_n^{T\text{-gr}}(A) \leq (\dim_F A)^{n+1}$. If T is finite, then $c_n^{T\text{-gr}}(A) \leq |T|^n c_n(A)$ for all $n \in \mathbb{N}$.*

Proof. Let $t_1, \dots, t_n \in T$. Denote by $P_{t_1,\dots,t_n}$ the vector space of multilinear T-graded polynomials in $x_1^{(t_1)}, \dots, x_n^{(t_n)}$. Then $P_n^{T\text{-gr}} = \bigoplus_{t_1,\dots,t_n \in T} P_{t_1,\dots,t_n}$. Let $\bar f_1, \dots, \bar f_{c_n(A)}$ be a basis in $\frac{P_n}{P_n \cap \mathrm{Id}(A)}$ where $f_i \in P_n$. Then, for every $\sigma \in S_n$, there exist $\alpha_{\sigma,i} \in F$ such that

$$x_{\sigma(1)} \dots x_{\sigma(n)} - \sum_{i=1}^{c_n(A)} \alpha_{\sigma,i} f_i(x_1, \dots, x_n) \in \mathrm{Id}(A).$$

Then for every $t_1, \dots, t_n \in T$ we have

$$x^{(t_1)}_{\sigma(1)} \dots x^{(t_n)}_{\sigma(n)} - \sum_{i=1}^{c_n(A)} \alpha_{\sigma,i} f_i \left(x_1^{(t_1)}, \dots, x_n^{(t_n)} \right) \in \mathrm{Id}^{T\text{-gr}}(A)$$

and

$$\frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)} = \left\langle \bar f_i \left(x_1^{(t_1)}, \dots, x_n^{(t_n)} \right) \;\middle|\; 1 \le i \le c_n(A),\ t_1, \dots, t_n \in T \right\rangle_F .$$

This implies the upper bound.

In order to get the lower bound, for a given n-tuple $(t_1, \dots, t_n) \in T^n$ we consider the map $\varphi_{t_1,\dots,t_n} \colon P_n \to \frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)}$, where $\varphi_{t_1,\dots,t_n}(f) = f\left(x_1^{(t_1)}, \dots, x_n^{(t_n)}\right)$ for $f = f(x_1, \dots, x_n) \in P_n$. Note that $f(x_1, \dots, x_n) \equiv 0$ is an ordinary polynomial identity if and only if

$$f\left(x_1^{(t_1)}, \dots, x_n^{(t_n)}\right) \equiv 0$$

is a graded polynomial identity for every $t_1, \dots, t_n \in T$. In other words, $P_n \cap \mathrm{Id}(A) = \bigcap_{(t_1,\dots,t_n) \in T^n} \ker \varphi_{t_1,\dots,t_n}$. Since P_n is a finite dimensional vector space, there exists a finite subset $\Lambda \subseteq T^n$ such that $P_n \cap \mathrm{Id}(A) = \bigcap_{(t_1,\dots,t_n) \in \Lambda} \ker \varphi_{t_1,\dots,t_n}$.

Consider the diagonal embedding

$$P_n \hookrightarrow P_n^{T\text{-gr}} = \bigoplus_{t_1,\dots,t_n \in T} P_{t_1,\dots,t_n},$$

where the image of $f(x_1, \dots, x_n) \in P_n$ equals $\sum_{(t_1,\dots,t_n) \in \Lambda} f\left(x_1^{(t_1)}, \dots, x_n^{(t_n)}\right)$. Then our choice of Λ implies that the induced map $\frac{P_n}{P_n \cap \mathrm{Id}(A)} \hookrightarrow \frac{P_n^{T\text{-gr}}}{P_n^{T\text{-gr}} \cap \mathrm{Id}^{T\text{-gr}}(A)}$ is an embedding and the lower bound follows.

The statement $c_n^{T\text{-gr}}(A) \le (\dim A)^{n+1}$ is a direct consequence of [Gor15b, Lemma 1] and [Gor13a, Lemma 4]. ∎

5.2 Overview of the Proof

Here we give a brief overview of Sections 5.3–5.5 and the main lines of the proof therein.

The context for the upper bound will be a finite dimensional T-graded algebra $A = \bigoplus_{t\in T} A^{(t)}$ over a field F of characteristic 0, for some semigroup T, such that $A/J(A) \cong M_k(F)$ for some $k \in \mathbb{N}$ and $A^{(t)} \cap J(A) = 0$ for all $t \in T$. For instance, when F is algebraically closed and A a T-graded-simple algebra for some $T = \mathcal{M}(\{e\}^0, n, m; P)$ these conditions are satisfied. For the lower bound we will restrict ourselves to the case where T is a right zero band and $k = 2$. At no moment in the proof we will need the full classification theorem obtained in Chapter 4, but only some lemmas preceding the classification.

The main steps of the proof follow those outlined in Section 1.3.2. So we start by decomposing $\frac{P_n^{T\text{-gr}}(A)}{P_n^{T\text{-gr}}(A)\cap \mathrm{Id}^{T\text{-gr}}(A)}$ as a semisimple FS_n-module and write

$$c_n^{T\text{-gr}}(A) = \sum_{\lambda\vdash n} m_\lambda^T \dim_F S^F(\lambda),$$

where m_λ^T is the multiplicity of $S^F(\lambda)$ in $\frac{P_n^{T\text{-gr}}(A)}{P_n^{T\text{-gr}}(A)\cap \mathrm{Id}^{T\text{-gr}}(A)}$.

The upper bound consists of two parts.

(a) First one needs a graded version of Berele-Regev Theorem 1.3.8, saying that the multiplicities are polynomially bounded, i.e there exist $a, b \in \mathbb{R}$ such that $\sum_{\lambda\vdash n} m_\lambda^T \leq an^b$. This is already known, see for example [Gor13b, Theorem 5].

(b) As in the ungraded case, it is sufficient to bound from above all $\dim_F S^F(\lambda)$ with $m_\lambda^T \neq 0$. By (1.4),

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n^{T\text{-gr}}(A))} \leq \sup_{\substack{\lambda\vdash n,\\ m_\lambda^T\neq 0}} \Phi\left(\frac{\lambda_1}{n_1}, \ldots, \frac{\lambda_q}{n_q}\right),$$

with $\Phi(x_1, \ldots, x_q) = \frac{1}{x_1^{x_1}\cdots\cdots x_q^{x_q}}$. Therefore the focus lies on restricting Φ to a compact region Ω such that if $\frac{\lambda}{n} = (\frac{\lambda_1}{n}, \ldots, \frac{\lambda_l}{n}) \notin \Omega$ then $m_\lambda^T = 0$. On the other hand, we also need that Ω does not contain too many partitions λ with $m_\lambda^T = 0$. No general techniques exist for this. In Section 5.3 we develop a special method that yields Lemma 5.3.2.

More precisely, let $f \in P_n^{(FT)^*}(F), \lambda \vdash n$ for some $n \in \mathbb{N}$ and $r = \dim_F A$. Then in Lemma 5.3.2 we prove that if

$$\sum_{i=1}^{r} \gamma_i\lambda_i \geq k \text{ or } \lambda_{r+1} > 0, \tag{5.2}$$

for some specific numbers γ_i defined in Section 5.3, then $e^*_{T_\lambda} f \in \mathrm{Id}^{(FT)^*}(A)$ for any Young tableau T_λ of shape λ. In particular $m^T_\lambda = 0$ for all partitions satisfying the conditions (5.2) and we may assume that Φ is defined in r variables.

Furthermore now we can restrict ourselves to the partitions $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ such that $\sum_{i=1}^{r} \gamma_i \frac{\lambda_i}{n} \leq \frac{k}{n}$. In particular, for large enough n, we may consider Φ on the following area

$$\Omega := \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1 \geq \dots \geq \alpha_r \geq 0,\ \alpha_1 + \alpha_2 + \dots + \alpha_r = 1,\ \gamma_1\alpha_1 + \dots + \gamma_r\alpha_r \leq 0\}.$$

Altogether we obtain the upper bound

$$\limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leq \max_{(\alpha_1,\dots,\alpha_r)\in\Omega} \Phi(\alpha_1, \dots, \alpha_r) =: d.$$

The formula for d is computed in Lemma 5.3.6 and the upper bound achieved in Theorem 5.3.7.

To obtain the lower bound, since $c_n^{T\text{-gr}}(A) \geq \dim_F S^F(\lambda)$ for all irreducible modules appearing in the decomposition, it is sufficient to find for each n a partition $\mu \vdash n$ such that $m^T_\mu \neq 0$ and having the appropriate dimension $\dim_F S^F(\mu) \gtrsim Cn^B d^n$ for some constants $B \in \mathbb{R}$ and $C > 0$. This will be done by using both methods explained in Section 1.3.2. So, on the one hand we will use the trick of making a partition μ from an extremal point after further describing the region Ω and on the other hand we construct for every $n \geq n_0$ a multilinear polynomial $f \in P_n^{T\text{-gr}}(F)$ with enough alternating sets such that $e^*_{T_\lambda} f \notin \mathrm{Id}^{T\text{-gr}}(A)$ for some tableau T_λ of a partition λ made out of μ.

All this is done in Section 5.4 and Section 5.5. More precisely the polynomial f is constructed in Lemmas 5.4.4 and 5.5.4. The exact exponent is computed in Theorem 5.4.5 and Theorem 5.5.5. We get $\exp^{T\text{-gr}}(A) = \dim_F A$ and

$$\exp^{T\text{-gr}}(A) = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < \dim_F A,$$

respectively. The numbers $|T_0|, |T_1|, |\bar{t}_0|$ are defined in the beginning of Section 5.4. In particular, any number $m + 1 + \sqrt{m}$, for any $m \in \mathbb{N}$, can be realized as the T-graded PI-exponent of some T-graded-simple finite dimensional algebra from our classification.

5.3 Upper bound for T-graded-simple algebras

Let $A = \bigoplus_{t\in T} A^{(t)}$ be a finite dimensional T-graded algebra over a field F of characteristic 0 for some semigroup T such that $A/J(A) \cong M_k(F)$ for some $k \in \mathbb{N}$ and

$A^{(t)} \cap J(A) = 0$ for all $t \in T$. In this section we prove an upper bound for T-graded codimensions of A.

For every $t \in T$ fix a basis $\mathcal{B}^{(t)}$ in $A^{(t)}$. Then $\mathcal{B} = \bigcup_{t\in T} \mathcal{B}^{(t)}$ is a basis in A. Fix also some isomorphism $\psi\colon A/J(A) \to M_k(F)$. Denote by $\pi\colon A \to A/J(A)$ the natural epimorphism. Define the function $\theta\colon \mathcal{B} \to \mathbb{Z}$ by $\theta(a) = \min\{i - j \mid \alpha_{ij} \neq 0,\ 1 \leq i, j \leq k\}$ if $\psi\pi(a) = \sum_{1\leq i,j\leq k} \alpha_{ij} e_{ij}$, $\alpha_{ij} \in F$.

The observation below plays a central role in the section.

Lemma 5.3.1. *Let $f \in P_n^{(FT)^*}(F)$ for some $n \in \mathbb{N}$ and let $a_i \in \mathcal{B}$, $1 \leq i \leq n$. If $f(a_1, \dots, a_n) \neq 0$, then $1 - k \leq \sum_{i=1}^n \theta(a_i) \leq k - 1$.*

Proof. Note that f is a linear combination of monomials $x^{h_1}_{\sigma(1)} x^{h_2}_{\sigma(2)} \dots x^{h_n}_{\sigma(n)}$, $h_i \in (FT)^*$, $\sigma \in S_n$. Denote by $A = \bigoplus_{t\in T} A^{(t)}$ the T-grading on A. Since $\operatorname{supp} A$ is finite, we may assume that f is a linear combination of monomials $x^{h_{t_1}}_{\sigma(1)} x^{h_{t_2}}_{\sigma(2)} \dots x^{h_{t_n}}_{\sigma(n)}$ with $t_i \in \operatorname{supp} A$, $\sigma \in S_n$ and h_{t_i} elements of the dual basis (i.e. $h_t(g) = \delta_{t,g}$.

Since all a_i are homogeneous, the value of $x^{h_{t_1}}_{\sigma(1)} x^{h_{t_2}}_{\sigma(2)} \dots x^{h_{t_n}}_{\sigma(n)}$ equals $a_{\sigma(1)} \dots a_{\sigma(n)}$ if $a_{\sigma(i)} \in A^{(t_i)}$ for all $1 \leq i \leq n$ and 0 otherwise. However $a_{\sigma(1)} \dots a_{\sigma(n)}$ is again a homogeneous element and $J(A) \cap A^{(t)} = 0$ for every $t \in T$. Thus $a_{\sigma(1)} \dots a_{\sigma(n)} \neq 0$ if and only if

$$\psi\pi(a_{\sigma(1)} \dots a_{\sigma(n)}) = \psi\pi(a_{\sigma(1)})\psi\pi(a_{\sigma(2)}) \dots \psi\pi(a_{\sigma(n)}) \neq 0.$$

Now we notice that $e_{i_1 j_1} e_{i_2 j_2} \dots e_{i_n j_n} \neq 0$ for some $1 \leq i_\ell, j_\ell \leq k$ only if $j_1 = i_2$, $j_2 = i_3$, $\dots$, $j_{n-1} = i_n$, and, in particular, $1 - k \leq \sum_{\ell=1}^n (i_\ell - j_\ell) = i_1 - j_k \leq k - 1$. Therefore, $a_{\sigma(1)} \dots a_{\sigma(n)} \neq 0$ only if $1 - k \leq \sum_{\ell=1}^n \theta(a_i) \leq k - 1$. ∎

Let $r := \dim A$. Define $\beta_\ell := \min\left\{ \sum\limits_{i=1}^{\ell} \theta(a_i) \;\middle|\; a_i \in \mathcal{B},\ a_i \neq a_j \text{ for } i \neq j \right\}$, $\gamma_\ell := \beta_\ell - \beta_{\ell-1}$, $1 \leq \ell \leq r$, $\beta_0 := 0$. Without loss of generality, we may assume that $\mathcal{B} = (a_1, \dots, a_r)$ where

$$\theta(a_1) \leq \theta(a_2) \leq \ldots \leq \theta(a_r).$$

Then $\beta_\ell = \sum\limits_{i=1}^{\ell} \theta(a_i)$ and $\gamma_\ell = \theta(a_\ell)$. In particular,

$$1 - k = \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_r.$$

The equality $\gamma_1 = 1 - k$ follows from the fact that e_{1k} has the minimal value of $(i - j)$ among all matrix units e_{ij} and the matrix unit e_{1k} must appear with a nonzero coefficient in the decomposition of $\varphi\pi(a)$ for some $a \in \mathcal{B}$.

Now we prove the main inequality for the shape of the partitions that may occur.

Lemma 5.3.2. *Let $f \in P_n^{(FT)^*}(F)$ and $\lambda \vdash n$ for some $n \in \mathbb{N}$. If $\sum_{i=1}^{r} \gamma_i \lambda_i \geq k$ or $\lambda_{r+1} > 0$, then $e^*_{T_\lambda} f \in \mathrm{Id}^{(FT)^*}(A)$ for any Young tableau T_λ of shape λ.*

Proof. Note that for each column of T_λ, the polynomial $e^*_{T_\lambda} f = b_{T_\lambda} a_{T_\lambda} f$ is alternating in the variables with indices from that column. Another remark is that, in order to determine whether a multilinear polynomial is a polynomial $(FT)^*$-identity of A, it is sufficient to substitute only elements from $\mathcal{B}$. If we substitute two coinciding elements for the variables of the same set of alternating variables, we get zero. Thus, if $\lambda_{r+1} > 0$, then the height of the first column is greater than or equal to $(r+1)$ and at least two elements coincide. Therefore, $e^*_{T_\lambda} f \in \mathrm{Id}^{(FT)^*}(A)$ and so we may assume that $\lambda_{r+1} = 0$.

Suppose $\sum_{i=1}^{r} \gamma_i \lambda_i \geq k$. We can rewrite this inequality in the form

$$\sum_{i=1}^{r} (\beta_i - \beta_{i-1}) \lambda_i = \sum_{i=1}^{r} \beta_i (\lambda_i - \lambda_{i+1}) \geq k. \tag{5.3}$$

Note that $(\lambda_i - \lambda_{i+1})$ equals the number of columns of height i in T_λ. Suppose $b_1, \dots, b_n \in \mathcal{B}$ are substituted for $x_1, \dots, x_n$. By the remark above, we may assume that for the variables of each column different basis elements are substituted. By the definition of β_i, $\sum_{i=1}^{n} \theta(b_i) \geq \sum_{i=1}^{r} \beta_i (\lambda_i - \lambda_{i+1})$. Combining this with (5.3), we get $\sum_{i=1}^{n} \theta(b_i) \geq k$. Now Lemma 5.3.1 implies $(e^*_{T_\lambda} f)(b_1, \dots, b_n) = 0$ and $e^*_{T_\lambda} f \in \mathrm{Id}^{(FT)^*}(A)$. ∎

As explained in the overview of the proof, we may now restrict the function $\Phi(\alpha_1, \dots, \alpha_r) = \frac{1}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \dots \alpha_r^{\alpha_r}}$ to the compact set

$$\Omega := \{(\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1 \geq \dots \geq \alpha_r \geq 0,\ \sum_{i=1}^{r} \alpha_i = 1,\ \gamma_1 \alpha_1 + \dots + \gamma_r \alpha_r \leq 0\}. \tag{5.4}$$

in such a way that

$$\limsup_{n \to \infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leq \max_{x \in \Omega} \Phi(x).$$

So in order to get Theorem 5.3.7 the maximum of Φ on Ω remains to be calculated. This is achieved in Lemma 5.3.6 and the rest of the section is devoted to its proof. The techniques are classical analysis, however the computations are tedious. So the reader willing to accept Lemma 5.3.6, may move immediately to Theorem 5.3.7. We begin with the most simple region.

Lemma 5.3.3. *Let* $r \in \mathbb{N}$ *and*

$$\Omega_0 := \left\{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \;\middle|\; \alpha_1, \ldots, \alpha_r \geq 0,\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 1\right\}.$$

Then $\max_{x \in \Omega_0} \Phi(x) = r$ *and* $\operatorname{argmax}_{x \in \Omega_0} \Phi(x) = \left(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right)$.

Proof. We prove the lemma by induction on r. The case $r = 1$ is trivial. Assume $r \geq 2$. First, we can express $\alpha_1 = 1 - \sum_{i=2}^r \alpha_i$ in terms of $\alpha_2, \alpha_3, \ldots, \alpha_r$, and study $\Phi_1(\alpha_2, \ldots, \alpha_r) = \frac{1}{\left(1-\sum_{i=2}^r \alpha_i\right)^{\left(1-\sum_{i=2}^r \alpha_i\right)} \alpha_2^{\alpha_2} \ldots \alpha_r^{\alpha_r}}$ on

$$\tilde{\Omega}_0 := \left\{(\alpha_2, \ldots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2, \ldots, \alpha_r \geq 0,\ 1 - \sum_{i=2}^r \alpha_i \geq 0\right\}.$$

Note that Φ_1 is continuous on the compact set $\tilde{\Omega}_0$ and differentiable at all inner points of $\tilde{\Omega}_0$. Thus Φ_1 can reach its extremal values only at inner critical points of Φ_1 or on $\partial\tilde{\Omega}_0$. By the induction assumption, $\Phi_1(x) \leq r - 1$ for all $x \in \partial\tilde{\Omega}_0$. Consider

$$\frac{\partial \Phi_1}{\partial \alpha_\ell}(\alpha_2, \ldots, \alpha_r) = \left(\ln\left(1 - \sum_{i=2}^r \alpha_i\right) - \ln \alpha_\ell\right) \Phi_1(\alpha_2, \ldots, \alpha_r).$$

Then $\frac{\partial \Phi}{\partial \alpha_\ell}(\alpha_2, \ldots, \alpha_r) = 0$ for all $2 \leq \ell \leq r$ only for $\alpha_2 = \ldots = \alpha_r = 1 - \sum_{i=2}^r \alpha_i = \frac{1}{r}$. Since $\Phi\left(\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right) = r > r - 1$, we get the lemma. ∎

The positive root ζ of the polynomial P, defined in the lemma below, will be used in the calculation of the upper bound of codimensions. Here we study the basic properties of P.

Lemma 5.3.4. *Let* $r \in \mathbb{N}$ *and* $\gamma_i \in \mathbb{Z}$, $1 \leq i \leq r$. *Suppose*

$$\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_r, \tag{5.5}$$

$\gamma_1 < 0$. *Consider the equation*

$$P(x) := \sum_{i=1}^r \gamma_i x^{\gamma_i - \gamma_1} = 0, \tag{5.6}$$

where x is the unknown variable. If $\sum_{i=1}^r \gamma_i \geq 0$, then (5.6) has a unique root, denoted ζ. Moreover $\zeta \in (0; 1]$. If $\sum_{i=1}^r \gamma_i < 0$, then $P(y) < 0$ for all $y \in [0; 1]$.

Proof. If $\gamma_r \le 0$, then all nonzero coefficients of P are negative and $P(y) < 0$ for all $y \in [0;1]$.

Suppose $\gamma_r > 0$. Inequality (5.5) implies that there is only one sign difference in the signs of coefficients of P. Therefore, by Descartes' rule of signs, (5.6) has a unique positive root which we call ζ. Define the positive integer m by

$$\gamma_1 = \ldots = \gamma_m < \gamma_{m+1}.$$

Note that $P(0) = m\gamma_1 < 0$ and $P(1) = \sum_{i=1}^r \gamma_i$. Therefore, if $\sum_{i=1}^r \gamma_i \ge 0$, we have $\zeta \in (0;1]$. If $P(1) = \sum_{i=1}^r \gamma_i < 0$, then $\zeta > 1$ and $P(y) < 0$ for all $y \in (0;1]$. ∎

It turns out that the root ζ of P is the extremal point of the function Ψ defined below. This will be used in the calculation of the maximum of Φ on our region Ω.

Lemma 5.3.5. *Let $r \in \mathbb{N}$ and $\gamma_i \in \mathbb{Z}$, $1 \le i \le r$. Suppose $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_r$, $\gamma_1 < 0$. Denote $\Psi(y) = \sum_{i=1}^r y^{\gamma_i}$. Then*

1. *if $\sum_{i=1}^r \gamma_i \ge 0$, then $\min_{y\in(0;1]} \Psi(y) = \Psi(\zeta)$ where $\zeta \in (0;1]$ is the positive root of (5.6);*

2. *if $\sum_{i=1}^r \gamma_i \le 0$, then $\min_{y\in(0;1]} \Psi(y) = r$.*

Proof. Note that $\Psi'(y) = \sum_{i=1}^r \gamma_i y^{\gamma_i - 1}$. Thus $\Psi'(y)$ has the same sign on $(0;1]$ as $P(y) = \sum_{i=1}^r \gamma_i y^{\gamma_i - \gamma_1}$. Also $\lim_{y\to 0+} \Psi(y) = +\infty$. Lemma 5.3.4 implies that if $\sum_{i=1}^r \gamma_i \ge 0$, then $\min_{y\in(0;1]} \Psi(y) = \Psi(\zeta)$, and if $\sum_{i=1}^r \gamma_i \le 0$, then $\min_{y\in(0;1]} \Psi(y) = \Psi(1) = r$. (In the case $\sum_{i=1}^r \gamma_i = 0$, we have $\zeta = 1$ and $\Psi(\zeta) = r$.) ∎

Now we are ready to calculate the maximum of Φ. For our convenience, we replace our region Ω with a larger region $\tilde{\Omega}$ and show that the maximum on both regions is the same.

Lemma 5.3.6. *Let $r \in \mathbb{N}$ and $\gamma_i \in \mathbb{Z}$, $1 \le i \le r$. Suppose $\gamma_1 \le \gamma_2 \le \ldots \le \gamma_r$, $\gamma_1 < 0$, $\sum_{i=1}^r \gamma_i \ge 0$. Let ω be as in (5.4). and let*

$$\tilde{\Omega} := \{(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{R}^r \mid \alpha_1, \ldots, \alpha_r \ge 0,\ \alpha_1 + \alpha_2 + \ldots + \alpha_r = 1,\ \gamma_1\alpha_1 + \ldots + \gamma_r\alpha_r \le 0\}.$$

Then $\max_{x\in\Omega} \Phi(x) = \max_{x\in\tilde{\Omega}} \Phi(x) = \sum_{i=1}^r \zeta^{\gamma_i}$ where $\zeta \in (0;1]$ is the positive root of (5.6).

Proof. Like in Lemma 5.3.3, we use induction on r. The conditions $\gamma_1 < 0$ and $\sum_{i=1}^r \gamma_i \geq 0$ imply that $r \geq 2$. We will not prove the induction base $r = 2$ separately, but the base will follow from the arguments below since for $r = 2$ we will not use the induction assumption.

Again, we express $\alpha_1 = 1 - \sum_{i=2}^r \alpha_i$ in terms of $\alpha_2, \alpha_3, \dots, \alpha_r$, and study $\Phi_1(\alpha_2, \dots, \alpha_r) = \frac{1}{\left(1-\sum_{i=2}^r \alpha_i\right)^{\left(1-\sum_{i=2}^r \alpha_i\right)} \alpha_2^{\alpha_2} \dots \alpha_r^{\alpha_r}}$ on

$$\Omega_1 := \left\{ (\alpha_2, \dots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2, \dots, \alpha_r \geq 0,\ 1 - \sum_{i=2}^r \alpha_i \geq 0,\ \gamma_1 + \sum_{i=2}^r (\gamma_i - \gamma_1)\alpha_i \leq 0 \right\}.$$

Now the proof of Lemma 5.3.3 implies that the only critical point of Φ_1 is $\left(\frac{1}{r}, \dots, \frac{1}{r}\right)$. This point belongs to Ω_1 if and only if $\sum_{i=1}^r \gamma_i \leq 0$. If indeed $\sum_{i=1}^r \gamma_i = 0$, then by Lemma 5.3.3 we have $\max_{x\in\Omega} \Phi(x) = r$. Since in this case $\zeta = 1$, the lemma is proven.

Suppose $\sum_{i=1}^r \gamma_i > 0$. Then the continuous function Φ_1 reaches its maximum on $\partial\Omega_1$. Note that $\partial\Omega_1 = \Omega_2 \cup \bigcup_{i=1}^r \Omega_{1i}$, where

$$\Omega_{11} = \left\{ (\alpha_2, \dots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2, \dots, \alpha_r \geq 0,\ 1 - \textstyle\sum_{i=2}^r \alpha_i = 0,\ \gamma_1 + \sum_{i=2}^r (\gamma_i - \gamma_1)\alpha_i \leq 0 \right\},$$

$$\Omega_{1\ell} = \left\{ (\alpha_2, \dots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2, \dots, \alpha_r \geq 0, \alpha_\ell = 0, 1 - \sum_{i=2}^r \alpha_i \geq 0,\ \gamma_1 + \sum_{i=2}^r (\gamma_i - \gamma_1)\alpha_i \leq 0 \right\},$$

for $2 \leq \ell \leq r$, and

$$\Omega_2 = \left\{ (\alpha_2, \dots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2, \dots, \alpha_r \geq 0,\ 1 - \sum_{i=2}^r \alpha_i \geq 0,\ \gamma_1 + \sum_{i=2}^r (\gamma_i - \gamma_1)\alpha_i = 0 \right\}.$$

We claim that

$$\max_{x\in\bigcup_{i=1}^r \Omega_{1i}} \Phi_1(x) < \sum_{j=1}^r \zeta^{\gamma_j}, \tag{5.7}$$

where $\zeta \in (0; 1]$ is the positive root of (5.6). Indeed, switching to the variables $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_r$, we get $\max_{x\in\Omega_{1i}} \Phi_1(x) = \max_{x\in\Omega_i'} \Phi(x)$, $1 \leq i \leq r$, where

$$\Omega_i' = \left\{ (\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_r \geq 0,\right.$$
$$\left. \alpha_1 + \dots + \hat{\alpha}_i + \dots + \alpha_r = 1,\ \gamma_1\alpha_1 + \dots + \widehat{\gamma_i\alpha_i} + \dots + \gamma_r\alpha_r \leq 0 \right\}.$$

(For the convenience, we denote the function $\Phi(\theta_1, \dots, \theta_m) = \frac{1}{\theta_1^{\theta_1} \dots \theta_m^{\theta_m}}$ by the same letter Φ for all i.)

If $i=1$ and $\gamma_2 \geq 0$, then

$$\Omega'_i = \Omega'_1 = \left\{ (\alpha_2,\dots,\alpha_r) \in \mathbb{R}^{r-1} \;\middle|\; \alpha_2,\dots,\alpha_r \geq 0, \alpha_2+\ldots+\alpha_r = 1, \right.$$
$$\left. \alpha_\ell = \alpha_{\ell+1} = \ldots = \alpha_r = 0 \right\},$$

where the number $2 \leq \ell \leq r$ is defined by the equality $\gamma_2 = \ldots = \gamma_{\ell-1} = 0$ and the inequality $\gamma_\ell > 0$. Then Lemma 5.3.3 implies $\max_{x\in\Omega'_1} \Phi(x) = \ell - 2$. Since $\sum_{j=1}^r \zeta^{\gamma_j} > \sum_{j=2}^{\ell-1} \zeta^{\gamma_j} = \ell - 2$, we get $\max_{x\in\Omega_{11}} \Phi_1(x) < \sum_{j=1}^r \zeta^{\gamma_j}$.

Suppose that either $i \geq 2$ or $\gamma_2 < 0$, and $\sum\limits_{\substack{\ell=1,\\ \ell\neq i}}^r \gamma_\ell \geq 0$. Then we apply the induction assumption for $(r-1)$. We have $\max_{x\in\Omega'_i} \Phi(x) = \sum\limits_{\substack{\ell=1,\\ \ell\neq i}}^r (\zeta')^{\gamma_\ell}$, where $\sum\limits_{\substack{\ell=1,\\ \ell\neq i}}^r \gamma_\ell\,(\zeta')^{\gamma_\ell-\gamma_1} = 0$ if $i>1$ and $\sum_{\ell=2}^r \gamma_\ell\,(\zeta')^{\gamma_\ell-\gamma_2} = 0$ if $i=1$. By Lemma 5.3.5,

$$\max_{x\in\Omega_{1i}} \Phi_1(x) = \max_{x\in\Omega'_i} \Phi(x) = \min_{y\in(0;1]} \sum_{\substack{\ell=1,\\ \ell\neq i}}^r y^{\gamma_\ell} < \min_{y\in(0;1]} \sum_{\ell=1}^r y^{\gamma_\ell} = \sum_{j=1}^r \zeta^{\gamma_j}. \tag{5.8}$$

Suppose that either $i \geq 2$ or $\gamma_2 < 0$, and $\sum\limits_{\substack{\ell=1,\\ \ell\neq i}}^r \gamma_\ell < 0$. Then $\left(\frac{1}{r-1}, \frac{1}{r-1}, \ldots, \frac{1}{r-1}\right) \in \Omega'_i$ and by Lemma 5.3.3 we have $\max_{x\in\Omega'_i} \Phi(x) = r-1$. Again, by Lemma 5.3.5, we get (5.8). Therefore (5.7) is proven.

We claim that $\max_{x\in\Omega_2} \Phi_1(x) = \sum_{i=1}^r \zeta^{\gamma_i}$ where $\zeta \in (0;1]$ is the positive root of (5.6). If $r=2$, then $\Omega_2 = \left\{-\frac{\gamma_1}{\gamma_2-\gamma_1}\right\}$, $\zeta = \left(-\frac{\gamma_1}{\gamma_2}\right)^{\frac{1}{\gamma_2-\gamma_1}}$,

$$\Phi_1\left(-\frac{\gamma_1}{\gamma_2-\gamma_1}\right) = \left(\frac{\gamma_2}{\gamma_2-\gamma_1}\right)^{-\frac{\gamma_2}{\gamma_2-\gamma_1}} \left(-\frac{\gamma_1}{\gamma_2-\gamma_1}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}}$$
$$= (\gamma_2-\gamma_1)\gamma_2^{-\frac{\gamma_2}{\gamma_2-\gamma_1}} (-\gamma_1)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} = \left(-\frac{\gamma_1}{\gamma_2}\right)^{\frac{\gamma_1}{\gamma_2-\gamma_1}} + \left(-\frac{\gamma_1}{\gamma_2}\right)^{\frac{\gamma_2}{\gamma_2-\gamma_1}} = \zeta^{\gamma_1} + \zeta^{\gamma_2}.$$

Therefore we may assume $r \geq 3$. Define $1 \leq m < r$ by $\gamma_1 = \ldots = \gamma_m < \gamma_{m+1}$. Then for all $(\alpha_2,\dots,\alpha_r) \in \Omega_2$ we have $\gamma_1 + \sum_{i=m+1}^r (\gamma_i - \gamma_1)\alpha_i = 0$ and

$$\alpha_{m+1} = -\frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^r (\gamma_i-\gamma_1)\alpha_i\right). \tag{5.9}$$

We express α_{m+1} and notice that $\max_{x\in\Omega_2} \Phi_1(x) = \max_{x\in\Omega_3} \Phi_2(x)$ where

$$\Phi_2(\alpha_2,\dots,\alpha_m,\alpha_{m+2},\dots,\alpha_r) = \Phi(\alpha_1,\dots,\alpha_r),$$

$$\Omega_3 := \left\{ (\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r) \in \mathbb{R}^{r-2} \;\middle|\; \alpha_1, \dots, \alpha_r \geq 0 \right\},$$
$$\alpha_1 = 1 - \sum_{\substack{i=2,\\ i\neq m+1}}^{r} \alpha_i + \frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i - \gamma_1)\alpha_i\right)$$

and α_{m+1} is defined by (5.9).

Consider

$$\frac{\partial \Phi_2}{\partial \alpha_i}(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r)$$
$$= \left((-\ln \alpha_1 - 1)\tfrac{\partial \alpha_1}{\partial \alpha_i} - \ln \alpha_i - 1 + (-\ln \alpha_{m+1} - 1)\tfrac{\partial \alpha_{m+1}}{\partial \alpha_i} \right) \Phi_2(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r).$$

Let $2 \leq i \leq m$. Then $\frac{\partial \alpha_1}{\partial \alpha_i} = -1$, $\frac{\partial \alpha_{m+1}}{\partial \alpha_i} = 0$, and

$$\frac{\partial \Phi_2}{\partial \alpha_i}(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r) = (\ln \alpha_1 - \ln \alpha_i)\Phi_2(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r).$$

Let $m+2 \leq i \leq r$. Then $\frac{\partial \alpha_1}{\partial \alpha_i} = \frac{\gamma_i - \gamma_{m+1}}{\gamma_{m+1}-\gamma_1}$, $\frac{\partial \alpha_{m+1}}{\partial \alpha_i} = \frac{\gamma_1 - \gamma_i}{\gamma_{m+1}-\gamma_1}$, and

$$\frac{\partial \Phi_2}{\partial \alpha_i}(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r)$$
$$= \left(-\frac{\gamma_i - \gamma_{m+1}}{\gamma_{m+1}-\gamma_1} \ln \alpha_1 - \ln \alpha_i - \frac{\gamma_1 - \gamma_i}{\gamma_{m+1}-\gamma_1} \ln \alpha_{m+1} \right) \Phi_2(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r).$$

Therefore, if $(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r) \in \Omega_3$ is a critical point for Φ_2, we have

$$\frac{\partial \Phi_2}{\partial \alpha_i}(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r) = 0$$

for all $2 \leq i \leq m$ and $m+2 \leq i \leq r$ which is equivalent to

$$\begin{cases} \alpha_i &= \alpha_1 & \text{for} \quad 2 \leq i \leq m, \\ \alpha_i &= \alpha_1^{\left(\frac{\gamma_{m+1}-\gamma_i}{\gamma_{m+1}-\gamma_1}\right)} \alpha_{m+1}^{\left(\frac{\gamma_i-\gamma_1}{\gamma_{m+1}-\gamma_1}\right)} & \text{for} \quad m+2 \leq i \leq r, \\ \alpha_1 &= 1 - \sum\limits_{\substack{i=2,\\ i\neq m+1}}^{r} \alpha_i + \frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i - \gamma_1)\alpha_i\right), & \\ \alpha_{m+1} &= -\frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i - \gamma_1)\alpha_i\right) & \end{cases}$$

and

$$\begin{cases} \alpha_i &= \alpha_1 & \text{for} \quad 2 \leq i \leq m, \\ \alpha_i &= \alpha_1 \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{\gamma_i-\gamma_1}{\gamma_{m+1}-\gamma_1}} & \text{for} \quad m+2 \leq i \leq r, \\ \alpha_1 &= 1 - \sum\limits_{\substack{i=2,\\ i\neq m+1}}^{r} \alpha_i + \frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i - \gamma_1)\alpha_i\right), & \\ \alpha_{m+1} &= -\frac{1}{\gamma_{m+1}-\gamma_1}\left(\gamma_1 + \sum_{i=m+2}^{r} (\gamma_i - \gamma_1)\alpha_i\right). & \end{cases}$$

Note that since we are looking for inner critical points of Φ_2 on $\Omega_3 \subset \mathbb{R}^{r-2}$, we may assume that all $\alpha_i > 0$.

Performing equivalent transformations, we get

$$\begin{cases} \alpha_i & = \alpha_1 \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{\gamma_i-\gamma_1}{\gamma_{m+1}-\gamma_1}} \quad \text{for } 1 \le i \le r, \\ \sum\limits_{i=1}^{r} \alpha_i & = 1, \\ \sum_{i=1}^{r} \gamma_i \alpha_i & = 0. \end{cases} \tag{5.10}$$

Now we introduce an additional variable $\zeta := \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{1}{\gamma_{m+1}-\gamma_1}}$ and get

$$\begin{cases} \zeta & = \left(\frac{\alpha_{m+1}}{\alpha_1}\right)^{\frac{1}{\gamma_{m+1}-\gamma_1}}, \\ \alpha_i & = \alpha_1 \zeta^{\gamma_i-\gamma_1} \quad \text{for } 1 \le i \le r, \\ \alpha_1 \sum\limits_{i=1}^{r} \zeta^{\gamma_i-\gamma_1} & = 1, \\ \sum_{i=1}^{r} \gamma_i \zeta^{\gamma_i-\gamma_1} & = 0. \end{cases} \tag{5.11}$$

Now the first equation is the consequence of the second one for $i = m+1$. Thus the original system is equivalent to

$$\begin{cases} \alpha_i & = \frac{\zeta^{\gamma_i-\gamma_1}}{\sum_{i=1}^{r} \zeta^{\gamma_i-\gamma_1}} \quad \text{for } 1 \le i \le r, \\ \sum_{i=1}^{r} \gamma_i \zeta^{\gamma_i-\gamma_1} & = 0. \end{cases} \tag{5.12}$$

By Lemma 5.3.4, the last equation has the unique solution $\zeta \in (0;1]$. Thus

$$(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r)$$

defined by (5.12) is the unique inner critical point of Φ_2. Using (5.10) and (5.11), we get

$$\Phi_2(\alpha_2, \dots, \alpha_m, \alpha_{m+2}, \dots, \alpha_r) = \Phi(\alpha_1, \dots, \alpha_r) = \frac{1}{\alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \dots \alpha_r^{\alpha_r}}$$
$$= \frac{1}{\alpha_1^{\alpha_1+\dots+\alpha_r} \zeta^{\alpha_1(\gamma_1-\gamma_1)} \dots \zeta^{\alpha_r(\gamma_r-\gamma_1)}} = \frac{1}{\alpha_1 \zeta^{-\gamma_1}} = \sum_{i=1}^{r} \zeta^{\gamma_i}.$$

Note that the values of Φ_2 on $\partial\Omega_3$ equal the values of Φ_1 at the corresponding points of $\bigcup_{i=1}^{r} \Omega_{1i}$. Therefore,

$$\max_{x \in \tilde{\Omega}} \Phi(x) = \max_{x \in \Omega_1} \Phi_1(x) = \max_{x \in \Omega_3} \Phi_2(x) = \sum_{i=1}^{r} \zeta^{\gamma_i}.$$

Since (5.12) implies $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r$, we get

$$\max_{x\in\Omega} \Phi(x) = \max_{x\in\tilde{\Omega}} \Phi(x) = \sum_{i=1}^{r} \zeta^{\gamma_i}.$$

∎

As explained in the overview of the proof, we now immediately get the desired upper bound.

Theorem 5.3.7. *Let $A = \bigoplus_{t\in T} A^{(t)}$ be a finite dimensional T-graded algebra over a field F of characteristic 0 for some semigroup T such that $A/J(A) \cong M_k(F)$ for some $k \in \mathbb{N}$ and $A^{(t)} \cap J(A) = 0$ for all $t \in T$. Let γ_ℓ, $1 \leq \ell \leq r$, $r := \dim A$, be the numbers defined before Lemma 5.3.2. Suppose $\sum_{i=1}^{r} \gamma_i \geq 0$. Let ζ be the positive root of (5.6). Then $\limsup\limits_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leq \sum_{i=1}^{r} \zeta^{\gamma_i}$.*

Proof. This is a consequence of equation (5.4) and Lemma 5.3.6. ∎

Remark 5.3.8. If A is a finite dimensional T-graded-simple algebra over an algebraically closed field F of characteristic 0 with $T = \mathcal{M}(\{e\}^0, n, m; P)$ (e.g. a zero band), see Section 4.1 for definitions, then by Lemmas 4.1.4 and 4.3.5 the algebra A satisfies the conditions of Theorem 5.3.7 and the upper bound above holds.

5.4 The case $A/J(A) \cong M_2(F)$ and $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$

From now on $A = \bigoplus_{t\in T} A^{(t)}$ is (still) a finite dimensional T-graded-simple algebra over a field F of characteristic 0, but with T *a right zero band.* Moreover, *we assume that $A/J(A) \cong M_2(F)$*. In the next two sections we show that

$$\liminf_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \geq \max_{x\in\Omega} \Phi(x)$$

and thus $\mathrm{PIexp}^{T\text{-gr}}(A)$ exists. For this we have to consider two cases, depending on some properties of the grading T, which will also yield two possibilities for $\mathrm{PIexp}^{T\text{-gr}}(A)$. In the first case, which we handle in this section, the graded PI-exponent will be equal to $\dim_F A$. In particular the result is the same as for a group-gradation.

Let $I_t := \pi(A^{(t)})$ for $t \in T$, where $\pi\colon A \to A/J(A)$ is the natural epimorphism.

Note that since $\dim A < +\infty$, only a finite number of I_t are nonzero. Let

$$T_0 := \{t \in T \mid \dim I_t = 2\} \text{ and } T_1 = \{t \in T \mid I_t = A/J(A)\}.$$

We have $I_t = 0$ for all $t \notin T_0 \sqcup T_1$. Moreover $A^{(t)} \cap \ker \pi = 0$ for all $t \in T$ implies $r := \dim A = 2|T_0| + 4|T_1|$. Define

$$t_1 \sim t_2 \text{ if } I_{t_1} = I_{t_2}.$$

Since $A/J(A) \cong M_2(F)$, all irreducible $A/J(A)$-modules are two-dimensional and isomorphic to each other. Thus $A/J(A) = I_{t_1} \oplus I_{t_2}$ for all $I_{t_1} \neq I_{t_2}$.

Now we show that if the cardinalities of equivalence classes satisfy some kind of triangle inequality, then we can combine the elements into pairs and, possibly, a triple such that the elements inside each pair or triple are non-equivalent.

Lemma 5.4.1. *Let T_0 be a finite non-empty set with an equivalence relation $\sim$. Suppose*

$$|\bar{t}_0| \leq \sum_{\substack{\bar{t} \in T_0/\sim, \\ \bar{t} \neq \bar{t}_0}} |\bar{t}| \text{ for all } \bar{t}_0 \in T_0/\sim. \tag{5.13}$$

Then we can choose $\{t_1, \dots, t_{|T_0|}\} = T_0$ such that

1. *if $2 \mid |T_0|$, we have $t_{2i-1} \nsim t_{2i}$ for all $1 \leq i \leq \frac{|T_0|}{2}$;*
2. *if $2 \nmid |T_0|$, we have $t_{2i-1} \nsim t_{2i}$ for all $1 \leq i \leq \frac{|T_0|-1}{2}$ and $t_{|T_0|-2}$, $t_{|T_0|-1}$, $t_{|T_0|}$ are pairwise non-equivalent.*

Remark 5.4.2. Note that if (5.13) does not hold, then there exists an equivalence class $\bar{t}_0 \in T_0/\sim$ such that $|\bar{t}_0| \geq \frac{|T_0|}{2}$.

Proof.[Proof of Lemma 5.4.1.] We give a proof by induction on $|T_0|$. Note that (5.13) implies $|T_0/\sim| \geq 2$.

Suppose $|T_0/\sim| = 2$. Then (5.13) implies $|\bar{t}_1| = |\bar{t}_2|$ where $T_0/\sim = \{\bar{t}_1, \bar{t}_2\}$. We can define $\{t_1, \dots, t_{|T_0|}\} = T_0$ by $\{t_1, t_3, \dots, t_{|T_0|-1}\} := \bar{t}_1$ and $\{t_2, t_4, \dots, t_{|T_0|}\} := \bar{t}_2$, and the lemma is proven.

Suppose $|T_0| = 3$. Then all the elements of T_0 are pairwise non-equivalent and again we get the lemma.

Now we assume that $|T_0| > 3$. Choose the classes $\bar{t}_1$ and $\bar{t}_2$ with the maximal number of elements. Choose some $t_1 \in \bar{t}_1$ and $t_2 \in \bar{t}_2$. Note that for the set $T_0 \backslash \{t_1, t_2\}$ and the same equivalence relation $\sim$, we still have (5.13). By the induction assumption we can choose $\{t_3, \ldots, t_{|T|}\} = T_0 \backslash \{t_1, t_2\}$ such that $t_1, \ldots, t_{|T_0|}$ satisfy the conditions of the lemma. ∎

The inequality (5.13) will be used to distinguish between the cases when $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$ and when $\mathrm{PIexp}^{T\text{-gr}}(A) < \dim A$.

First, we need the following technical lemma.

Lemma 5.4.3. *Let $\alpha, \beta, \tilde\alpha, \tilde\beta \in F$. Then*

1.

$$[\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}] = \begin{pmatrix} \alpha\beta & \beta^2 \\ -\alpha^2 & -\alpha\beta \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix};$$

2.

$$\begin{aligned} &[\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}][\tilde\alpha e_{11} + \tilde\beta e_{12}, \tilde\alpha e_{21} + \tilde\beta e_{22}] \\ &\quad + [\tilde\alpha e_{11} + \tilde\beta e_{12}, \tilde\alpha e_{21} + \tilde\beta e_{22}][\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}] \\ &\quad = -\begin{vmatrix} \alpha & \beta \\ \tilde\alpha & \tilde\beta \end{vmatrix}^2 (e_{11} + e_{22}) \end{aligned}$$

Proof. The first equality is verified by explicit calculations. In order to prove the second one, we notice that

$$\begin{aligned} &[\alpha e_{11} + \beta e_{12}, \alpha e_{21} + \beta e_{22}][\tilde\alpha e_{11} + \tilde\beta e_{12}, \tilde\alpha e_{21} + \tilde\beta e_{22}] \\ &= \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde\beta & 0 \\ -\tilde\alpha & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \alpha\tilde\beta - \beta\tilde\alpha & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} \\ &= \begin{vmatrix} \alpha & \beta \\ \tilde\alpha & \tilde\beta \end{vmatrix} \begin{pmatrix} \beta & 0 \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} \tilde\alpha & \tilde\beta \\ 0 & 0 \end{pmatrix} = \begin{vmatrix} \alpha & \beta \\ \tilde\alpha & \tilde\beta \end{vmatrix} \begin{pmatrix} \tilde\alpha\beta & \beta\tilde\beta \\ -\alpha\tilde\alpha & -\alpha\tilde\beta \end{pmatrix}. \end{aligned}$$

∎

Now we can prove the existence of a multilinear polynomial $(FT)^*$-non-identity with sufficiently numerous alternations.

Lemma 5.4.4. *Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 5.4. Suppose also that (5.13) holds or $T_0 = \emptyset$. Then there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exist disjoint subsets $X_1, \ldots, X_{2k} \subseteq \{x_1, \ldots, x_n\}$, $k = \lfloor \frac{n-n_0}{2 \dim A} \rfloor$, $|X_1| = \ldots = |X_{2k}| = \dim A$ and a polynomial $f \in P_n^H \setminus \mathrm{Id}^H(A)$ alternating in the variables of each set X_j.*

Proof. We start by considering the Regev polynomial of $M_2(F)$, cf. (1.7),

$$f_0(x_1, \ldots, x_4, y_1, \ldots, y_4) = \sum_{\sigma, \rho \in S_4} \mathrm{sign}(\sigma\rho) x_{\sigma(1)}\; y_{\rho(1)}\; x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\; y_{\rho(2)} y_{\rho(3)} y_{\rho(4)}$$

which was proven by Formanek [For87] to be a non-identity taking only central values on $M_2(F)$. Let

$$f_{t_1,t_2}(x_{t_1,1}, x_{t_1,2}, x_{t_2,1}, x_{t_2,2}) := \left[x_{t_1,1}^{h_{t_1}}, x_{t_1,2}^{h_{t_1}}\right] \left[x_{t_2,1}^{h_{t_2}}, x_{t_2,2}^{h_{t_2}}\right] + \left[x_{t_2,1}^{h_{t_2}}, x_{t_2,2}^{h_{t_2}}\right] \left[x_{t_1,1}^{h_{t_1}}, x_{t_1,2}^{h_{t_1}}\right]$$

and

$$\begin{aligned} f_{t_1,t_2,t_3}(x_{t_1,1}, x_{t_1,2}, x_{t_2,1}, x_{t_2,2}, x_{t_3,1}, x_{t_3,2}) := {} & \left[x_{t_1,1}^{h_{t_1}}, x_{t_1,2}^{h_{t_1}}\right] \left[x_{t_3,1}^{h_{t_3}}, x_{t_3,2}^{h_{t_3}}\right] \left[x_{t_2,1}^{h_{t_2}}, x_{t_2,2}^{h_{t_2}}\right] \\ & - \left[x_{t_2,1}^{h_{t_2}}, x_{t_2,2}^{h_{t_2}}\right] \left[x_{t_3,1}^{h_{t_3}}, x_{t_3,2}^{h_{t_3}}\right] \left[x_{t_1,1}^{h_{t_1}}, x_{t_1,2}^{h_{t_1}}\right]. \end{aligned}$$

Let $\{t_1, \ldots, t_{|T_0|}\} = T_0$ be the elements from Lemma 5.4.1. If $2 \mid |T_0|$, then we define

$$\begin{aligned} f_1 = z_1 \ldots z_{n-(\dim A)2k} \prod_{i=1}^{k} \Bigg(& \prod_{t \in T_1} f_0(x_{i,t,1}^{h_t}, \ldots, x_{i,t,4}^{h_t}, y_{i,t,1}^{h_t}, \ldots, y_{i,t,4}^{h_t}) \\ & \times \left(\prod_{\ell=1}^{\frac{|T_0|}{2}} f_{t_{2\ell-1},t_{2\ell}}(x_{i,t_{2\ell-1},1}, x_{i,t_{2\ell-1},2}, x_{i,t_{2\ell},1}, x_{i,t_{2\ell},2}) \right. \\ & \left. \times f_{t_{2\ell-1},t_{2\ell}}(y_{i,t_{2\ell-1},1}, y_{i,t_{2\ell-1},2}, y_{i,t_{2\ell},1}, y_{i,t_{2\ell},2}) \right) \Bigg). \end{aligned}$$

If $2 \nmid |T_0|$, then we define

$$\begin{aligned} f_1 = z_1 \dots z_{n-(\dim A)2k} \prod_{i=1}^{k} & \left(\prod_{t \in T_1} f_0(x^{h_t}_{i,t,1}, \dots, x^{h_t}_{i,t,4}, y^{h_t}_{i,t,1}, \dots, y^{h_t}_{i,t,4}) \right) \\ & \times \left(\prod_{\ell=1}^{\frac{|T_0|-3}{2}} f_{t_{2\ell-1},t_{2\ell}}(x_{i,t_{2\ell-1},1}, x_{i,t_{2\ell-1},2}, x_{i,t_{2\ell},1}, x_{i,t_{2\ell},2}) \right. \\ & \qquad \left. \times f_{t_{2\ell-1},t_{2\ell}}(y_{i,t_{2\ell-1},1}, y_{i,t_{2\ell-1},2}, y_{i,t_{2\ell},1}, y_{i,t_{2\ell},2}) \right) \\ \times f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}} & (x_{i,t_{|T_0|-2},1}, x_{i,t_{|T_0|-2},2};\ x_{i,t_{|T_0|-1},1}, x_{i,t_{|T_0|-1},2};\ x_{i,t_{|T_0|},1}, x_{i,t_{|T_0|},2}) \\ \times f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}} & (y_{i,t_{|T_0|-2},1}, y_{i,t_{|T_0|-2},2};\ y_{i,t_{|T_0|-1},1}, y_{i,t_{|T_0|-1},2};\ y_{i,t_{|T_0|},1}, y_{i,t_{|T_0|},2}). \end{aligned}$$

We claim that $f_1 \notin \mathrm{Id}^{(FT)^*}(A)$. If $2 \mid |T_0|$, then we take any isomorphism

$$\psi \colon A/J(A) \xrightarrow{\sim} M_2(F).$$

If $2 \nmid |T_0|$, we define ψ as follows. First, we notice that $t_{|T_0|-2} \nsim t_{|T_0|-1}$ implies $A/J(A) = I_{t_{|T_0|-2}} \oplus I_{t_{|T_0|-1}}$. By Theorem 4.2.2, there exists an isomorphism $\psi \colon A/J(A) \xrightarrow{\sim} M_2(F)$ such that $\psi(I_{t_{|T_0|-2}}) = \langle e_{11}, e_{21} \rangle_F$ and $\psi(I_{t_{|T_0|-1}}) = \langle e_{12}, e_{22} \rangle_F$. If $2 \nmid |T_0|$, then we take this isomorphism ψ.

Lemma 4.2.3 implies that for every $t \in T_0$ there exist $\alpha_t, \beta_t \in F$ such that

$$\psi(I_t) = \mathrm{span}_F\{\alpha_t e_{i1} + \beta_t e_{i2} \mid i = 1, 2\}.$$

If $2 \nmid |T_0|$, by our choice of ψ, we may assume that

$$(\alpha_{t_{|T_0|-2}}, \beta_{t_{|T_0|-2}}) = (1, 0) \text{ and } (\alpha_{t_{|T_0|-1}}, \beta_{t_{|T_0|-1}}) = (0, 1).$$

Note that $I_{t_1} = I_{t_2}$ if and only if the rows $(\alpha_{t_1}, \beta_{t_1})$ and $(\alpha_{t_2}, \beta_{t_2})$ are proportional.

Fix some element $e \in A$ such that $\psi\pi(e)$ is the identity matrix. We substitute $z_1 = \dots = z_{n-(\dim A)2k} = e$,

$$x_{i,t,1} = y_{i,t,1} = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{11}),\ x_{i,t,2} = y_{i,t,2} = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{12}),$$

$$x_{i,t,3} = y_{i,t,3} = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{21}),\ x_{i,t,4} = y_{i,t,4} = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{22})$$

for all $t \in T_1$ and $1 \le i \le k$ and

$$x_{i,t,1} = y_{i,t,1} = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(\alpha_t e_{11} + \beta_t e_{12}),$$

$$x_{i,t,2} = y_{i,t,2} = \left(\pi\Big|_{A^{(t)}}\right)^{-1} \psi^{-1}(\alpha_t e_{21} + \beta_t e_{22})$$

for all $t \in T_0$ and $1 \leq i \leq k$.

In order to show that f_1 does not vanish under this evaluation, we apply $\psi\pi$ to the result of the substitution. The value of $f_{t_{2\ell-1},t_{2\ell}}$ is nonzero since by Lemma 5.4.3,

$$\begin{aligned}
&[\alpha_{t_{2\ell-1}} e_{11} + \beta_{t_{2\ell-1}} e_{12},\ \alpha_{t_{2\ell-1}} e_{21} + \beta_{t_{2\ell-1}} e_{22}][\alpha_{t_{2\ell}} e_{11} + \beta_{t_{2\ell}} e_{12},\ \alpha_{t_{2\ell}} e_{21} + \beta_{t_{2\ell}} e_{22}] \\
&+[\alpha_{t_{2\ell}} e_{11} + \beta_{t_{2\ell}} e_{12},\ \alpha_{t_{2\ell}} e_{21} + \beta_{t_{2\ell}} e_{22}][\alpha_{t_{2\ell-1}} e_{11} + \beta_{t_{2\ell-1}} e_{12},\ \alpha_{t_{2\ell-1}} e_{21} + \beta_{t_{2\ell-1}} e_{22}] \\
&= - \begin{vmatrix} \alpha_{t_{2\ell-1}} & \beta_{t_{2\ell-1}} \\ \alpha_{t_{2\ell}} & \beta_{t_{2\ell}} \end{vmatrix}^2 (e_{11} + e_{22}) \neq 0.
\end{aligned}$$

The polynomial $f_{t_{|T_0|-2},t_{|T_0|-1},t_{|T_0|}}$ does not vanish under this evaluation because

$$\begin{aligned}
&[e_{11}, e_{21}][\alpha_{t_{|T_0|}} e_{11} + \beta_{t_{|T_0|}} e_{12}, \alpha_{t_{|T_0|}} e_{21} + \beta_{t_{|T_0|}} e_{22}][e_{12}, e_{22}] \\
&-[e_{12}, e_{22}][\alpha_{t_{|T_0|}} e_{11} + \beta_{t_{|T_0|}} e_{12}, \alpha_{t_{|T_0|}} e_{21} + \beta_{t_{|T_0|}} e_{22}][e_{11}, e_{21}] = -\alpha_{t_{|T_0|}} \beta_{t_{|T_0|}} (e_{11} + e_{22}) \neq 0.
\end{aligned}$$

by Lemma 5.4.3. Thus $f_1 \notin \mathrm{Id}^{(FT)^*}(A)$. Now we define $f = \mathrm{Alt}_1 \ldots \mathrm{Alt}_{2k} f_1$, where Alt_i is the operator of alternation on the set X_i, as defined after (1.7), where

$$X_{2i-1} = \{x_{i,t,j} \mid t \in T,\ 1 \leq j \leq 2 \text{ for } t \in T_0,\ 1 \leq j \leq 4 \text{ for } t \in T_1\},$$

$$X_{2i} = \{y_{i,t,j} \mid t \in T,\ 1 \leq j \leq 2 \text{ for } t \in T_0,\ 1 \leq j \leq 4 \text{ for } t \in T_1\},$$

$1 \leq i \leq 2k$. Note that f does not vanish under the same substitution that we used for f_1 since if the alternation replaces x_{i_1,t_1,j_1} with x_{i_2,t_2,j_2} for $t_1 \neq t_2$, then the value of $x_{i_2,t_2,j_2}^{h_{t_1}}$ is zero and the corresponding item vanishes.

For our convenience, we rename the variables of f to $x_1, \ldots, x_n$. Then f satisfies all the conditions of the lemma. ∎

Now we have all the ingredients to conclude that in the case (5.13) is satisfied, one has $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim A$.

Theorem 5.4.5. *Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 5.4. Suppose also that (5.13) holds or $T_0 = \varnothing$. Then there exist $C > 0$, $D \in \mathbb{R}$, such that*

$$Cn^D (\dim_F A)^n \leq c_n^{T\text{-gr}}(A) \leq (\dim_F A)^{n+1}.$$

In particular, $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim_F A$.

Proof. First note that by Lemma 5.1.4, $c_n^{T\text{-gr}}(A) = c_n^{(FT)^*}(A)$ for all $n \in \mathbb{N}$. The upper bound follows from Proposition 5.1.6. In order to get the lower bound, remark that the proof of Proposition 1.3.11 also works, word by word, for graded polynomials. So by Lemma 5.4.4 and Lemma 5.3.2 we see that the proof of Proposition 1.3.11 yields that for all $n \geq n_0$ there exists a partition $\lambda = (\lambda_1, \dots, \lambda_{\dim A}) \vdash n$ such that $\lambda_i > 2k$ for every $1 \leq i \leq \dim_F A$ and moreover $m_\lambda^T \neq 0$.

Now as shown after Proposition 1.3.11, since $m_\lambda^T \neq 0$ we know that $c_n(A) \geq \dim_F S^F(\lambda)$. Further by Corollary 1.2.22 and Example 1.2.21,as $k \to \infty$,

$$\dim_F S^F(\lambda) \geq \dim_F S^F((2k)^{\dim A}) \simeq C k^{\frac{1-(\dim A)^2}{\dim A}} (\dim A)^{2k \dim A},$$

for some constant $C \in \mathbb{R}$. Since $k = \lfloor \frac{n-n_0}{2 \dim A} \rfloor$ this finishes the proof. ∎

5.5 The case $A/J(A) \cong M_2(F)$ and $\mathrm{PIexp}^{T\text{-gr}}(A) < \dim A$

Assume A and T are as in the previous section. *Now we handle the second case, i.e. we suppose that $T_0 \neq \varnothing$ and the inequality (5.13) does not hold in A.* This is equivalent to the existence of $t_0 \in T_0$ such that $|\bar{t}_0| > \frac{|T_0|}{2}$. Using Theorem 4.2.2, we fix an isomorphism $\psi \colon A/J(A) \to M_2(F)$ such that $\psi(I_{t_0}) = \operatorname{span}_F\{e_{11}, e_{21}\}$. By Lemma 4.2.3, for every $t \in T_0$ one can choose $\alpha_t, \beta_t \in F$ such that $(\alpha_t e_{i1} + \beta_t e_{i2} \mid 1 \leq i \leq 2)$ is a basis of $\psi(I_t)$. We may assume that $(\alpha_t, \beta_t) = (1, 0)$ for $t \sim t_0$. Note that $\beta_t \neq 0$ if $I_t \neq I_{t_0}$.

Now we fix the basis

$$\left(\left(\pi\big|_{A^{(t)}} \right)^{-1} \psi^{-1}(\alpha_t e_{i1} + \beta_t e_{i2}) \;\Big|\; 1 \leq i \leq 2 \right)$$

in $A^{(t)}$ for each $t \in T_0$ and

$$\left(\left(\pi\big|_{A^{(t)}} \right)^{-1} \psi^{-1}(e_{ij}) \;\Big|\; 1 \leq i, j \leq 2 \right)$$

in $A^{(t)}$ for each $t \in T_1$. Further let $\mathcal{B}$ be the basis of A consisting of the union of the bases in $A^{(t)}$ chosen above.

Now we calculate the numbers γ_i introduced at the beginning of Section 5.3. We notice that

$$\theta \left(\left(\pi\big|_{A^{(t)}} \right)^{-1} \psi^{-1}(\alpha_t e_{i1} + \beta_t e_{i2}) \right) = i - 2$$

for $1 \le i \le 2$ and $t \in T_0$, $t \nsim t_0$. Then

$$(\gamma_1, \dots, \gamma_r) = (\underbrace{-1, \dots, -1}_{|T_0|+|T_1|-|\bar{t}_0|}, \underbrace{0, \dots, 0}_{|T_0|+2|T_1|}, \underbrace{1, \dots, 1}_{|T_1|+|\bar{t}_0|}). \tag{5.14}$$

Recall also that the partitions λ with $m_\lambda^T \ne 0$ satisfy the inequality from Lemma 5.3.2.

Below we prove three lemmas which enable us to choose elements $b_1, \dots, b_m$ that we will substitute for the variables corresponding to the numbers in a column of a given Young diagram. Hereby it is important to control the sum $\sum_{j=1}^m \theta(b_i)$.

Lemma 5.5.1. *Let $1 \le m \le r$. Then*

$$m - \sum_{j=1}^m \gamma_j \le 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

Proof. If $m \le |T_0| + |T_1| - |\bar{t}_0|$, then $\gamma_j = -1$ for $1 \le j \le m$ and $\sum_{j=1}^m \gamma_j = -m$. Hence

$$m - \sum_{j=1}^m \gamma_j = 2m \le 2|T_0| + 2|T_1| - 2|\bar{t}_0| \le 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

If $|T_0| + |T_1| - |\bar{t}_0| \le m \le 2|T_0| + 3|T_1| - |\bar{t}_0|$, then $\gamma_j = 0$ for $|T_0| + |T_1| - |\bar{t}_0| < j \le m$ and $\sum_{j=1}^m \gamma_j = -(|T_0| + |T_1| - |\bar{t}_0|)$. Hence

$$m - \sum_{j=1}^m \gamma_j = m + (|T_0| + |T_1| - |\bar{t}_0|) \le 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

If $m \ge 2|T_0| + 3|T_1| - |\bar{t}_0|$, then $\gamma_j = 1$ for $j > 2|T_0| + 3|T_1| - |\bar{t}_0|$ implies

$$m - \sum_{j=1}^m \gamma_j = 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

∎

Lemma 5.5.2. *Let $\sum_{j=1}^m \gamma_j > 0$ for some $m \in \mathbb{N}$. Then there exists $b_1, \dots, b_m \in \mathcal{B}$, $b_i \ne b_j$ for $i \ne j$, such that $\sum_{j=1}^m \theta(b_j) = \sum_{j=1}^m \gamma_j$ and*

- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i\}$ for some $1 \le i \le m$ and $t \in T_0 \sqcup T_1$, then $t \in T_0$, $t \sim t_0$ and $b_i = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{11})$;*
- *if $\{b_1, \dots, b_m\} \cap A^{(t)} = \{b_i, b_j\}$ for some $1 \le i, j \le m$ and $t \in T_0 \sqcup T_1$, then either $\theta(b_i) \ne 0$ or $\theta(b_j) \ne 0$.*

Proof. By the definition of γ_i (see the beginning of Section 5.3), there exist $b_1, \ldots, b_m \in \mathcal{B}$, $b_i \neq b_j$ for $i \neq j$, such that $\sum_{j=1}^m \theta(b_j) = \sum_{j=1}^m \gamma_j$ and $\sum_{j=1}^m \theta(a_j) \geq \sum_{j=1}^m \gamma_j$ for all $a_1, \ldots, a_m \in \mathcal{B}$ where $a_i \neq a_j$ for $i \neq j$. Since $\sum_{j=1}^m \gamma_j > 0$, the minimality of $\sum_{j=1}^m \theta(b_j)$ implies that the set $\{b_1, \ldots, b_m\}$ contains all elements $b \in \mathcal{B}$ with $\theta(b) \leq 0$. Now the choice of $\mathcal{B}$ implies the lemma. ∎

Lemma 5.5.3. *Let $\sum_{j=1}^m \gamma_j \leq q \leq 0$ for some $m \in \mathbb{N}$, $q \in \mathbb{Z}$. Then there exist $b_1, \ldots, b_m \in \mathcal{B}$, $b_i \neq b_j$ for $i \neq j$, such that $\sum_{j=1}^m \theta(b_j) = q$ and*

- *if $\{b_1, \ldots, b_m\} \cap A^{(t)} = \{b_i\}$, $\theta(b_i) = 0$, for some $1 \leq i \leq m$ and $t \in T_0 \sqcup T_1$, then $t \in T_0$, $t \sim t_0$ and $b_i = \left(\pi\big|_{A^{(t)}}\right)^{-1} \psi^{-1}(e_{11})$;*
- *if $\{b_1, \ldots, b_m\} \cap A^{(t)} = \{b_i, b_j\}$ for some $1 \leq i, j \leq m$ and $t \in T_0 \sqcup T_1$, then either $\theta(b_i) \neq 0$ or $\theta(b_j) \neq 0$.*

Proof. Recall that

$$|\{b \in \mathcal{B} \mid \theta(b) = -1\}| = |T_0| + |T_1| - |\bar{t}_0|,$$

$$|\{b \in \mathcal{B} \mid \theta(b) = 0\}| = |T_0| + 2|T_1|,$$

$$|\{b \in \mathcal{B} \mid \theta(b) = 1\}| = |T_1| + |\bar{t}_0|.$$

Let

$$\{t_1, \ldots, t_{|T_1|}\} := T_1, \{\tilde{t}_1, \ldots, \tilde{t}_{|T_0|-|\bar{t}_0|}\} := T_0 \backslash \bar{t}_0 \text{ and } \{\hat{t}_1, \ldots, \hat{t}_{|\bar{t}_0|}\} := \bar{t}_0.$$

Now we consider two main cases:

(1) Suppose $m < 2(|T_0| + |T_1| - |\bar{t}_0|) + q$. Let $\ell = \lfloor \frac{m+q}{2} \rfloor$. Note that

$$\ell - q \leq \frac{m-q}{2} < |T_0| + |T_1| - |\bar{t}_0|$$

and $\ell \leq \ell - q < |T_0| + |T_1| - |\bar{t}_0| < |T_1| + |\bar{t}_0|$. These inequalities imply that below we have enough elements from $(T_0 \backslash \bar{t}_0) \sqcup T_1$ and $\bar{t}_0 \sqcup T_1$, respectively.

Suppose first that $m = 2\ell - q$. If $\ell \leq |T_1| + q$, then we take

$$\{b_1, \ldots, b_m\} = \left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{12}) \mid 1 \leq i \leq \ell - q \right\} \cup \left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq \ell \right\}.$$

If $|T_1| + q < \ell \leq |T_1|$, then we take

$$\begin{aligned}\{b_1, \ldots, b_m\} = &\left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{12}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi\big|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leq i \leq \ell - q - |T_1| \right\} \\ &\cup \left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq \ell \right\}.\end{aligned}$$

If $\ell > |T_1|$, then we take

$$\begin{aligned}\{b_1, \ldots, b_m\} = &\left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{12}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi\big|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leq i \leq \ell - q - |T_1| \right\} \\ &\cup \left\{ \left(\pi\big|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi\big|_{A^{(\hat{t}_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq \ell - |T_1| \right\}.\end{aligned}$$

If $m = 2\ell - q + 1$, then we add the element $\left(\pi\big|_{A^{(\hat{t}_1)}}\right)^{-1} \psi^{-1}(e_{11})$ in each of the three cases above.

(2) Suppose $m \geq 2(|T_0| + |T_1| - |\bar{t}_0|) + q$.

By Lemma 5.5.1,

$$m - \sum_{j=1}^{m} \gamma_j \leq 3|T_0| + 4|T_1| - 2|\bar{t}_0|.$$

Hence

$$m - q - 2(|T_0| + |T_1| - |\bar{t}_0|) \leq |T_0| + 2|T_1|$$

and we can choose

$$0 \leq k, \ell \leq |T_1|, \qquad 0 \leq s \leq |\bar{t}_0|, \qquad 0 \leq u \leq |T_0| - |\bar{t}_0|,$$

such that $2(|T_0| + |T_1| - |\bar{t}_0|) + q + k + \ell + s + u = m$.

Note that

$$(|T_0| + |T_1| - |\bar{t}_0|) + q \geq (|T_0| + |T_1| - |\bar{t}_0|) + \sum_{j=1}^{m} \gamma_j \geq 0.$$

If $(|T_0| + |T_1| - |\bar{t}_0|) + q \leq |T_1|$, we define

$$\begin{aligned}\{b_1, \ldots, b_m\} = &\left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{12}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{11}) \mid 1 \leq i \leq k \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq (|T_0| + |T_1| - |\bar{t}_0|) + q \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{22}) \mid 1 \leq i \leq \ell \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leq i \leq |T_0| - |\bar{t}_0| \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{21} + \beta_{\tilde{t}_i} e_{22}) \mid 1 \leq i \leq u \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\hat{t}_i)}}\right)^{-1} \psi^{-1}(e_{11}) \mid 1 \leq i \leq s \right\}.\end{aligned}$$

If $(|T_0| + |T_1| - |\bar{t}_0|) + q > |T_1|$, we define

$$\begin{aligned}\{b_1, \ldots, b_m\} = &\left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{12}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{11}) \mid 1 \leq i \leq k \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq |T_1| \right\} \\ &\cup \left\{ \left(\pi|_{A^{(t_i)}}\right)^{-1} \psi^{-1}(e_{22}) \mid 1 \leq i \leq \ell \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{11} + \beta_{\tilde{t}_i} e_{12}) \mid 1 \leq i \leq |T_0| - |\bar{t}_0| \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\tilde{t}_i)}}\right)^{-1} \psi^{-1}(\alpha_{\tilde{t}_i} e_{21} + \beta_{\tilde{t}_i} e_{22}) \mid 1 \leq i \leq u \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\hat{t}_i)}}\right)^{-1} \psi^{-1}(e_{11}) \mid 1 \leq i \leq s \right\} \\ &\cup \left\{ \left(\pi|_{A^{(\hat{t}_i)}}\right)^{-1} \psi^{-1}(e_{21}) \mid 1 \leq i \leq (|T_0| - |\bar{t}_0|) + q \right\}.\end{aligned}$$

∎

In Section 1.3.2 we explained that if we can reduce the region Ω to a region Ω_0 having the properties

$$\max_{\Omega} \Phi = \max_{\Omega_0} \Phi \text{ and if } \lambda \vdash n \text{ such that } (\frac{\lambda_1}{n}, \ldots, \frac{\lambda_d}{n}) \in \Omega_0 \text{ then } m_\lambda \neq 0$$

then the lower bound would follow immediately from it. Unfortunately we are not able to do this. However, the next lemma is quite close to reaching this. Due to this we will see in the proof of Theorem 5.5.5 that the idea of considering the partition $\mu = (n - \sum_{i=2}^{r} \lfloor \alpha_i n \rfloor, \lfloor \alpha_2 n \rfloor, \dots, \lfloor \alpha_r n \rfloor)$ associated with an extremal point in order to finish the lower bound also 'almost works'. By this we mean that the existence of the needed partition will follow from the form of μ and Lemma 5.5.4.

Lemma 5.5.4. *Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ for some $n \in \mathbb{N}$. If $\sum_{i=1}^{r} \gamma_i \lambda_i \le 0$ and $2 \mid \lambda_i$ for $i \ge 2$, then there exists a Young tableau T_λ of shape λ and an $(FT)^*$-polynomial $f \in P_n^{(FT)^*}(F)$ such that $b_{T_\lambda} f := \sum_{\sigma \in C_{T_\lambda}} (\operatorname{sign} \sigma) \sigma f \notin \mathrm{Id}^{(FT)^*}(A)$.*

Proof. For ease of notation write μ for the conjugate partition λ' of λ. In particular μ_i is the number of boxes in the i-th column of the tableau T_λ. Further let $m_i := \sum_{j=1}^{\mu_i} \gamma_j$ with $1 \le i \le \lambda_1$.

Note that since $(\lambda_i - \lambda_{i+1})$ equals the number of columns of height i (here $\lambda_{r+1} := 0$) and λ_1 equals the total number of columns, the inequality $\sum_{i=1}^{r} \gamma_i \lambda_i \le 0$ can be rewritten as

$$\sum_{i=1}^{r} \left(\sum_{j=1}^{i} \gamma_j - \sum_{j=1}^{i-1} \gamma_j \right) \lambda_i = \sum_{i=1}^{r} \left(\sum_{j=1}^{i} \gamma_j \right) (\lambda_i - \lambda_{i+1}) = \sum_{i=1}^{\lambda_1} m_i \le 0. \tag{5.15}$$

By Lemma 5.3.1, if the sum of the values of θ on basis elements substituted for the variables of a multilinear $(FT)^*$-polynomial is less than -1 or greater than 1, then the $(FT)^*$-polynomial vanishes. We will choose elements $b_{itj} \in A^{(t)} \cap \mathcal{B}$ such that the sum of the values of θ on them equals 0. If $m_i > 0$ for some i, we have to make the sum of values of θ for some other columns negative.

Define the number $1 \le \ell \le \lambda_1$ by the conditions $m_\ell > 0$ and $m_{\ell+1} \le 0$.

Since $2 \mid \lambda_i$ for all $i \ge 2$, we have $m_{2j-1} = m_{2j}$ for $1 \le j \le \frac{\lambda_2}{2}$. Hence $2 \mid \ell$ and $2 \mid \sum_{i=1}^{\ell} m_i$. Recall that $m_i = -1$ for $\lambda_2 + 1 \le i \le \lambda_1$. By (5.15),

$$2 \sum_{i=1}^{\lambda_2/2} m_{2i} - (\lambda_1 - \lambda_2) = \sum_{i=1}^{\lambda_1} m_i \le 0$$

and

$$\sum_{i=1}^{\lambda_2/2} m_{2i} - \left[\frac{\lambda_1 - \lambda_2}{2} \right] \le 0.$$

Thus one can choose integers N and q_{2i} for $\ell < 2i \le \lambda_2$, such that $0 \le N \le \left[\frac{\lambda_1 - \lambda_2}{2} \right]$, $m_{2i} \le q_{2i} \le 0$ and $\sum_{i=1}^{\ell/2} m_{2i} + \sum_{i=\ell/2+1}^{\lambda_2} q_{2i} - N = 0$. Define $q_{2i-1} := q_{2i}$ for $\frac{\ell}{2} < i \le \frac{\lambda_2}{2}$, $q_i := -1$ for $\lambda_2 + 1 \le i \le \lambda_2 + 2N$, $q_i := 0$ for $\lambda_2 + 2N + 1 \le i \le \lambda_1$. Then

$$\sum_{i=1}^{\ell} m_i + \sum_{i=\ell+1}^{\lambda_1} q_i = 0. \tag{5.16}$$

Now we use Lemma 5.5.2 for every $1 \leq i \leq \ell$ (there $m = \mu_i$) and Lemma 5.5.3 for every $\ell + 1 \leq i \leq \lambda_1$ (there $m = \mu_i$ and $q = q_i$). We sort the obtained elements b_j in accordance with the homogeneous components $A^{(t)}$ they belong to. For a fixed $1 \leq i \leq \lambda_1$ we get elements $b_{itj} \in A^{(t)} \cap \mathcal{B}$ where $t \in T_0 \sqcup T_1$, $1 \leq j \leq n_{it}$, the total number of b_{itj} equals $n_{it} \geq 0$,

$$\sum_{t \in T_0 \sqcup T_1} n_{it} = \mu_i,$$

$$b_{it_1j_1} \neq b_{it_2j_2} \text{ if } (t_1, j_1) \neq (t_2, j_2),$$

$$\sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) = m_i \text{ if } m_i > 0$$

and

$$m_i \leq \sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) = q_i \leq 0 \text{ if } m_i \leq 0.$$

By the virtue of (5.16),

$$\sum_{i=1}^{\lambda_1} \sum_{t \in T_0 \sqcup T_1} \sum_{j=1}^{n_{it}} \theta(b_{itj}) = 0. \tag{5.17}$$

Since $q_{2i-1} = q_{2i}$ and $\mu_{2i-1} = \mu_{2i}$ if $2i \leq \lambda_2$, we may assume that $n_{2i-1,t} = n_{2i,t}$, $b_{2i-1,t,j} = b_{2i,t,j}$ for all $1 < 2i \leq \lambda_2$, $t \in T_0 \sqcup T_1$, $1 \leq j \leq n_{2i,t}$.

We will substitute elements b_{itj} for the variables with the indices from the i-th column.

Denote by W_{-1} the set of all pairs (i, t) where $1 \leq i \leq \lambda_1$, $t \in T_0 \sqcup T_1$, such that $\theta(b_{itj}) = -1$ for some $1 \leq j \leq n_{ti}$ and $\theta(b_{itj}) \leq 0$ for all $1 \leq j \leq n_{ti}$. Denote by W_1 the set of all pairs (i, t) where $1 \leq i \leq \lambda_1$, $t \in T_0 \sqcup T_1$, such that $\theta(b_{itj}) = 1$ for some $1 \leq j \leq n_{ti}$ and $\theta(b_{itj}) \geq 0$ for all $1 \leq j \leq n_{ti}$. Let $W_1^{(i)} := \{t \in T_0 \sqcup T_1 \mid (i, t) \in W_1\}$ for $1 \leq i \leq \lambda_1$ and $W_0^{(i)} := \{t \in T_0 \sqcup T_1 \mid (i, t) \notin W_{-1} \sqcup W_1, n_{it} > 0\}$.

By (5.17), $|W_{-1}| = |W_1|$. Therefore, there exist maps $\varkappa \colon W_1 \to \{1, \ldots, \lambda_1\}$ and $\rho \colon W_1 \to T_0 \sqcup T_1$ such that $(i, t) \mapsto (\varkappa(i, t), \rho(i, t))$ is a bijection $W_1 \to W_{-1}$.

Define polynomials f_{it} and $\tilde{f}_{it}$, $1 \leq i \leq \lambda_2$, $t \in T_0 \sqcup T_1$, as follows.

If $n_{it} = 1$, then $f_{it}(x_1) = x_1$.

If $n_{it} = 2$, then $f_{it}(x_1, x_2) = x_1x_2 - x_2x_1$.

If $n_{it} = 3$, then $f_{it}(x_1, x_2, x_3) = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$.

If $1 \leq n_{it} \leq 3$, then

$$\tilde{f}_{it}(x_1, \ldots, x_{n_{it}}; y_1, \ldots, y_{n_{it}}) = f_{it}(x_1, \ldots, x_{n_{it}}) f_{it}(y_1, \ldots, y_{n_{it}}).$$

If $n_{it} = 4$, we consider Regev's central polynomial

$$\tilde{f}_{it}(x_1, \ldots, x_4, y_1, \ldots, y_4) = \sum_{\sigma,\tau \in S_4} \operatorname{sign}(\sigma\tau) x_{\sigma(1)}\ y_{\tau(1)}\ x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}\ y_{\tau(2)} y_{\tau(3)} y_{\tau(4)}.$$

Let $X_i := \{x_{itj} \mid 1 \leq j \leq n_{it},\ t \in T_0 \sqcup T_1\}$, $1 \leq i \leq \lambda_1$. Define

$$\begin{aligned} f := {} & \mathrm{Alt}_{X_1} \mathrm{Alt}_{X_2} \ldots \mathrm{Alt}_{X_{\lambda_1}} \\ & \prod_{i=1}^{\lambda_2/2} \left(\prod_{t \in W_0^{(2i-1)}} \tilde{f}_{2i-1,t}(x^{h_t}_{2i-1,t,1}, \ldots, x^{h_t}_{2i-1,t,n_{2i-1,t}};\ x^{h_t}_{2i,t,1}, \ldots, x^{h_t}_{2i,t,n_{2i,t}}) \right. \\ & \times \left(\prod_{t \in W_1^{(2i-1)}} f_{\varkappa(2i-1,t)\rho(2i-1,t)} \left(x^{h_{\rho(2i-1,t)}}_{\varkappa(2i-1,t)\rho(2i-1,t)1}, \ldots, x^{h_{\rho(2i-1,t)}}_{\varkappa(2i-1,t)\rho(2i-1,t)n_{\varkappa(2i-1,t)\rho(2i-1,t)}} \right) \right. \\ & \times f_{2i-1,t}(x^{h_t}_{2i-1,t,1}, \ldots, x^{h_t}_{2i-1,t,n_{2i-1,t}}) \\ & \times f_{\varkappa(2i,t)\rho(2i,t)} \left(x^{h_{\rho(2i,t)}}_{\varkappa(2i,t)\rho(2i,t)1}, \ldots, x^{h_{\rho(2i,t)}}_{\varkappa(2i,t)\rho(2i,t)n_{\varkappa(2i,t)\rho(2i,t)}} \right) \\ & \left.\left. \times f_{2i,t}(x^{h_t}_{2i,t,1}, \ldots, x^{h_t}_{2i,t,n_{2i,t}}) \right)\right) \prod_{\substack{i=\lambda_2+1, \\ t \in W_0^{(i)}}}^{\lambda_1} x^{h_t}_{it1}. \end{aligned}$$

Note that by Lemma 5.5.2 and Lemma 5.5.3, if $(i,t) \in W_{-1}$, then

$$\{\psi\pi(b_{it1}), \ldots, \psi\pi(b_{itn_{it}})\}$$

coincides with one of the following sets: $\{e_{12}\}$, $\{\alpha_t e_{11} + \beta_t e_{12}\}$, $\{e_{12}, e_{22}\}$, $\{e_{11}, e_{12}\}$, $\{\alpha_t e_{11} + \beta_t e_{12}, \alpha_t e_{21} + \beta_t e_{22}\}$, $\{e_{11}, e_{12}, e_{22}\}$.

If $t \in W_0^{(i)}$, then $\{\psi\pi(b_{it1}), \ldots, \psi\pi(b_{itn_{it}})\}$ coincides with one of the following sets: $\{e_{11}\}$, $\{e_{12}, e_{21}\}$, $\{e_{12}, e_{22}, e_{21}\}$, $\{e_{11}, e_{12}, e_{21}\}$, $\{e_{11}, e_{12}, e_{22}, e_{21}\}$.

If $t \in W_1^{(i)}$, then $\{\psi\pi(b_{it1}), \ldots, \psi\pi(b_{itn_{it}})\}$ coincides with one of the following sets: $\{e_{21}\}$, $\{e_{22}, e_{21}\}$, $\{e_{21}, e_{11}\}$, $\{e_{22}, e_{21}, e_{11}\}$.

By Lemma 5.4.3 and the remarks above, the image of the value of f under the substitution $x_{itj} = b_{itj}$, $1 \leq i \leq \lambda_1$, $t \in T_0 \sqcup T_1$, $1 \leq j \leq n_{it}$, does not vanish under the homomorphism $\psi\pi$ since if the alternation replaces x_{i_1,t_1,j_1} with x_{i_2,t_2,j_2} for $t_1 \neq t_2$, then the value of $x^{h_{t_1}}_{i_2,t_2,j_2}$ is zero and the corresponding item vanishes. For our convenience, we rename the variables of f to $x_1, \ldots, x_n$. Then f satisfies all the conditions of the lemma. ∎

Now we are ready to calculate $\mathrm{PIexp}^{T\text{-gr}}(A)$ in the second case, i.e. when (5.13) does not hold.

Theorem 5.5.5. *Let A be a finite dimensional T-graded-simple algebra over a field F of characteristic 0 for a right zero band T. Suppose $A/J(A) \cong M_2(F)$. Let $T_0, T_1 \subseteq T$ and $\sim$ be, respectively, the subsets and the equivalence relation defined at the beginning of Section 5.4. Suppose also that $|\bar{t}_0| > \frac{|T_0|}{2}$ for some $\bar{t}_0 \in T_0/\sim$. Then,*

$$\exp^{T\text{-gr}}(A) = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < 2|T_0| + 4|T_1| = \dim A.$$

Proof. Recall that at the beginning of Section 5.5 we chose the basis $\mathcal{B}$ in A. Equation (5.14) implies that $0 < \zeta = \sqrt{\frac{|T_0|+|T_1|-|\bar{t}_0|}{|T_1|+|\bar{t}_0|}} < 1$ is the root of (5.6).

Let

$$\Omega = \left\{ (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \;\middle|\; \sum_{i=1}^{r} \alpha_i = 1,\ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r \geq 0,\ \sum_{i=1}^{r} \gamma_i \alpha_i \leq 0 \right\}.$$

By Lemma 5.3.6,

$$d := \max_{x \in \Omega} \Phi(x) = \sum_{i=1}^{r} \zeta^{\gamma_i} = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}. \qquad (5.18)$$

Denote by $(\alpha_1, \ldots, \alpha_r) \in \Omega$ such a point that $\Phi(\alpha_1, \ldots, \alpha_r) = d$.

For every $n \in \mathbb{N}$ define $\mu \vdash n$ by $\mu_i = 2\left[\frac{\alpha_i n}{2}\right]$ for $2 \leq i \leq r$ and $\mu_1 = n - \sum_{i=2}^{r} \mu_i$.

By (5.10), $\sum_{i=1}^{r} \gamma_i \alpha_i = 0$. Since

$$\gamma_1 = \ldots = \gamma_{|T_0|+|T_1|-|\bar{t}_0|} = -1,$$

$$\gamma_{|T_0|+|T_1|-|\bar{t}_0|+1} = \ldots = \gamma_{2|T_0|+3|T_1|-|\bar{t}_0|} = 0,$$

and

$$\gamma_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \ldots = \gamma_r = 1.$$

By (5.11),

$$\alpha_1 = \ldots = \alpha_{|T_0|+|T_1|-|\bar{t}_0|},$$

$$\alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} = \ldots = \alpha_{2|T_0|+3|T_1|-|\bar{t}_0|},$$

$$\alpha_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \ldots = \alpha_r,$$

and we have

$$\alpha_1 n - 2 \leq \mu_2 = \ldots = \mu_{|T_0|+|T_1|-|\bar{t}_0|} \leq \alpha_1 n,$$

$$\alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} n - 2 \leq \mu_{|T_0|+|T_1|-|\bar{t}_0|+1} = \ldots = \mu_{2|T_0|+3|T_1|-|\bar{t}_0|} \leq \alpha_{|T_0|+|T_1|-|\bar{t}_0|+1} n,$$

$$\alpha_r n - 2 \leq \mu_{2|T_0|+3|T_1|-|\bar{t}_0|+1} = \ldots = \mu_r \leq \alpha_r n.$$

Now $\sum_{i=1}^{r} \alpha_i = 1$ implies $\alpha_1 n \le \mu_1 \le \alpha_1 n + 2r$.

Note that

$$\begin{aligned}\sum_{i=1}^{r} \gamma_i \mu_i &= -\left(n - \sum_{i=2}^{r} \mu_i\right) - \sum_{i=2}^{|T_0|+|T_1|-|\bar t_0|} \mu_i + \sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r} \mu_i \\ &= \left(\sum_{i=|T_0|+|T_1|-|\bar t_0|+1}^{2|T_0|+3|T_1|-|\bar t_0|} \mu_i\right) + 2\left(\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r} \mu_i\right) - n \\ \le n \left(\sum_{i=|T_0|+|T_1|-|\bar t_0|+1}^{2|T_0|+3|T_1|-|\bar t_0|} \alpha_i\right) &+ 2n\left(\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r} \alpha_i\right) - n\sum_{i=1}^{r} \alpha_i = n\sum_{i=1}^{r} \gamma_i\alpha_i = 0.\end{aligned} \tag{5.19}$$

Analogously,

$$\begin{aligned}\sum_{i=1}^{r} \gamma_i \mu_i &= \left(\sum_{i=|T_0|+|T_1|-|\bar t_0|+1}^{2|T_0|+3|T_1|-|\bar t_0|} \mu_i\right) + 2\left(\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r} \mu_i\right) - n \\ &\ge n \left(\sum_{i=|T_0|+|T_1|-|\bar t_0|+1}^{2|T_0|+3|T_1|-|\bar t_0|} \alpha_i\right) + 2n\left(\sum_{i=2|T_0|+3|T_1|-|\bar t_0|+1}^{r} \alpha_i\right) - n\sum_{i=1}^{r} \alpha_i - 4r \qquad (5.20) \\ &= n\sum_{i=1}^{r} \gamma_i\alpha_i - 4r = -4r.\end{aligned}$$

By (5.19) and Lemma 5.5.4, $b_{T_\mu} f \notin \operatorname{Id}^{(FT)^*}(A)$ for some $f \in P_n^{(FT)^*}(F)$. Let $\bar f$ be the image of f in $\frac{P_n^{(FT)^*}(F)}{P_n^{(FT)^*}(F) \cap \operatorname{Id}^{(FT)^*}(A)}$. Consider $FS_n b_{T_\mu} \bar f \subseteq \frac{P_n^{(FT)^*}(F)}{P_n^{(FT)^*}(F) \cap \operatorname{Id}^{(FT)^*}(A)}$. By Theorem 1.2.8 all S_n-representations over fields of characteristic 0 are completely decomposable into Specht modules, thus

$$FS_n b_{T_\mu} \bar f \cong FS_n e_{T_{\lambda^{(1)}}} \oplus \ldots \oplus FS_n e_{T_{\lambda^{(s)}}}$$

for some $\lambda^{(i)} \vdash n$ and some Young tableaux $T_{\lambda^{(i)}}$ of shape $\lambda^{(i)}$, $1 \le i \le s$, $s \in \mathbb{N}$. In particular, $e^*_{T_{\lambda^{(1)}}} FS_n b_{T_\mu} \ne 0$.

Now we notice that $e^*_{T_{\lambda^{(1)}}} FS_n b_{T_\mu} \ne 0$ implies $a_{T_{\lambda^{(1)}}} \sigma b_{T_\mu} = \sigma a_{\sigma^{-1} T_{\lambda^{(1)}}} b_{T_\mu} \ne 0$ for some $\sigma \in S_n$. Since $a_{\sigma^{-1} T_{\lambda^{(1)}}}$ is the operator of symmetrization in the numbers from the rows of the Young tableau $\sigma^{-1} T_{\lambda^{(1)}}$ and b_{T_μ} is the operator of alternation in the numbers from the columns of the Young tableau T_μ, all numbers from the first row of $\sigma^{-1} T_{\lambda^{(1)}}$ must be in different columns of T_μ. Thus $\left(\lambda^{(1)}\right)_1 \le \mu_1$. Moreover, all numbers from each of the first μ_r columns of T_μ must be in different rows of $\sigma^{-1} T_{\lambda^{(1)}}$. Since by Lemma 5.3.2, $\left(\lambda^{(1)}\right)_{r+1} = 0$, we have $\left(\lambda^{(1)}\right)_r \ge \mu_r$.

Thus (5.20) implies

$$\begin{aligned}\sum_{i=1}^{r}\gamma_i\left(\lambda^{(1)}\right)_i &\geq \sum_{i=1}^{r}\gamma_i\left(\lambda^{(1)}\right)_i - \sum_{i=1}^{r}\gamma_i\mu_i - 4r\\ &= \sum_{i=1}^{|T_0|+|T_1|-|\bar{t}_0|}\left(\mu_i - \left(\lambda^{(1)}\right)_i\right) + \sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\left(\left(\lambda^{(1)}\right)_i - \mu_i\right) - 4r\\ &\geq \sum_{i=1}^{|T_0|+|T_1|-|\bar{t}_0|}\left(\mu_1 - \left(\lambda^{(1)}\right)_i\right) + \sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\left(\left(\lambda^{(1)}\right)_i - \mu_r\right) - 6r - 2r^2\end{aligned}$$

Since by Lemma 5.3.2 we have $\sum_{i=1}^{r}\gamma_i\left(\lambda^{(1)}\right)_i \leq 1$ and both $\left(\mu_1 - \left(\lambda^{(1)}\right)_i\right)$ and $\left(\left(\lambda^{(1)}\right)_i - \mu_r\right)$ are nonnegative, we get $\mu_1 - (2r^2+6r+1) \leq \left(\lambda^{(1)}\right)_i \leq \mu_1$ for all $1 \leq i \leq |T_0|+|T_1|-|\bar{t}_0|$ and $\mu_r \leq \left(\lambda^{(1)}\right)_i \leq \mu_r + (2r^2+6r+1)$ for all $2|T_0|+3|T_1|-|\bar{t}_0|+1 \leq i \leq r$.

Recall that $a_{\sigma^{-1}T_{\lambda^{(1)}}}b_{T_\mu} \neq 0$ implies that all numbers from each column of T_μ are in different rows of $\sigma^{-1}T_{\lambda^{(1)}}$. Applying this to the first $\mu_{2|T_0|+3|T_1|-|\bar{t}_0|}$ columns, we obtain that in the last $|\bar{t}_0|+|T_1|+1$ rows of $T_{\lambda^{(1)}}$ we have at least $\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|}^{r}\mu_i$ boxes and

$$\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|}^{r}\left(\lambda^{(1)}\right)_i \geq \sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|}^{r}\mu_i = (|\bar{t}_0|+|T_1|)\mu_r + \mu_{2|T_0|+3|T_1|-|\bar{t}_0|}.$$

Thus

$$\begin{aligned}\lambda^{(1)}_{2|T_0|+3|T_1|-|\bar{t}_0|} &\geq (|\bar{t}_0|+|T_1|)\mu_r + \mu_{2|T_0|+3|T_1|-|\bar{t}_0|} - \textstyle\sum_{i=2|T_0|+3|T_1|-|\bar{t}_0|+1}^{r}\left(\lambda^{(1)}\right)_i\\ &\geq \mu_{2|T_0|+3|T_1|-|\bar{t}_0|} - (2r^3+6r^2+r)\end{aligned}$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, where

$$\lambda_i = \begin{cases} \mu_1 - (2r^2+6r+1) & \text{for} \quad 1 \leq i \leq |T_0|+|T_1|-|\bar{t}_0|,\\ \mu_{2|T_0|+3|T_1|-|\bar{t}_0|} - (2r^3+6r^2+r) & \text{for} \quad |T_0|+|T_1|-|\bar{t}_0|+1 \\ & \qquad\qquad \leq i \leq 2|T_0|+3|T_1|-|\bar{t}_0|,\\ \mu_r & \text{for} \quad 2|T_0|+3|T_1|-|\bar{t}_0|+1 \leq i \leq r.\end{cases}$$

Let $n_1 = \sum_{i=1}^{r}\lambda_i$. Note that $n - (2r^4+6r^3+r^2) \leq n_1 \leq n$.

For every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $\lambda_1 \geq \ldots \geq \lambda_r$ and $\Phi\left(\frac{\lambda_1}{n_1}, \ldots, \frac{\lambda_r}{n_1}\right) > d - \varepsilon$. Since D_λ is a subdiagram of $D_{\lambda^{(1)}}$, we have $c_n^{(FT)^*}(A) \geq \dim_F S^F(\lambda)$ and by the hook and the Stirling formulas, there exist $C_1 > 0$ and $r_1 \in \mathbb{R}$

such that we have

$$\begin{aligned} c_n^{(FT)^*}(A) \geq \dim_F S^F(\lambda) = \frac{n_1!}{\prod_{i,j} h_\lambda(i,j)} \geq \frac{n_1!}{(\lambda_1 + r - 1)! \dots (\lambda_r + r - 1)!} \\ \geq \frac{n_1!}{n_1^{r(r-1)} \lambda_1! \dots \lambda_r!} \geq \frac{C_1 n_1^{r_1} \left(\frac{n_1}{e}\right)^{n_1}}{\left(\frac{\lambda_1}{e}\right)^{\lambda_1} \dots \left(\frac{\lambda_r}{e}\right)^{\lambda_r}} \\ \geq C_1 n_1^{r_1} \left(\frac{1}{\left(\frac{\lambda_1}{n_1}\right)^{\frac{\lambda_1}{n_1}} \dots \left(\frac{\lambda_r}{n_1}\right)^{\frac{\lambda_r}{n_1}}} \right)^{n_1} \geq C_1 n_1^{r_1} (d - \varepsilon)^{n - (2r^4 + 6r^3 + r^2)}. \end{aligned} \tag{5.21}$$

Hence $\liminf_{n\to\infty} \sqrt[n]{c_n^{(FT)^*}(A)} \geq d - \varepsilon$. Since $\varepsilon > 0$ is arbitrary,

$$\liminf_{n\to\infty} \sqrt[n]{c_n^{(FT)^*}(A)} \geq d.$$

By Lemma 5.1.4 and (5.18),

$$\liminf_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \geq |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}.$$

Theorem 5.3.7 and (5.18) implies

$$\limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} \leq |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)}.$$

The condition $|\bar{t}_0| > \frac{|T_0|}{2}$ implies $|T_1| + |\bar{t}_0| > |T_0| + |T_1| - |\bar{t}_0|$,

$$2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < (|T_1| + |\bar{t}_0|) + (|T_0| + |T_1| - |\bar{t}_0|)$$

and

$$\lim_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(A)} = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < 2|T_0| + 4|T_1| = \dim_F A.$$

∎

Remark 5.5.6. A finite dimensional T-graded-simple algebra A with any given $|T_0|$, $|T_1|$, $\frac{|T_0|}{2} < |\bar{t}_0| \leq |T_0|$ exists by Proposition 4.4.2. Also the PI-exponent is either $\dim_F A$ or an irrational number. Since we proved that $\exp^{T\text{-gr}}(A)$ is equal to a certain maximum whose value is given by integer equations, the only rational numbers that could occur as $\exp^{T\text{-gr}}(A)$ for one of our T-graded simple algebras A are integer numbers.

5.6 Lie Algebras

In this section we consider finite dimensional Lie algebras, semigroup-gradings, their polynomial identities and invariants associated to them in a similar way as we did for associative algebras. In the first part we recall briefly the appropriate definitions and the state of the art. This part is based on [Gor15a, Section 1] and [GZ05, Chapter 12]. For the basic theory on Lie algebras we refer to [Ser65]. Next we construct the first example of a semigroup-graded Lie algebra with an irrational graded PI-exponent. In contrast to the associative case, this example can not be simple with respect to the grading as follows from results in [PZ89, EK13, Gor15a]. Along the way we note a graded version of Ado's Theorem which asserts the existence of a faithful finite dimensional Lie representation, in order to extract a polynomial bound on the multiplicities from the associative case (cf. Theorem 1.3.8).

In the third part we prove that if L is semisimple and a direct sum of H-simple subalgebras for some generalized action by a finite dimensional unital associative algebra H, then the H-PI-exponent exists and equals the maximum of the dimensions of the H-simple components.

5.6.1 Survey on Graded Lie Polynomials and their Codimensions

Let L be a finite dimensional Lie algebra over some field F of characteristic 0 and T an arbitrary semigroup. In the sequel we introduce, along the lines of [Gor15a], the necessary background, the Lie algebra variant of Amitsur's conjecture and state the most recent results concerning this problem.

We consider polynomials over F in the indeterminates $X^{T\text{-gr}} := \bigcup_{t\in T} X^{(t)}$ with $X^{(t)} = \{x_1^{(t)}, x_2^{(t)}, \ldots\}$. Let $F\{X^{T\text{-gr}}\}$ be the free non-associative algebra on $X^{T\text{-gr}}$. The algebra $F\{X^{T\text{-gr}}\}$ is naturally T-graded by defining the T-degree of $x_{i_1}^{(t_1)} \cdots x_{i_n}^{(t_n)}$ to be $t_1 \ldots t_n$, opposed to its total degree which is defined to be n. Thus,

$$F\{X^{T\text{-gr}}\} = \bigoplus_{t\in T} F\{X^{T\text{-gr}}\}^{(t)}$$

where, similar to before, $F\{X^{T\text{-gr}}\}^{(t)}$ is the subspace spanned by all the monomials having T-degree t and $F\{X^{T\text{-gr}}\}^{(t)} F\{X^{T\text{-gr}}\}^{(s)} \subseteq F\{X^{T\text{-gr}}\}^{(ts)}$.

Let J be the intersection of all graded ideals of $F\{X^{T\text{-gr}}\}$ containing the set

$$\{u(vw) + v(wu) + w(uv) \mid u, v, w \in F\{X^{T\text{-gr}}\}\} \cup \{u^2 \mid u \in F\{X^{T\text{-gr}}\}\}. \tag{5.22}$$

Then

$$L\left(X^{T\text{-gr}}\right) = F\{X^{T\text{-gr}}\}/J$$

is the free object on $X^{T\text{-gr}}$ in the category of T-graded Lie algebras, i.e. for any T-graded Lie algebra $L = \bigoplus_{t\in T} L^{(t)}$ and map $\psi : X^{T\text{-gr}} \to L$ such that $\psi(X^{(t)}) \subseteq L^{(t)}$ there exists a unique homomorphism $\overline{\psi} : L\left(X^{T\text{-gr}}\right) \to L$ of graded Lie algebras such that $\overline{\psi}\restriction_{X^{T\text{-gr}}} = \psi$.

In the sequel we use the commutator $[\cdot,\cdot]$ notation for multiplication in Lie algebras. Moreover, as before, all commutators will be left-normed, i.e. $[x_1,\dots,x_n] = [[x_1,\dots,x_{n-1}],x_n]$. In this notation, an F-vector space direct sum decomposition $L = \bigoplus_{t\in T} L^{(t)}$ of L is a T-grading if $\left[L^{(s)}, L^{(t)}\right] \subseteq L^{(st)}$ for all $s,t \in T$.

Remark. In case T is a commutative semigroup one can also first consider the free associative T-graded algebra $F\langle X^{T\text{-gr}}\rangle$, as in Section 5.1, and replace the multiplication with the additive bracket $[x,y] = xy - yx$. In this way we get a Lie algebra, denoted $F\langle X^{T\text{-gr}}\rangle^{[-]}$. Then by the Poincarré-Birkhoff-Witt theorem $F\langle X^{T\text{-gr}}\rangle^{[-]}$ is isomorphic to the ordinary free Lie algebra with free generators from $X^{T\text{-gr}}$ which in turn is isomorphic to $L\left(X^{T\text{-gr}}\right)$, as defined above, since then the ideal generated by the elements (5.22) is already graded. However if $ts \neq st$ for some $s,t \in T$ then, by the anti-commutativity, $[x_i^{(s)}, x_j^{(t)}] = 0$ for all i,j. This explains why $L\left(X^{T\text{-gr}}\right)$ is defined as above for non-abelian gradings.

As usual, we can now define polynomial identities and codimensions.

Definition 5.6.1. A polynomial $f(x_{i_1}^{(t_1)},\dots,x_{i_n}^{(t_n)}) \in L\left(X^{T\text{-gr}}\right)$ is called a *T-graded polynomial identity* if $f(l_{i_1}^{(t_1)},\dots,l_{i_n}^{(t_n)}) = 0$ for all $l_{i_j}^{(t_j)} \in L^{(t_j)}$. Let $\mathrm{Id}^{T\text{-gr}}(L)$ be the T-ideal of T-graded identities of L,

$$V_n^{T\text{-gr}}(F) = \mathrm{span}_F\{[x_{\sigma(1)}^{(t_1)},\dots,x_{\sigma(n)}^{(t_n)}] \mid t_i \in T, \sigma \in S_n\},$$

the multilinear Lie polynomials and $c_n^{T\text{-gr}}(L) = \mathrm{span}_F \frac{V_n^{T\text{-gr}}(F)}{V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(L)}$, called the n^{th} *graded codimension of L*.

If L satisfies an ordinary polynomial identity, e.g. L is finite dimensional over F, and T is finite, then one can prove for graded and ungraded codimensions an inequality analogous to Proposition 5.1.6 where we considered associative algebras.

Proposition 5.6.2. *Let L be an T-graded PI Lie algebra for some finite semigroup T. Then*

$$c_n(L) \leq c_n^{T\text{-gr}}(L) \leq |T|^n c_n(L)$$

Proof. Let $f(x_1, \ldots, x_n) \in V_n(F)$. Remark that $f(x_1, \ldots, x_n) \equiv 0$ is a polynomial identity if and only if $f(x_1^{(t_1)}, \ldots, x_n^{(t_n)}) \equiv 0$ is a graded polynomial identity for all $(t_1, \ldots, t_n) \in T^n$. Also $V_n^{T\text{-gr}}(F) = \bigoplus_{(t_1,\ldots,t_n)\in T^n} V_{t_1,\ldots,t_n}$ where $V_{t_1,\ldots,t_n}$ is the subspace spanned by the monomials in $x_1^{(t_1)}, \ldots, x_n^{(t_n)}$. Then the map $\phi : V_n(F) \to V_n^{T\text{-gr}}(F)$ defined by $\phi(f) = \bigoplus_{(t_1,\ldots,t_n)\in T^n} f(x_1^{(t_1)}, \ldots, x_n^{(t_n)})$ is an embedding inducing an embedding $\frac{V_n(F)}{V_n(F)\cap Id(L)} \to \frac{V_n^{T\text{-gr}}(F)}{V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(L)}$. This implies the lower bound.

For the upper bound let $\overline{f}_1, \ldots, \overline{f}_{c_n(L)}$ be a basis of $\frac{V_n(F)}{V_n(F)\cap \mathrm{Id}(L)}$. Then for all Lie polynomials $[x_{\sigma(1)}^{(t_1)}, \ldots, x_{\sigma(n)}^{(t_n)}]$ there exists $a_{\sigma,j} \in F$ such that

$$[x_{\sigma(1)}^{(t_1)}, \ldots, x_{\sigma(n)}^{(t_n)}] - \sum_{j=1}^{c_n(L)} a_{\sigma,j} f_j \left(x_1^{(t_1)}, \ldots, x_n^{(t_n)}\right) \in \mathrm{Id}^{T\text{-gr}}(L).$$

Thus $\frac{V_n^{T\text{-gr}}(F)}{V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(L)} = \mathrm{span}_F\{\overline{f}_j \left(x_1^{(t_1)}, \ldots, x_n^{(t_n)}\right) \mid 1 \leq j \leq n,\ (t_1, \ldots, t_n) \in T^n\}$ which yields the upper bound. ∎

However in contrast to the associative case $c_n(L)$ is not necessarily exponentially bounded for a PI Lie algebra, see [GZ05, Theorem 12.3.20] or [Pet98]. If L has a faithful Lie representation, then by [GZ05, Theorem 12.3.11] its codimension sequence is exponentially bounded. By the theorem of Ado finite dimensional Lie algebras possess such a representation. In the sequel we only consider finite dimensional Lie algebras. So in this case $c_n(L)$ and consequently, by Proposition 5.6.2, $c_n^{T\text{-gr}}(L)$ is exponentially bounded and it makes sense to formulate a graded Lie version of Amitsur conjecture.

Conjecture 4. *Let L be a finite dimensional Lie algebra over a field of characteristic* 0 *and T a finite semigroup. Then $\lim_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(L)}$ exists.*

When $T = \{e\}$, i.e. in the classical case, the Amitsur conjecture was solved in its full generality by Zaicev in [Zai02]. Previously, it had been solved in case L was solvable, semisimple or its solvable radical coincides with the nilpotent radical in respectively [MP99, GRZ00, GRZ99]. Before formulating Zaicev's theorem in detail we need to recall some definitions and the Levi decomposition.

An ideal I of L is called *solvable* if the series $I^{(i)} = [I^{(i-1)}, I^{(i-1)}]$ with $I^{(2)} = [I, I]$ descends to zero. It is easy to prove that the sum of solvable ideals is again a solvable ideal and therefore there exists a maximal solvable ideal in I, called the solvable radical and is denoted $R(L)$. A Lie algebra is *semisimple* if $R(L) = 0$. Now the *Levi decomposition* asserts that, if F is algebraically closed, there exists a unique, up to conjugation, maximal semisimple subalgebra B such that

$$L = B \oplus R(L).$$

Note that this decomposition will play the role that the Wedderburn-Malcev decomposition plays in the associative case.

Let $I_1, I_2, \ldots, I_m, J_1, J_2, \ldots, J_m$ be ideals of L. We say that *they satisfy condition* $(\star)$ if

1. I_k/J_k is an irreducible L-module for all $1 \leq k \leq m$.
2. for any B-submodules T_k such that $I_k = J_k \oplus T_k$, there exist numbers $q_i \geq 0$ such that
$$\left[[T_1, \underbrace{L, \ldots, L}_{q_1}], [T_2, \underbrace{L, \ldots, L}_{q_2}], \ldots, [T_m, \underbrace{L, \ldots, L}_{q_m}] \right] \neq 0$$

Now we have all ingredients to formulate Zaicev's Theorem [Zai02].

Theorem 5.6.3 (Zaicev). *Let L be a non-nilpotent finite dimensional Lie algebra over a field F with char$(F) = 0$. Then there exist constants $C_1, C_2 > 0, r_1, r_2 \in \mathbb{R}, d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leq c_n(L) \leq C_2 n^{r_2} d^n$ for all n. If F is algebraically closed then the number d is equal to*

$$\exp(L) := \max \left(\dim_F \frac{L}{Ann(I_1/J_1) \cap \ldots \cap Ann(I_m/J_m)} \right)$$

where the maximum is found among all $m \in \mathbb{N}$ and all $I_1, \ldots, I_m, J_1, \ldots, J_m$ satisfying property $(\star)$.

Afterwards Gordienko generalized the previous theorem by, amongst others, including gradations of finite abelian groups [Gor12] and, most recently, to H-module Lie algebras where H is some finite dimensional semisimple Hopf algebra [Gor15a, Theorem 10]. More generally, he proved a generalized version of Zaicev's theorem for Lie algebras who have a 'nice structure' with respect to some action by a Hopf algebra, not necessarily semisimple or finite dimensional.

Theorem 5.6.4 (Gordienko)**.** *Let L be a non-nilpotent H-nice Lie algebra over an algebraically closed field F of characteristic* 0 *for a Hopf algebra H. Then there exist constants $C_1, C_2 > 0, r_1, r_2 \in \mathbb{R}, d \in \mathbb{N}$ such that $C_1 n^{r_1} d^n \leq c_n^H(L) \leq C_2 n^{r_2} d^n$.*

As explained in [Gor15a], in case of a finitely generated abelian group gradation the 'H-nice' assumption is satisfied. Moreover, it is proven that [Gor15a, Lemma 26] the case of an arbitrary group gradation can be reduced to finite generated abelian groups. In particular, the graded exponent PI-exponent for any group G exists and is an integer. By a careful analysis of the proof of [Gor15a, Theorem 10] one can prove that this even holds if L is graded by some cancellative semigroup. However, we will not explain this further and instead concentrate on a counterexample for a general semigroup S, which is obtained in Theorem 5.7.1.

5.7 A graded non-integer Exponent

To start, note that the associative algebras A constructed in Section 5.5 can not be used since their T-gradation does not yield a gradation on $A^{[-]}$ (the algebra A viewed as Lie algebra with the additive bracket). We now define the main protagonist of this section. Let

$$t := e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v := e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, u := e_{11} - e_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $I = \text{span}_F\{u, v, t\}$. Note that $[u, v] = -2v$, $[u, t] = 2t$ and $[v, t] = -u$. Thus $I \cong \mathfrak{sl}_2(F)$, the Lie algebra consisting of the trace zero matrices over F. This is a simple Lie algebra of dimension 3. Denote by $\langle u, v\rangle_L$ the Lie subalgebra of I generated by $\{u, v\}$. Then we define

$$L = I \oplus \langle u, v\rangle_L,$$

with the usual bracket $[(a, b); (c, d)] = ([a, c], [b, d])$. Further we grade $L = L_0 \oplus L_1$ by the semigroup $(\mathbb{Z}_2, .)$ with $L_0 = (\mathfrak{sl}_2(F), 0)$ and $L_1 = \{(a, a) \mid a \in \langle u, v\rangle_L\}$.

We prove in this section that the graded exponent of L is irrational. For ease of notation, we write in the remainder of the section $\exp^{\mathbb{Z}_2}(L)$ and $c_n^{\mathbb{Z}_2}(L)$ instead of respectively $\exp^{\mathbb{Z}_2\text{-gr}}(L)$ and $c_n^{\mathbb{Z}_2\text{-gr}}(L)$.

Theorem 5.7.1. *Let L be the Lie algebra with $(\mathbb{Z}_2, \cdot)$-grading as above. Then* $\exp^{\mathbb{Z}_2}(L) = \lim\limits_{n\to\infty} \sqrt[n]{c_n^{\mathbb{Z}_2}(L)} = 2 + 2\sqrt{2}$.

Remark 5.7.2. One verifies easily that $\mathrm{Rad}(L)$, the solvable radical, equals $(0, \langle u, v\rangle_L)$. Therefore, L is not semisimple. Moreover, since the only graded ideals of L are $0, L$ and $(I, 0)$, we see that L also is not $(\mathbb{Z}_2, \cdot)$-semisimple (i.e. it is not the sum of graded-simple subalgebras). Later on, in remark 5.8.4, we will note that if L is graded-semisimple then the exponent is an integer. Finally, note that $\mathrm{Rad}(L)$ is not graded, which is an important difference with the group-graded case [PRZ13, Prop. 3.3]. Actually it is this lack of structure theory that enables the current counterexample to the graded version of Amitsur's Conjecture.

From now on, in order to avoid confusion with $(\mathbb{Z}_2, +)$-gradings, we denote $T = (\mathbb{Z}_2, \cdot)$.

The proof of Theorem 5.7.1 follows the same path as the one explained in Section 1.3.2 and Section 5.2, with the only difference that now we decompose

$$\frac{V_n^S(L)}{V_n^S(L) \cap \mathrm{Id}^S(L)}$$

as FS_n-module, into a direct sum of Specht modules. Thus,

$$c_n^{T\text{-gr}}(L) = \sum_{\lambda \vdash n} m_\lambda^T(L) \dim_F S^F(\lambda),$$

where $m_\lambda^T(L)$ is the multiplicity of $S^F(\lambda)$. We give a summary of the proof which consists of the following three parts.

(a) First, we have to prove that the multiplicities $\sum_{\lambda \vdash n} m_\lambda^T(L)$ are bounded by a polynomial function. This will be proven in Corollary 5.7.5 as a consequence of a graded version of Ado's Theorem 5.7.3.

(b) Now, as before, using (a) and (1.4) we have that

$$\limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(L))} \leq \sup_{\substack{\lambda \vdash n, \\ m_\lambda^T(L) \neq 0}} \Phi\left(\frac{\lambda_1}{n_1}, \ldots, \frac{\lambda_q}{n_q}\right). \tag{5.23}$$

By restricting Φ to a region Ω having the property that "if $\frac{\lambda}{n} = (\frac{\lambda_1}{n}, \ldots, \frac{\lambda_q}{n}) \notin \Omega$, then $m_\lambda^T(L) = 0$" we can lower the upper bound to $\limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(L))} \leq \max_{\vec{\alpha}\in\Omega} \Phi(\vec{\alpha})$. Proposition 5.7.6 shows that if $\lambda_6 > 0$ and $\lambda_1 + 1 < \lambda_5 + \lambda_4$, then $m_\lambda^T(L) = 0$. In particular we may take

$$\Omega := \left\{ (\alpha_1, \ldots, \alpha_5) \in \mathbb{R}^5 \mid \sum_{1\leq i\leq 5} \alpha_i = 1,\ \alpha_1 \geq \ldots \geq \alpha_5 \geq 0,\ \alpha_4 + \alpha_5 \leq \alpha_1 \right\}.$$

The value of d is given in Lemma 5.7.9.

(c) For the lower bound the first method explained in Section 1.3.2 will work. We show, namely in Lemma 5.7.11, that we can restrict Ω further to a region Ω_0 such that $\max_\Omega \Phi = \max_{\Omega_0} \Phi$ and if $\frac{\lambda}{n} \in \Omega_0$ with $\lambda \vdash n$, then $m_\lambda^T(L) \neq 0$. So, in this case, if $(\alpha_1, \ldots, \alpha_5)$ is an extremal point of Φ on Ω_0, then the partition $\mu = (\mu_1, \ldots, \mu_k)$ with

$$\begin{cases} \mu_i = \lfloor \alpha_i n \rfloor & \text{for } 2 \leq i \leq k \\ \mu_1 = 1 - \sum_{i=2}^k \mu_i \end{cases}$$

will have the right asymptotics and $m_\mu^T(L) \neq 0$, thus finishing the lower bound.

5.7.1 Upper bound

Recall that, by the Theorem of Ado, any finite dimensional Lie algebra has a finite dimensional faithful representation, i.e. there exists a Lie monomorphism $\rho : L \to \mathrm{End}_F(V)$ into the associated Lie algebra $\mathfrak{gl}_n(V) = \mathrm{End}_F(V)^{[-]}$, with V a finite dimensional F-vector space. We prove that $A = \mathrm{End}_F(V)$ can be chosen such that a given gradation on L 'is induced' from a gradation on$A$.

Theorem 5.7.3. *Let $L = \bigoplus_{t \in T} L^{(t)}$ be a finite dimensional Lie algebra graded by a finite abelian semigroup T. Then there exist a finite dimensional T-graded associative algebra $A = \bigoplus_{t \in T} A^{(t)}$ and a Lie monomorphism $\rho^{gr} : L \to A$ such that $\rho^{gr}(L^{(t)}) \subseteq A^{(t)}$ for all $t \in T$.*

Proof. As mentioned there exists a finite dimensional faithful Lie-representation $\rho : L \to \mathrm{End}_F(V)$. Further fix a vector space isomorphism $\psi_t : V \to V^{(t)}$ for each $t \in T$ and define the finite dimensional T-graded vector space $V^T = \bigoplus_{t \in T} V^{(t)}$. Also denote $\mathrm{End}_F(V^T)^{(t)} = \left\{ f \in \mathrm{End}_F(V^T) \mid f(V^{(s)}) \subseteq V^{(st)} \text{ for all } s \in T \right\}$ for all $t \in T$. The desired monomorphism is

$$\rho^{gr} : L \longrightarrow \bigoplus_{t \in T} \mathrm{End}_F(V^T)^{(t)},$$

a map from L to the outer direct sum $\bigoplus_{t \in T} \mathrm{End}_F(V^T)^{(t)}$ that sends an arbitrary homogeneous element $l^{(t)} \in L^{(t)}$ to the linear map $\rho^{gr}(l^{(t)}) : \bigoplus\limits_{s \in T} V^{(s)} \to \bigoplus\limits_{s \in T} V^{(s)}$ defined by the commutative diagram

$$\begin{tikzcd} V^{(s)} \arrow[r, "\rho^{gr}(l^{(t)})"] \arrow[d, "\psi_s"'] & V^{(st)} \\ V \arrow[r, "\rho(l^{(t)})"'] & V \arrow[u, "(\psi_{st})^{-1}"'] \end{tikzcd}$$

One easily checks that ρ^{gr} inherits from ρ the faithfulness and property to be a Lie map. Clearly ρ^{gr} satisfies the extra property $\rho^{gr}(L^{(t)}) \subseteq A^{(t)}$ where $A = \bigoplus_{t\in T} A^{(t)} = \bigoplus_{t\in T} \mathrm{End}_F(V^T)^{(t)}$. ∎

For a grading by the group $(\mathbb{Z}, +)$ the theorem above was proven in [Ros65].

Remark 5.7.4.
- In general $\mathrm{End}(W) \neq \bigoplus_{t\in T} \mathrm{End}(W)^{(t)}$ for a T-graded vector space W. This is the reason why we use the outer direct sum $\bigoplus_{t\in T} \mathrm{End}_F(W)^{(t)}$ in the proof of Theorem 5.7.3.
- If S is abelian, then the gradation of A induces also a gradation on $A^{[-]}$. Moreover in this case ρ is a graded lie morphism, i.e $\rho(L^{(t)}) \subseteq (A^{[-]})^{(t)}$ for all $t \in T$.

As a direct consequence we get now that the multiplicities $\sum_{\lambda\vdash n} m_\lambda^T(L)$ are polynomially bounded.

Corollary 5.7.5. *Let L be a T-graded Lie algebra for some finite abelian semigroup T. Then there exist constants $C, d \in \mathbb{N}$ such that $\sum_{\lambda\vdash n} m_\lambda^T(L) \leq Cn^d$ for all $n \in \mathbb{N}$.*

Proof. By Theorem 5.7.3 there exists a finite dimensionsal associative algebra A and a Lie monomorphism $\rho : L \to A^{[-]}$ such that $\rho(L^{(t)}) \subseteq A^{(t)}$ for $t \in T$. In particular $\sum_{\lambda\vdash n} m_\lambda^T(L) \leq \sum_{\lambda\vdash n} m_\lambda^T(A^{[-]})$, where $\frac{V_n^{T\text{-gr}}(F)}{V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(A^{[-]})} = \bigoplus_{\lambda\vdash n} m_\lambda^T(A^{[-]}) S^F(\lambda)$. Let $m_\lambda^T(A)$ be the multiplicity of $S^F(\lambda)$ in $\frac{P_n^{T\text{-gr}}(F)}{P_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(A)}$. Note that

$$\sum_{\lambda\vdash n} m_\lambda^T(A^{[-]}) \leq \sum_{\lambda\vdash n} m_\lambda^T(A)$$

since $V_n^{T\text{-gr}}(F)$ is an FS_n-submodule of $P_n^{T\text{-gr}}(F)$ and $V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(A^{[-]}) = V_n^{T\text{-gr}}(F)\cap \mathrm{Id}^{T\text{-gr}}(A)$. Now, by [Gor13b, Theorem 5], there exist constants $C, d \in \mathbb{N}$ such that $\sum_{\lambda\vdash n} m_\lambda^T(A) \leq Cn^d$. This finishes the proof. ∎

Corollary 5.7.5 can also be proven without making use of Theorem 5.7.3. Actually one can rewrite word by word the proof of [Gor15a, Theorem 12] for $H = (FT)^*$. However this is lengthier.

Due to the strategy explained before, the upper bound will be a direct consequence of the following proposition.

Proposition 5.7.6. *Let* $L = \mathfrak{sl}_2(\mathbb{C}) \oplus \langle u, v\rangle_L$ *be the* $T = (\mathbb{Z}_2, .)$*-graded Lie algebra defined at the beginning of Section 5.7. Assume* $m_\lambda^T(L) \neq 0$ *for some partition* $\lambda \vdash n$*. Then* $\lambda_6 = 0$ *and* $\lambda_1 + 1 \geq \lambda_5 + \lambda_4$.

Remark 5.7.7. Denote the F-basis of L by $\mathcal{B}_L = \{(u,0),(u,u),(v,0),(v,v),(t,0)\}$. In the sequel we will always assume that the evaluations are from elements in $\mathcal{B}_L$.

Proof. Since $m_\lambda^T(L) \neq 0$ there exists a multilinear polynomial $f \in V_n^{T\text{-gr}}(F)$ such that $e_\lambda f \notin \mathrm{Id}^{T\text{-gr}}(L)$.

Recall that $e_\lambda^* = \sum\limits_{\substack{\sigma\in R_\lambda\\ \tau\in C_\lambda}} \mathrm{sgn}(\tau)\ \tau\circ\sigma$. Thus $e_\lambda^* f$ is alternating in the sets of variables corresponding to the numbers of each column of T_λ and symmetric in those corresponding to the rows of T_λ. Thus, since $\dim_F L = 5$ and $m_\lambda^T(L) \neq 0$, we must have that $\lambda_6 = 0$ which we assume for the sequel of the proof.

Now define the function $\theta : L \to \mathbb{Z}$ first on the basis elements by

$$\theta(u,u) = \theta(u,0) = 0, \qquad \theta(v,v) = \theta(v,0) = 1, \quad \text{and} \quad \theta(t,0) = -1.$$

and on an arbitrary element we take the maximum. Suppose $[b_1,\dots,b_m] \neq 0$ for some basis elements $b_i \in \mathcal{B}_L$. One easily proves that

$$-1 \leq \sum_{1\leq i\leq m} \theta(b_i) = \theta([b_1,\dots,b_m]) \leq 1.$$

Also $\sum\limits_{b\in\mathcal{B}_L} \theta(b) = 1$ and $\sum\limits_{b\in\mathcal{B}_L\setminus\{d\}} \theta(b) \geq 0$ for any $d \in \mathcal{B}_L$. Since $e_\lambda^* f \notin \mathrm{Id}^{T\text{-gr}}(L)$ there exist some basis elements $b_1,\dots,b_m \in \mathcal{B}_L$ such that $[b_1,\dots,b_m] \neq 0$. By the previous inequalities we know that the λ_4 first columns of T_λ give an altogether θ-value of at least λ_5. Since the total θ-value of $[b_1,\dots,b_m]$ does not exceed 1, there must remain at least $\lambda_5 - 1$ columns. Since the number of remaining columns is equal to $\lambda_1 - \lambda_4$ we get that $\lambda_1 - \lambda_4 \geq \lambda_5 - 1$ as desired. ∎

Remark 5.7.8. By interchanging the θ-values of u and t one can prove analogously the above result for $L = \mathfrak{sl}_2(\mathbb{C}) \oplus \langle u, t\rangle_L$.

As explained in the overview of the proof we have to compute the maximum of $\Phi(x_1,\dots,x_q) = \frac{1}{x_1^{x_1}\dots x_q^{x_q}}$ on the region

$$\Omega = \left\{ (x_1,\dots,x_q) \in \mathbb{R}^q \mid \sum_{1\le i\le q} x_i = 1,\ x_1 \ge \ldots \ge x_q \ge 0,\ x_{q-1}+x_q \le x_1 \right\} \tag{5.24}$$

for $q = 5$. This was already done in [Gor15b, Lemma 3].

Lemma 5.7.9. *Let $q \in \mathbb{N}_{\ge 4}$. Then* $\max_{\vec{x}\in\Omega} \Phi(\vec{x}) = (q-3) + 2\sqrt{2} \approx q - 0.1716\ldots$.

Corollary 5.7.10. $\limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(L)} \le 2 + 2\sqrt{2}$.

5.7.2 Lower bound

Note that $\max_\Omega \Phi$, with Ω as in (5.24), is reached at a point $(\alpha_1,\dots,\alpha_5)$ with $\alpha_5 \ne 0$. Now we prove that $m_\lambda^T(L) \ne 0$ for all partitions $\lambda \vdash n$ with $\lambda_5 \ne 0$ and $\frac{\lambda}{n} \in \Omega$. So in this way we obtain the region Ω_0 mentioned in the overview of the proof.

Lemma 5.7.11. *Suppose $\lambda_5 + \lambda_4 \le \lambda_1$ and $\lambda_5 > 0$, then there exists a multilinear polynomial f such that $e^*_{T_\lambda} f \notin \mathrm{Id}^{(FT)^*}(L)$ for a Young tableau T_λ constructed in the proof.*

Proof. Since $\lambda_5 + \lambda_4 \le \lambda_1$ we can define numbers $\beta_2,\dots,\beta_8 \in \mathbb{N}$ such that $\beta_2 = \lambda_4 - \lambda_5, \beta_3 + \beta_4 = \lambda_3 - \lambda_4, \beta_5 + \beta_6 = \lambda_2 - \lambda_3$, $\beta_7 + \beta_8 = \lambda_1 - \lambda_2$ and $\beta_3 + \beta_5 + \beta_7 = \lambda_5$. We introduce these numbers to subdivide the columns of T_λ in order to get more control of the different θ-values of each column. Recall that, by the proof of Proposition 5.7.6, we know that the total θ-value has to be between -1 and 1 for a non-zero valuation. Remark also that we need the condition $\lambda_5 + \lambda_4 \le \lambda_1$ in order to be able to assume that all β_i are greater than or equal to zero.

Now we define alternating multilinear $(FT)^*$-polynomials corresponding respectively to the $\lambda_5, \beta_2, \dots, \beta_8$ first columns. Recall that h_t, $t \in T$, denotes the dual basis of FT, i.e. $h_t(s) = 1$ if $t = s$ and zero otherwise and $T = (\mathbb{Z}_2, \cdot)$.

$$f_1 := \sum_{\sigma\in \mathrm{Sym}\{i_1,\dots,i_5\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_2)}, x^{h_0}_{\sigma(i_4)}, x^{h_1}_{\sigma(i_3)}, x^{h_0}_{\sigma(i_1)}, x^{h_1}_{\sigma(i_5)}],$$

$$f_2 := \sum_{\sigma\in \mathrm{Sym}\{i_1,\dots,i_4\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_2)}, x^{h_0}_{\sigma(i_4)}, x^{h_1}_{\sigma(i_3)}, x^{h_0}_{\sigma(i_1)}],$$

$$f_3 := \sum_{\sigma\in \mathrm{Sym}\{i_1,i_2,i_3\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_2)}, x^{h_0}_{\sigma(i_1)}, x^{h_1}_{\sigma(i_3)}],$$

$$f_4 := \sum_{\sigma\in \mathrm{Sym}\{i_1,i_2,i_3\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_1)}, x^{h_1}_{\sigma(i_3)}, x^{h_1}_{\sigma(i_2)}],$$

$$f_5 := \sum_{\sigma\in \mathrm{Sym}\{i_1,i_2\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_1)}, x^{h_0}_{\sigma(i_2)}], \qquad f_6 := \sum_{\sigma\in \mathrm{Sym}\{i_1,i_2\}} (\operatorname{sign}\sigma)[x^{h_0}_{\sigma(i_1)}, x^{h_1}_{\sigma(i_2)}],$$

$$f_7 := x^{h_0}_{i_1}, \qquad f_8 := x^{h_1}_{i_1}.$$

Finally, if $\beta_7 \neq 0$, define the polynomial

$$f = [(f_1 f_3)^{\beta_3}, (f_1 f_5)^{\beta_5}, (f_1 f_7)^{\beta_7 - 1}, f_1, f_2^{\beta_2}, f_4^{\beta_4}, f_6^{\beta_6}, f_8^{\beta_8}, f_7] \in V_n^{(FT)^*}(F),$$

where by $[x, (ab)^c]$ we denote the polynomial $[x, \underbrace{a, b}_{c-\text{times}}, \dots, a, b]$.

If $\beta_7 = 0$ and $\beta_5 \neq 0$ then we define the polynomial

$$f' = [(f_1 f_3)^{\beta_3}, (f_1 f_5)^{\beta_5 - 1}, f_1, f_2^{\beta_2}, f_4^{\beta_4}, f_6^{\beta_6}, f_8^{\beta_8}, f_5] \in V_n^{(FT)^*}(F)$$

and

$$f'' = [(f_1 f_3)^{\beta_3 - 1}, f_1, f_2^{\beta_2}, f_4^{\beta_4}, f_6^{\beta_6}, f_8^{\beta_8}, f_3] \in V_n^{(FT)^*}(F)$$

if $\beta_5 = \beta_7 = 0$. Note that $\beta_3 \neq 0$ as $\lambda_5 = \beta_3 + \beta_5 + \beta_7 > 0$. Note that here different copies of f_i depend on different variables. Thus:

The copies of f_1 are alternating polynomials of degree 5 corresponding to the first λ_5 columns of height 4.

The copies of f_2 are alternating polynomials of degree 4 corresponding to the next β_2 columns of height 5.

...

The copies of f_8 are polynomials of degree 1 corresponding to the last β_8 columns of height 1.

However, the same values will be substituted.

Consider now the Young tableau T_λ given by the figure below. We prove that $e^*_{T_\lambda} f$ does not vanish on L. First remark that $f \notin \mathrm{Id}^{(FT)^*}(L)$. Indeed the following substitution in f is equal to a multiple of the element $(u, 0)$.

(Here in the i-th block we have β_i columns with the same values in all cells of a row. For shortness, we depict each value for each block only once. The tableau T_λ is still of the shape λ.)

Figure 5.1:

$$
T_\lambda = \begin{array}{cccccccc}
\lambda_5 & \beta_2 & \beta_3 & \beta_4 & \beta_5 & \beta_6 & \beta_7 & \beta_8 \\
\hline
\multicolumn{1}{|c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(t,0)} & \multicolumn{1}{c|}{(u,u)} \\
\hline
\multicolumn{1}{|c|}{(u,0)} & \multicolumn{1}{c|}{(u,0)} & \multicolumn{1}{c|}{(u,0)} & \multicolumn{1}{c|}{(v,v)} & \multicolumn{1}{c|}{(u,0)} & \multicolumn{1}{c|}{(v,v)} & & \\
\cline{1-6}
\multicolumn{1}{|c|}{(u,u)} & \multicolumn{1}{c|}{(u,u)} & \multicolumn{1}{c|}{(u,u)} & \multicolumn{1}{c|}{(u,u)} & & & & \\
\cline{1-4}
\multicolumn{1}{|c|}{(v,0)} & \multicolumn{1}{c|}{(v,0)} & & & & & & \\
\cline{1-2}
\multicolumn{1}{|c|}{(v,v)} & & & & & & & \\
\cline{1-1}
\end{array}
$$

In fact one easily checks that after substitution f_i, for $i = 3, 5, 7$, yields respectively $-8(t, 0), 4(t, 0)$ and $(t, 0)$, for $i = 2, 4, 6$, respectively $16(u, 0), 2(u, 0)$ and $2(u, 0)$ and f_1 gives $-64(v, 0)$.

We claim that the substitution in $e^*_{T_\lambda} f$ as in figure (5.1) is a non-zero multiple of the evaluated value of f. First remark, by construction of f, that $e^*_{T_\lambda} f = C a_{T_\lambda} f$ with $C = (5!)^{\lambda_5}(4!)^{\beta_2}(3!)^{\beta_3+\beta_4}(2!)^{\beta_5+\beta_6}$ and a_{T_λ} symmetrizes f corresponding to the rows of T_λ. Since $(t, 0)$ and (u, u) are in different homogeneous components all terms where $a_{T_\lambda} f$ interchanges a $(t, 0)$ with (u, u) will be zero. Similarly if a $(u, 0)$ is interchanged with a (v, v) in the second row, then this term is zero. So the claim and therefore the proposition are proven. ∎

Corollary 5.7.12. *With L and T as before, we have that*

$$\exp^{T\text{-gr}}(L) := \limsup_{n\to\infty} \sqrt[n]{c_n^{T\text{-gr}}(L)} = 2 + 2\sqrt{2}.$$

Proof. For the sake of completeness, we write how Lemma 5.7.11 implies the lower bound, even though this was already sketched before. Let $(\alpha_1, \dots, \alpha_5) \in \mathbb{R}^5$ be an extremal point of the function $\Phi(x_1, \dots, x_5) = \frac{1}{x_1^{x_1} \cdots\cdots x_5^{x_5}}$ on the polytope

$$\Omega := \left\{ (\alpha_1, \dots, \alpha_5) \in \mathbb{R}^5 \mid \sum_{1\le i\le 5} \alpha_i = 1,\ \alpha_1 \ge \dots \ge \alpha_5 > 0,\ \alpha_4 + \alpha_5 \le \alpha_1 \right\}.$$

By Lemma 5.7.9, $\Phi(\alpha_1, \dots, \alpha_5) = 2 + 2\sqrt{2}$. Define now the partition $\mu \vdash n$ by

$$\begin{cases} \mu_i = \lfloor n\alpha_i \rfloor & \text{for } 2 \le i \le 5 \\ \mu_1 = 1 - \sum_{i=2}^5 \mu_i. & \end{cases}$$

Since $(\alpha_1,\ldots,\alpha_5)\in\Omega$, the partition μ satisfies $\mu_4+\mu_5\le\mu_1$ and $\mu_5>0$. Thus, by Lemma 5.7.11, $m^T_\mu(L)\neq 0$. Moreover, for every $\epsilon>0$ there exists a n_0 such that $\Phi(\frac{\mu_1}{n},\ldots,\frac{\mu_5}{n})\ge 2+2\sqrt{2}-\epsilon$ for all $n\ge n_0$. Now, for some constants $C_1,B_1\in\mathbb{R}$

$$\dim_F(S^F_\mu)\ge\frac{n!}{n^{5.4}\mu_1!\ldots\mu_5!}\ge C_1n^{B_1}\left(\frac{1}{(\frac{\mu_1}{n})^{\frac{\mu_1}{n}}\cdot\cdots\cdot(\frac{\mu_5}{n})^{\frac{\mu_5}{n}}}\right)^n\ge C_1n^{B_1}(d-\epsilon)^n,$$

which yields the lower bound $\liminf_{n\to\infty}\sqrt[n]{c^{T\text{-gr}}_n(L)}\ge 2+2\sqrt{2}$. Together with Corollary 5.7.10 we get that $\exp^{T\text{-gr}}(L)=\lim_{n\to\infty}\sqrt[n]{c^{T\text{-gr}}_n(L)}=2+2\sqrt{2}$. ∎

5.8 Semisimple Lie algebras with generalized action

In this section H will always be a finite dimensional associative algebra with 1 and L a finite dimensional Lie algebra on which H is acting in a generalized way, i.e. indexgeneralized action

$$h.[l_1, l_2] = \sum_{i=1}^{k} [h_i' l_1, h_i'' l_2] \tag{5.25}$$

for some $h_i', h_i'' \in H$. We refer to [Gor15a, Gor13b] for examples of a generalized action and for all basic definitions such as H-polynomials and H-codimensions.

In the sequel we first prove that if L is semisimple and H-semisimple then $\exp^H(L) \in \mathbb{Z}$ and more precisely is equal to the the maximal H-exponent of an H-simple component of L, see Corollary 5.8.5. This will follow readily from the case that L is H-simple semisimple. In this case, we construct in Theorem 5.8.1 a non-identity with enough alternating sets such that, in the classical way, we can deduce that $\exp^H(L) = \dim_F L$. Remark that, by the example from previous section, the condition H-semisimple can not be dropped in order to get an integer exponent. On the other hand we do not know if it is necessary for L to be semisimple.

5.8.1 H-simple

We say that L is H-simple if it is non-abelian and the only H-invariant ideals of L are 0 and L. In particular since $[L, L]$ is H-invariant, implicitly, we assume that $[L, L] = L$.

Recall that the adjoint representation, $\mathrm{ad} : L \to \mathrm{End}_F(L)$, of L is defined as $\mathrm{ad}(l)(l') = [l, l']$ for all $l, l' \in L$. We will sometimes write $\mathrm{ad}_l := \mathrm{ad}(l)$. Further, denote the map corresponding to the H-action by $\rho : H \to \mathrm{End}_F(L)$. Remark that by (5.25) the following equality holds

$$\rho(h)\,\mathrm{ad}(l) = \sum_i \mathrm{ad}(h_i' l)\rho(h_i''). \tag{5.26}$$

Finally, by $Q^H_{t,k,n} \subseteq V^H_n$ we denote the subspace spanned by all multilinear H-polynomials alternating in k disjoint sets $\{x_1^i, \dots, x_t^i\} \subseteq \{x_1, \dots, x_n\}$ of size t.

We will prove that $\exp^H(L) = \dim_F L$ if moreover L is semisimple (without consideration of the H-action).

In the sequel of this section we fix an F-basis $\mathcal{B}(L) = \{l_1, \dots, l_t\}$ of L.

First we prove that the necessary H-polynomial with sufficiently numerous alternations exists. The following is an analog of [GSZ11, Theorem 1].

Theorem 5.8.1. *Let L be a H-simple semisimple Lie algebra endowed with a generalized action of a finite dimensional associative algebra H with 1. Then there exist a non-zero positive integer constant C and $\overline{z}_1, \dots, \overline{z}_C \in L$ such that for any k there exists*

$$f = f(x_1^1, \dots, x_t^1; \dots; x_1^{2k}, \dots, x_t^{2k}; z_1, \dots, z_C; z) \in Q^H_{t,2k,2kt+C+1}$$

such that for any $\overline{z} \in L$ we have $f(l_1, \dots, l_t; \dots; l_1, \dots, l_t; \overline{z_1}, \dots, \overline{z_C}; \overline{z}) = \overline{z}$.

First we do the case $k = 1$ separately.

Lemma 5.8.2. *For some $C \in \mathbb{N}$ there exists a polynomial*

$$f = f(x_1, \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z) \in V^H_{2t+C+1}$$

alternating in $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$ satisfying the property that there exist $\overline{z_1}, \dots, \overline{z_C} \in L$ such that for any $\overline{z} \in L$ we have $f(l_1, \dots, l_t, l_1, \dots, l_t, \overline{z_1}, \dots, \overline{z_C}, \overline{z}) = \overline{z}$.

Proof.
One can consider L as module over its multiplication algebra $M(L) = \operatorname{span}_F\{\rho(H), \operatorname{ad}(L)\}$. Since L is H-simple it is moreover an irreducible faithful module over $M(L)$ and so, as L is finite dimensional, by the Density theorem $\operatorname{End}_F(L) = \operatorname{span}_F\{\rho(H), \operatorname{ad}(L)\}$.
Note that due to (5.26) we can always move the $\rho(h)$ to the right in any expression in $\operatorname{span}_F\{\rho(H), \operatorname{ad}(L)\}$. Thus $\operatorname{End}_F(L) = \operatorname{span}_F\{\operatorname{ad}_l \circ \rho(h) \mid l \in L, h \in H\}$ and of course $\operatorname{End}_F(L) \cong M_t(F)$ as vector spaces, since $t = \dim L$. Let

$$\mathcal{B}(\operatorname{End}_F(L)) = \{\operatorname{ad}_{l_1}, \dots, \operatorname{ad}_{l_t}; \operatorname{ad}(l_{i_1})\rho(h_1), \dots, \operatorname{ad}(l_{i_s})\rho(h_s)\}$$

be a basis of $\operatorname{End}_F(L)$ with $i_j \in \{1, \dots, t\}$ appropriate indices. Recall that by [For87] the Regev polynomial

$$\begin{aligned} f_t(x_1, \dots, x_{t^2}; y_1, \dots, y_{t^2}) \quad = \quad & \sum_{\sigma,\tau \in S_{t^2}} \operatorname{sign}(\sigma\tau) x_{\sigma(1)} y_{\tau(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)} y_{\tau(2)} y_{\tau(3)} y_{\tau(4)} \cdots \\ & x_{\sigma(t^2-2t+2)} \cdots x_{\sigma(t^2)} y_{\tau(t^2-2t+2)} \cdots y_{\tau(t^2)} \end{aligned}$$

is a central polynomial of $M_t(F)$. Replace now each $x_1, \dots, x_t$ by the respective ad_{x_i}, $y_1, \dots, y_t$ by ad_{y_i}, x_{t+j} by $\operatorname{ad}_{z_j} \circ \rho(h_j)$ and y_{t+j} by $\operatorname{ad}_{v_{s+j}} \circ \rho(h_j)$ for $1 \le j \le s$, where x_i, y_i, z_i are new variables that take values in L. Denote the polynomial that we get after this substitution by $\tilde{f}_t$. Note that if we evaluate $\tilde{f}_t$ by $x_i = y_i = l_i$ and $z_j = z_{s+j} = l_{i_j}$ then we get $K\mathrm{id}_L$ for some non-zero constant $K \in F$. Finally put $C = 2s$, then clearly $f := K^{-1}\tilde{f}_t(z) \in V_n^H$ satisfies the needed properties. ∎

In order to obtain more alternating sets it became traditional, but crucial, to use a trick by Razmyslov [Raz94, Chapter III]. The proof is completely similar to those of Lemma 3 and Theorem 1 in [GSZ11] where non-associative algebras without H-action are considered or also similar to the proof of [Gor13a, Theorem 7] where associative algebras endowed with a generalized action are considered. Since we have not used Razmyslov's trick yet in this thesis, we give a sketch for the interested reader.

Proof.[Proof of Theorem 5.8.1] For $k = 1$ it is proven in Lemma 5.8.2. Denote this polynomial by $f_1(x_1, \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z)$. We will now add alternating sets. Start by defining the following polynomial:

$$f_1^{(1)}(u_1, v_1, x_1, \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z) = \sum_{i=1}^{t} f_1(x_1, \dots, [u_1, v_1, x_i], \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z).$$

This polynomial satisfies the following properties:

(i) $f_1^{(1)}$ is still alternating in $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$.

(ii) For all evaluations on L we have that

$$f_1^{(1)}(\overline{u_1}, \overline{v_1}, \overline{x_1}, \dots, \overline{x_t}; \overline{y_1}, \dots, \overline{y_t}; \overline{z_1}, \dots, \overline{z_C}; \overline{z}) = \kappa(\overline{u}_1, \overline{v}_1) f_1(\overline{x_1}, \dots, \overline{x_t}; \overline{y_1}, \dots, \overline{y_t}; \overline{z_1}, \dots, \overline{z_C}; \overline{z})$$

since by (i) in any non-zero evaluation all evaluated elements must be different basis elements. Here $\kappa(\overline{u}_1, \overline{v}_1) = \mathrm{Tr}(\mathrm{ad}(\overline{u}_1) \circ \mathrm{ad}(\overline{v}_1))$ denotes the Killing form of L.

We continue to do this. Thus, define

$$f_1^{(j)}(u_1, \dots, u_j; v_1, \dots, v_j; x_1, \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z) = \sum_{i=1}^{t} f_1^{(j-1)}(u_1, \dots, u_{j-1}; v_1, \dots, v_{j-1}; x_1, \dots, u_j v_j x_i, \dots, x_t; y_1, \dots, y_t; z_1, \dots, z_C; z).$$

This polynomial satisfies

(i) $f_1^{(j)}$ is still alternating in $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_t\}$.

(ii) For all substitutions from L,

$$f_1^{(j)}(\overline{u_1},\dots,\overline{u_j};\overline{v_1},\dots,\overline{v_j};\overline{x_1},\dots,\overline{x_t};\overline{y_1},\dots,\overline{y_t};\overline{z_1},\dots,\overline{z_C};\overline{z})$$
$$= \kappa(\overline{u}_1,\overline{v}_1)\dots\kappa(\overline{u}_j,\overline{v}_j)f_1(\overline{x_1},\dots,\overline{x_t};\overline{y_1},\dots,\overline{y_t};\overline{z_1},\dots,\overline{z_C};\overline{z}).$$

We continue this procedure till $j=t$. Then, define

$$f_2(u_1,\dots,u_t;v_1,\dots,v_t;x_1,\dots,x_t;y_1,\dots,y_t;z_1,\dots,z_C;z)$$
$$= \frac{1}{t!\det(\kappa(a_i,a_j))_{i,j=1}^t}\sum_{\sigma,\tau,S_t}\operatorname{sign}(\sigma\tau)f_1^{(t)}(u_{\sigma(1)},\dots,u_{\sigma(t)};v_{\tau(1)},\dots,v_{\tau(t)};x_1,\dots,x_t;$$
$$y_1,\dots,y_t;z_1,\dots,z_C;z)$$

where $\det(\kappa(a_i,a_j))_{i,j=1}^t$ is not zero because κ is non-degenerate since L is semisimple. Note that this is the only moment where we use the semisimplicity of L. The former polynomial satisfies the following properties.

(i) $f_2\in Q_{t,4,4t+C+1}$,

(ii) $f_2(a_1,\dots,a_t;a_1,\dots,a_t;a_1,\dots,a_t;a_1,\dots,a_t;\overline{z_1},\dots,\overline{z_C};\overline{z})=\overline{z}$ where $\overline{z_1},\dots,\overline{z_C}$ are chosen such that $f_1(a_1,\dots,a_t;a_1,\dots,a_t;\overline{z_1},\dots,\overline{z_C};\overline{z})=\overline{z}$ for all $\overline{z}\in L$.

Now continue this procedure by using f_2 instead of f_1 and $\{u_1,\dots,u_t\}$ instead of $\{x_1,\dots,x_t\}$. We can do this till f_k for any k. This finishes the proof ∎

As outlined in Section 1.3.2, the polynomial f from Theorem 5.8.1 delivers now that the H-exponent equals the dimension for H-simple semisimple Lie algebras.

Theorem 5.8.3. *Let L be an H-simple semisimple Lie algebra endowed with a generalized action of a finite dimensional unital associative algebra H. Then* $\exp^H(L)=\dim L$.

Proof. Let $n\geq 2t+c+1$ and $k=\lfloor\frac{n-(2t+c+1)}{2t}\rfloor$. By Theorem 5.8.1 there exists a $f\in Q^H_{t,2k,n}$ which is a non-identity of L. We start by proving that there exists a $\lambda=(\lambda_1,\dots,\lambda_h)\vdash n$ with $\lambda_i\geq 2k$ for $1\leq i\leq h=t=\dim(L)$ such that $e_\lambda f\notin \operatorname{Id}^H(L)$ and in particular $m^H_\lambda(L)\neq 0$.

Recall that by Theorem 1.2.23 we can write $FS_n=\bigoplus_{\substack{\lambda\vdash n,\\ T_\lambda \text{ standard}}} FS_n e^*_{T_\lambda}$. Consequently, as $f\notin \operatorname{Id}^H(L)$, there exists a $\lambda\vdash n$ such that $e^*_\lambda f\notin\operatorname{Id}^H(L)$. Moreover $\lambda_i\geq\lambda_t\geq 2k$. Indeed, $e^*_\lambda=b_{T_\lambda}a_{T_\lambda}$ and a_{T_λ} is symmetrizing the variables of each row of T_λ, so each row of T_λ may contain at most one variable from each $X_i=\{x_1^{(i)},\dots,x_t^{(i)}\}$ since otherwise,

f being alternating in X_i, $a_{T_\lambda} f = 0$. Thus $\sum_{i=1}^{t-1} \lambda_i \leq 2k(t-1) + (n-2kt) = n-2k$ and $\lambda_t = \sum_{i=1}^{t} \lambda_i - \sum_{i=1}^{t-1} \lambda_i = n - (n-2k) = 2k$ as claimed.

For this partition $c_n^H(L) \geq \dim_F S^F(\lambda)$. Also $((2k)^t) \leq \lambda$, i.e D_λ contains the $t \times 2k$-box. By applying the Branching rule, theorem 1.2.32, $n-2kt$ times we see that $\dim_F S^F(\lambda) \geq \dim_F S^F((2k)^t))$. Finally

$$\dim_F S^F((2k)^t)) \geq \frac{2kt!}{((2k+t)!)^t} \simeq \frac{\sqrt{4\pi tk}(\frac{2kt}{e})^{2kt}}{(\sqrt{2\pi(2k+t)}(\frac{2k+t}{e})^{2k+t})^t} \simeq C_1 k^{C_2} t^{2kt},$$

for some constants $C_1 \geq 0,\ C_2 \in \mathbb{Q}$ as $k \to \infty$. This finishes the under bound.

The upper bound is also classical. For this consider H-polynomials as n-linear maps from L to L. Then the map $V_n^H \to \mathrm{Hom}_F(L^{\otimes n}, L)$ has kernel $V_n^H \cap \mathrm{Id}^H(L)$. Thus $c_n^H(L) \leq \dim \mathrm{Hom}_F(L^{\otimes n}, L) = (\dim L)^{n+1}$. ∎

Remark 5.8.4. *(i):* Suppose $H = (FS)^*$ for some semigroup S. In this case, Theorem 5.8.3 follows immediately from well known results and, moreover, one has **not** to assume L to be semisimple as ungraded algebra. Indeed, in [EK13, Prop. 1.12] it is proven that if L is S-graded-simple then S is actually a commutative group. Moreover in [PRZ13, Prop. 3.1] it is proven that L is semisimple (as ungraded algebra) with isomorphic simple components whenever L is group-graded-simple. Finally, by [Gor15a, Theorem 1] a finite dimensional Lie algebra graded by an arbitrary group satisfies the graded version of Amitsur conjecture and more precisely $\exp^G L = \dim_F L$ if L is G-graded simple.

(ii): We do not know if, as for group-gradings, one can drop the assumption semisimple in Theorem 5.8.3. Recall that the Lie algebra L from Section 5.7, with a non-integer exponent, is neither semisimple nor graded-semisimple.

Corollary 5.8.5. *Let $L = L_1 \oplus \ldots \oplus L_m$ be a H-semisimple Lie algebra, where L_i is a H-simple semisimple algebra. Then $\exp^H(L) = \max_{1 \leq i \leq m}\{\dim_F L_i\}$.*

Proof. Since L_i is a subalgebra of L, $\mathrm{Id}^H(L) \subseteq \mathrm{Id}^H(L_i)$ and thus

$$\max_{1 \leq i \leq t} \exp^H(L_i) \leq \liminf_{n \to \infty} \sqrt[n]{c_n^H(L)}.$$

Now, note that since $L = \bigoplus_{i=1}^m L_i$ is a direct sum of Lie algebras, the Lie bracket can be seen as being the component-wise Lie bracket, i.e. $[(l_1, \cdots, l_m), (l'_1, \cdots, l'_m)] = ([l_1, l'_1], \cdots, [l_m, l'_m])$. Let $\mathcal{B}$ be a basis of L consisting of the union of a fixed basis of each

L_i. For a multilinear polynomial it is enough to evaluate basis elements in order to check whether it is a polynomial identity. Thus $V_n^H(F) \cap \mathrm{Id}^H(L) = V_n^H(F) \cap \bigcap_{i=1}^m \mathrm{Id}^H(L_i)$.

By [GZ05, Th. 12.2.13] the statement now follows if L satisfies $Q_{\max_i \dim L_i, k} = \bigcup_{n \in \mathbb{N}} Q_{\max_i \dim L_i, k, n}$, i.e. the Capelli identity of rank $\max_i \dim_F(L_i) + 1$. However, by above remark this is clear. ∎

Inleiding (Nederlands)

Motivatie en oorsprong

Voor twee gegegeven algebra's A en B is de meest natuurlijke maar moeilijkste vraag die men kan stellen, of A en B isomorf zijn als algebra. Een manier om A en B te onderscheiden, is door een eigenschap te vinden vervuld door A maar niet door B of, met andere woorden, door een invariant te associëren met elke algebra onder isomorfisme, waarvan de waarde anders zou zijn op A dan op B. Zelfs indien men een invariant vindt die A onderscheidt van B, betekent dit niet dat het ook zou toelaten een onderscheid te maken tussen A en een derde algebra C. Om het probleem in haar volledige algemeenheid op te lossen, moet men een zogenaamde volledige lijst van invarianten hebben. In volledige algemeenheid is dit natuurlijk niet goed te doen. Dit doet echter geen afbreuk aan het feit dat bepaalde invarianten van belang kunnen zijn. In dit proefschrift zullen we aan elke eindig dimensionale algebra A, over een veld met gelijk welke karakteristiek, een reeks getallen $(c_n(A))_n$ koppelen, codimensies genaamd. Op haar beurt zal deze reeks een aantal concrete invarianten met concrete informatie over de structuur van A opleveren.

Om precies te zijn, we associëren een dergelijke rij met een veel grotere klasse van algebra's, namelijk de klasse van algebra's die voldoen aan een polynoomidentiteit, kortom PI-algebra's. Een polynoomidentiteit van A is een niet-nul polynoom $f(x_1, \dots, x_n)$ in niet-commutatieve variabelen $x_1, \dots, x_n$ zodat het identiek verdwijnt wanneer het berekend wordt op A, d.w.z. $f(a_1, \dots, a_n) = 0$ voor alle $a_i \in A$. Zoals in Hoofdstuk 1 uitgelegd zal worden, voldoet elke eindigdimensionale algbera aan een dergelijke 'universele relatie'. Alvorens de resultaten van dit proefschrift te bespreken, laten we eerst zien waar dit onderzoeksgebied haar inspiratie haalt. Het veld van de PI-theorie begint namelijk rond het einde van de jaren 1940, bij het begin van de niet-commutatieve ring theorie, met werk van Jacobson [Jac45], Kaplansky [Kap48] and Levitzki [Lev46] waarin zij het begrensde Kurosh probleem bewijzen

Stelling (begrensde Kurosh probleem)**.** *Een eindig voortgebrachte associatieve algebra A over een veld F waarin iedere element $x \in A$ voldoet aan een polynoom $x^m + c_1 x^{m-1} + \ldots + c_1 x + c_0, \quad c_i \in F$ met m uniform begrensd door een vast natuurlijk getal n, is eindig dimensionaal over F.*

In het voorkomend geval dat A gewoon een eindig voortgebrachte algebraïsche algebra is, zonder de uniform begrensde veronderstelling, werd een tegenvoorbeeld gegeven in 1964 door Golod en Shafarevich [GS64, Gol64]. Interessant is dat ze in dezelfde artikels een tegenvoorbeeld geven voor het algemene Burnside probleem in de groepentheorie, waarvan de begrensde tegenhanger echter niet waar blijkt te zijn, zoals bewezen in het werk van Adian en Novikov [AN68a, AN68b, AN68c].

Een eerste stap om het begrensde Kurosh probleem op te lossen, is, zoals bewezen door Jacobson, dat een algebraïsche algebra van begrensde graad voldoet aan een polynoomidentiteit. Vervolgens is, zoals blijkt uit Kaplansky en Levitzki, een eindig voortgebrachte algebraïsche PI-algebra eindig dimensionaal. Dit geeft al de indruk dat de PI-eigenschap een voorwaarde is die op een of andere manier 'de oneindige-dimensionaliteit beperkt/controleert' van een bepaalde PI-algebra. De lezer zou het recht hebben om dat te denken. In feite heeft een PI-algebra slechts eindig dimensionale eenvoudige representaties. Daardoor is de PI-theorie strikt verbonden met de studie van eindig dimensionale representaties van algebras, een theorie die een sterk geometrische kleur heeft, zoals aangetoond werd door Artin en Procesi in de jaren 1960. Dit kan worden gezien als een tweede ontwikkeling van de PI-theorie.

Hierdoor kan commutatieve geometrie in zekere mate worden gebruikt in de niet-commu-
tatieve opzet. Dus hoewel PI-algebra's a priori zeer niet-commutatief kunnen zijn, combineert het methodes van commutatieve algebra met methodes van eindig dimensionale algebras, samengebracht door de representatietheorie. Een ander probleem dat de PI-theorie in een tweede fase heeft ontwikkeld is het inbeddingsprobleem.

Vraag (inbeddingsprobleem)**.** *Karakteriseer associatieve ringen die ingebed kunnen worden in een matrix ring $M_n(C)$ over een commutatieve ring C.*

Een baanbrekend resultaat is de stelling van Amitsur-Levitzki, waarin gesteld wordt dat $M_n(C)$ voldoet aan een polynoomidentiteit, namelijk de standaard polynoom. Een ring die dus voldoet aan het inbeddingsprobleem moet voldoen aan alle polynoomidentiteiten van een matrix algebra. Helaas is dit in het algemeen niet voldoende. Zoals we zullen zien in Hoofdstuk 1, is het inbeddingsprobleem echter waar, zoals bewezen door Kemer in de jaren 1980, voor bepaalde 'universele PI-algebra's'. De theorie ontwikkeld in deze oplossing zal een centrale rol spelen in Hoofdstuk 2.

Hopelijk gelooft de lezer op dit punt dat polynoomidentiteiten zich op het raakvlak bevinden van de niet-commutatieve algebra, algebraïsche meetkunde en representatietheorie.

Overzicht van de bereikte resultaten

Laat ons nu terugkeren naar onze oorspronkelijke motivatie, nl. het associëren van interessante invarianten met PI-algebra's, in het bijzonder eindig voortgebrachte algebra's. In dit proefschrift doen we dit door middel van het T-ideaal $\mathrm{Id}(A)$, bestaande uit alle polynoomidentiteiten van A. Hierdoor onderzoeken wij eerder PI-equivalentie klassen dan isomorfisme klassen. Van twee algebra's A en B wordt gezegd dat ze PI-gelijkwaardig zijn indien $\mathrm{Id}(A) = \mathrm{Id}(B)$.

Indien F karakteristiek 0 heeft, wordt het T-ideaal $\mathrm{Id}(A)$ voortgebracht (als T-ideaal) door multilineaire polynomen. Dus alle informatie over A geleverd door polynoomidentiteiten moet ook worden geleverd door de multilineaire. We noteren door $P_n(F) = \mathrm{span}_F\{x_{\sigma(1)} \dots x_{\sigma(n)} \mid \sigma \in S_n\}$ de multilineaire polynomen van graad n. Vervolgens wordt $c_n(A) = \dim_F \frac{P_n(F)}{P_n(F)\cap \mathrm{Id}(A)}$ de n-de codimensie van A genoemd en $(c_n(A))_n$ zijn codimensie rij. In dit proefschrift zijn we geïnteresseerd in het begrijpen van het asymptotisch gedrag van deze rij in termen van algebraïsche data. In zuiver analytische termen werd het gedrag voorspeld door Regev. We zeggen dat twee functies f en g asymptotisch hetzelfde groeien, aangeduid $f \simeq g$, indien $\lim_{n\to\infty} \frac{f}{g} = 1$.

Conjectuur (Regev)**.** *Zij A een F-algebra met char$(F) = 0$, dan*

$$c_n(A) \simeq cn^t d^n,$$

voor constanten $c \in \mathbb{Q}[\sqrt{2\pi}, \sqrt{v}], v \in \mathbb{N}, t \in \frac{\mathbb{Z}}{2}$ en $d \in \mathbb{Z}$.

Deze conjectuur werd bevestigd door Berele en Regev [BR08] voor unitale algebra's en voor willekeurige algebra's toonden ze aan dat $c_n(A) \simeq \Theta(n^t d^n)$. Dankzij dit kunnen we twee invarianten koppelen aan elke PI-algebra, namelijk $'t'$ en $'d'$. Verder verwijzen we naar deze getallen als respectievelijk het polynomiale en exponentiële deel van A. Merk op dat de integraliteit van d echt een opvallend resultaat is. Het geeft aan dat deze groeifunctie zeer verschillend is van andere zoals de Gelfand-Kirillov of de woord-groeifunctie in de groepentheorie waar bijna elk reëel getal als exponentiële groei kan verschijnen.

Het is duidelijk dat de getallen niet intrinsiek zijn aan A, maar aan de PI-equivalentie-klasse waarvoor ze potentieel interessante invarianten zijn. Dus de volgende vraag is welke algebraïsche informatie aanwezig is in deze getallen, indien die er is? Betreffende het exponentiële deel, ook wel de PI-exponent genaamd, bewezen Giambruno en Zaicev in hun baanbrekende paper [GZ98] dat de PI-exponent van een eindig dimensionale algebra A als volgt verbonden is met de Wedderburn-Malcev ontbinding:

$$d = \max\{dim_F(A_{i_1} \oplus \cdots \oplus A_{i_r}) \mid A_{i_1} J A_{i_2} \cdots J A_{i_r} \neq 0 \text{ with } i_j \neq i_k \text{ for } j \neq k\},$$

waarbij $A \cong A_{ss} \bigoplus J(A)$ en $A_{ss} \cong A_1 \oplus \ldots \oplus A_q$ een maximale semisimple deelalgebra van A is. Merk op dat dit resultaat werd verkregen voor het voornoemde resultaat van Berele en Regev en in feite bewezen Giambruno-Zaicev [GZ98, GZ99] het bestaan en de integraliteit van $\lim\limits_{n\to\infty} \sqrt[n]{c_n(A)}$ voor iedere PI-algebra, zelfs niet noodzakelijkerwijs eindig voortgebracht.

Om over te gaan tot de niet-eindige dimensionale context gebruikten de auteurs Kemer theorie, maar deze leidt echter tot het verliezen van een concrete interpretatie. In tegenstelling tot het bewijs van het resultaat van Giambruno-Zaicev, is het bewijs van Berele en Regev's Stelling van nature geometrisch, wat het mogelijk maakt om de asymptotische groei te begrijpen. Het geeft echter geen inzicht in de algebraïsche kant. Een van de doelen van dit proefschrift is om deze leemte te vullen.

Dit proefschrift kan worden onderverdeeld in twee verschillende delen. In het eerste deel werken wij alleen met de algebra A zelf en zijn we gericht op het begrijpen van het veeltermdeel t en onderzoeken we of dergelijke invarianten ook kunnen worden geïntroduceerd en gebruikt voor hoofdideaaldomeinen en vooral in $\mathbb{Z}$ (d.w.z. voor ringen) en $\mathbb{F}_p$. In het tweede deel houden we rekening met het feit dat acties op een object vaak interessante informatie bevatten over het object in kwestie. We zullen ons vooral richten op acties van de duale van een semigroup algebra FS, wat kan worden geherformuleerd in de taal van gradaties. Dit gebeurt met behulp van gegradeerde S-polynomen en analogen van de codimensie rij en het polynomiale en exponentiële gedeelte. Hoewel we informatie verkrijgen, zal het verhaal minder transparant blijken te zijn.

Het klassieke niet-gegradeerde deel

Laten we nu de voornaamste resultaten in het niet-gegradeerde deel van dit proefschrift bespreken.

Het doel is het polynomiale deel t te verbinden met de algebraïsche structuur van A. Om dit te realiseren moet men eerst het probleem reduceren tot bepaalde algebra's die als bouwblokken dienen als men op PI-equivalentie na werkt. Dit zijn de zogenaamde basic algebra's en die worden geïntroduceerd door Kemer in zijn oplossing van het Specht probleem. We herhalen nu de definitie ervan.

Hiervoor ontbinden we de algebra $A \cong A_{ss} \oplus J(A)$ zoals in de Wedderburn-Malcev stelling. Het tuple $\mathrm{Par}(A) = (\dim_F A_{ss}, J(A))$ noemen we de parameter van A. Een eindig dimensionale algebra heet basic als hij niet PI-equivalent is met de directe som van algebra's $C_1 \oplus \ldots \oplus C_l$ met $\mathrm{Par}(C_i) < \mathrm{Par}(A)$ voor alle i. Volgens het werk van Kemer is ieder eindig dimensionale algebra A PI-equivalent met een eindige directe som $B_1 \oplus \ldots \oplus B_t$ van basic algebra's. Omdat $t(A) = \max_i\{t(B_i) \mid d(B_i) = d(A)\}$, zie Gevolg 1.3.7, moet men eerst een interpretatie vinden voor het polynomiale deel van een basic algebra. Een vermoeden van Giambruno levert algebraïsche interpretatie. In Hoofdstuk 2 bewijzen we dit vermoeden. Dit is het resultaat van gezamenlijk werk met Aljadeff en Karasik.

Stelling 2.2.13. *[AJK17] Zij A een basic algebra met Wedderburn-Malcev ontbinding $A \cong M_{d_1}(F) \oplus \cdots \oplus M_{d_q}(F) \oplus J(A)$ en $Par(A) = (d, s)$. Dan is*

$$c_n(A) = \Theta(n^{\frac{q-d}{2}+s} d^n).$$

In het bijzonder geval dat A een eenheidselement heeft, geldt bovendien dat

$$c_n(A) \simeq C n^{\frac{q-d}{2}+s} d^n,$$

voor een constante $0 < C \in \mathbb{R}$.

Om een interpretatie voor t intern aan A te vinden, moet men bijgevolg nog een constructief algoritme vinden om A in basic algebra's te ontbinden. Deze weg hebben we echter niet gevolgd. Een logische volgende stap is dan dat we ofwel deze invariant gebruiken om PI-equivalentie klassen te onderscheiden ofwel onderzoeken hoe de condities op het grondveld kunnen worden verzwakt. Dit laatst is het onderwerp van Hoofdstuk 3 en werd gezamenlijk met Gordienko uitgewerkt.

Zij R een ring. In dit geval beschouwen we $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R)}$ als een eindig voortgebrachte abelse groep. Zijn decompositie als abelse groep geeft aanleiding tot verschillende codimensie rijen $(c_n(R,\mathbb{Z},p^k))_n$, één voor iedere priem-macht p^k voorkomend in de decompositie. In sectie 1.1.2 en sectie 3.1 onderzoeken we hoe deze codimensie rijen zich gedragen ten opzichte van extensie en restrictie van scalairen. De resultaten leveren Regevs vermoeden voor torsievrije ringen met eenheidselement.

Stelling 3.1.3. *[GJ13] Zij R een torsievrije ring die aan een polynoomidentiteit voldoet. Dan*

1. *als $p^k \neq 0$, dan $c_n(R,\mathbb{Z},p^k) = 0$.*
2. *of $c_n(R,\mathbb{Z},0) = 0$ voor alle $n \geqslant n_0$, $n_0 \in \mathbb{N}$, of er bestaat een $d \in \mathbb{N}, t \in \frac{\mathbb{Z}}{2}$ en $C_1, C_2 > 0$, zodat $C_1 n^t d^n \leqslant c_n(R,\mathbb{Z},0) \leqslant C_2 n^t d^n$ voor alle $n \in \mathbb{N}$; in het bijzonder bestaat $\lim_{n\to\infty} \sqrt[n]{c_n(R,\mathbb{Z},0)} \in \mathbb{N}$;*
3. *als R een eenheidselement bevat, dan bestaat er een $C > 0$ en $t \in \frac{\mathbb{Z}}{2}$ zodat $c_n(R,\mathbb{Z},0) \simeq C n^t d^n$ as $n \to \infty$.*

Helaas, indien R additieve torsie bevat, dan zal men deze informatie verliezen onder extensie van scalairen $R \otimes_{\mathbb{Z}} \mathbb{Q}$. Dus in dit geval, in tegenstelling tot het torsievrije geval, kan men niet hopen dat het voldoende is om de klassieke theorie voor velden van karakteristiek 0 te gebruiken. Verder zijn, in het algemeen, codimensies voor verschillende priem machten p^k niet-nul voor ringen met additieve torsie . Een eerste probleem als men over $\mathbb{Z}$ werkt is dat de modulen niet meer semi-eenvoudig zijn. Daarom onderzoeken we in Hoofdstuk 3 het bestaan van 'mooie' $\mathbb{Z}S_n$-filtraties die de rol van directe som decomposities kunnen overnemen. Om preciezer te zijn, stellen we ons de volgende vraag.

Question. *Zij R een ring. Heeft $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \mathrm{Id}(R,\mathbb{Z})}$ een keten van deelmodulen met factoren die isomorf zijn met $S(\lambda)/mS(\lambda)$, waarbij λ een partitie van n is, en $m \in \mathbb{Z}$ verbonden met de torsie van R?*

De $\mathbb{Z}S_n$-modulen $S(\lambda)$ heten Specht modulen. In sectie 1.2 geven we een overzicht van de nodige S_n-representatie theorie. In de volgende stelling reduceren we het probleem tot zogenaamde *proper* polynomen. Dit zijn producten van lange commutatoren en de vectorruimte dat deze bevat noteren we met $\Gamma_n(\mathbb{Z})$.

Stelling 3.2.1. *[GJ13] Zij R een ring met eenheid,* $\operatorname{char} R = \ell$, $\ell \in \mathbb{Z}_+$. *Beschouw voor iedere $n \in \mathbb{N}$ de keten bestaande uit $\mathbb{Z}S_n$-deelmodulen*

$$M_0 := \frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R, \mathbb{Z})} \supsetneqq M_2 \supseteq M_3 \supseteq \cdots \supseteq M_n \cong \frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R, \mathbb{Z})}$$

waar iedere M_k het beeld is van $\bigoplus_{t=k}^{n} \mathbb{Z}S_n(x_{t+1} \dots x_n \Gamma_t(\mathbb{Z}))$ en $M_{n+1} := 0$. Dan $M_0/M_2 \cong \mathbb{Z}/\mathbb{Z}_\ell$ (triviale S_n-actie) en

$$\begin{array}{rcl} M_t/M_{t+1} & \cong & \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \uparrow S_n \\ & := & \mathbb{Z}S_n \otimes_{\mathbb{Z}(S_t \times S_{n-t})} \left(\frac{\Gamma_t(\mathbb{Z})}{\Gamma_t(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Z}\right) \end{array}$$

voor alle $2 \leqslant t \leqslant n$ waar S_{n-t} de variabelen $x_{t+1}, \dots, x_n$ permuteert en $\mathbb{Z}$ is de triviale $\mathbb{Z}S_{n-t}-$moduul.

In het geval dat $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ 'redelijk mooi' is, d.w.z. van de vorm $S(\lambda)/mS(\lambda)$, levert een veralgemening van Youngs regel, zie Stelling 3.3.1, een positief antwoord op de eerder vermelde vraag. Bijgevolg moeten we onderzoeken wanneer $\frac{\Gamma_n(\mathbb{Z})}{\Gamma_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ 'redelijk mooi' is. We bewijzen dit voor twee belangrijke voorbeelden. Het eerste voorbeeld is de veralgemeende bovendriehoeks matrix ring $R = \begin{pmatrix} R_1 & M \\ 0 & R_2 \end{pmatrix}$, met M een (R_1, R_2)-bimoduul over commutatieve unitale ringen R_1 en R_2. Het tweede voorbeeld is de Grassmann algebra G_S over een commutatieve unitale ring S met oneven karakteristiek l. Herinnert u zich dat G_S voortgebracht is door de aftelbare verzameling $\{e_n \mid n \in \mathbb{N}\}$ en voldoet aan $e_i e_j = -e_j e_i$ voor $i \neq j$. In sectie 3.4 en sectie 3.5 bewijzen we dat de *proper* niet-polynomoomidentiteiten inderdaad 'redelijk mooi' zijn. Meer precies,

Stelling 3.4.8. *[GJ13] Zij R, ℓ, en m, respectievelijk, de ring en de getallen uit Subsectie 3.4.1. Dan bestaat er een keten van $\mathbb{Z}S_n$-deelmodulen in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z}) \cap \operatorname{Id}(R,\mathbb{Z})}$ waarbij de verzameling factoren bestaat uit één kopie van $\mathbb{Z}_\ell$ en $(\lambda_1 - \lambda_2 + 1)$ kopieën van $S(\lambda_1, \lambda_2, \lambda_3)/mS(\lambda_1, \lambda_2, \lambda_3)$ met $(\lambda_1, \lambda_2, \lambda_3) \vdash n$, $\lambda_2 \geqslant 1$, $\lambda_3 \in \{0, 1\}$.*

In het geval van de Grassmann algebra verkijgen wij

Stelling 3.5.4. *[GJ13] Zij G_S de Grassmann algebra over de commutatieve unitale ring S met $\ell = \operatorname{char} S$. Dan bestaat er een keten van $\mathbb{Z}S_n$-deelmodulen in $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \operatorname{Id}(G_R,\mathbb{Z})}$ met factoren $S(n-k,1^k)/\ell S(n-k,1^k)$ voor iedere $0 \leqslant k \leqslant n-1$ (iedere factor komt exact 1 keer voor).*

Het gegradeerde gedeelte

Laten we nu de resultaten verkregen in het gegradeerde deel van deze thesis bespreken.

In Hoofdstuk 4 bespreken we gezamenlijk werk met Jespers en Gordienko. Hierbij klasseren we alle eindig dimensionale S-gegradeerd-simpele algebra's gegradeerd door een volledige 0-deelgroep met triviale maximale deelgroep, i.e. voor $S = \mathcal{M}(\{e\}^0, n, m, P)$. A priori kan dit klinken als een beperkte klasse van semigroepen, maar in sectie 4.1 tonen we aan dat als A een S-gegradeerd-simpele algebra is, S gereduceerd kan worden tot semigroepen van de vorm $\mathcal{M}(\{G\}^0, n, m, P)$. Intuïtief bestaat deze semigroep uit (bepaalde) $n \times m$-matrices met waarden in $G \cup \{0\}$. Het geval dat we behandelen, i.e. $G = \{e\}$, is in een zekere zin het tegenovergestelde geval van een groepsgradatie (in onze notatie komt dit overeen met $n = m = 1$). Voor groepsgradaties werd een classificatie worden bereikt door Bahturin, Zaicev en Sehgal [BZ02, BZS08]. Hopelijk zullen in de toekomst beide gevallen samengesmolten kunnen worden om tenslotte het algemene antwoord te verkrijgen.

De classificatie bestaat uit twee delen. Eerst beschrijven we eindig dimensionale S-gegradeerd-simpele algebra's met $S = \mathcal{M}(\{e\}^0, n, m, P)$ en dan bewijzen we dat iedere algebra die voldoet aan de beschrijving aanleiding geeft tot een $\mathcal{M}(\{e\}^0, n, m, P)$-gegradeerd-simpele structuur. De beschrijving doen we door eerst A te ontbinden als $B \oplus J(A)$ met B als een maximale semi-eenvoudige gegradeerde deelalgebra. Nadien ontbinden we $J(A)$ in linkse B-modulen die allemaal isomorf zijn met concrete delen van B. Vooraleer de classificatie te citeren wensen we op te merken dat door de definitie van $\mathcal{M}(\{e\}^0, n, m, P)$ de algebra A ontbonden kan worden in deelruimten als volgt

$$A = \bigoplus_{\substack{1\leq i\leq n,\\ 1\leq j\leq m}} A_{ij},$$

met $A_{ij}A_{k\ell} \subseteq A_{i\ell}$. We noteren de 'kolommen' en de 'rijen' als

$$L_i := \bigoplus_{k=1}^{n} A_{ki} \text{ and } R_i := \bigoplus_{k=1}^{m} A_{ik}. \tag{5.27}$$

Nu vermelden we de gegradeerde versie van de Wedderburn-Malcev decompositie die geldig is in ons kader. Het bewijs hiervan is constructief.

Stelling 4.3.2. *[GJJ17] Zij $A = \bigoplus_{i,j} A_{ij}$ een eindig dimensionale S-gegradeerde F-algebra over een veld F zodat $AJ(A)A = 0$ en $S = \mathcal{M}(\{e\}^0, n, m, P)$. Dan bestaan er orthogonale idempotenten $f_1, \ldots, f_m$ en orthogonale idempotenten $f'_1, \ldots, f'_n$ (sommige eventueel nul) zodat*

$$B = \bigoplus_{i,j} f'_i A f_j = \bigoplus_{i,j} (B \cap A_{ij})$$

een S-gegradeerde maximale semi-eenvoudige deelalgebra van A is, $f'_i \in B \cap R_i$ voor $1 \leq i \leq n$, $f_j \in B \cap L_j$ voor $1 \leq j \leq m$, $\sum_{i=1}^n f'_i = \sum_{j=1}^m f_j = 1_B$, en $A = B \oplus J(A)$, een directe som van deelruimten.

Op dit punt geldt dat $A = B \oplus J(A)$ waarbij B een gegradeerde deelalgebra is die constructief opgesteld kan worden. Vooraleer de ontbinding van het radicaal $J(A)$ te beschrijven moeten we nog enkele notaties invoeren. Beschouw voor iedere $1 \leq i \leq n$ en $1 \leq j \leq m$ de deelruimten

$$J_{ij}^{10} := f'_i L_j (1 - 1_B) \quad \text{and} \quad J_{ij}^{01} := (1 - 1_B) R_i f_j.$$

Verder zij

$$J_{*j}^{10} := \sum_{1 \leq i \leq n} J_{ij}^{10} = 1_B L_j (1 - 1_B) \quad \text{and} \quad J_{i*}^{01} := \sum_{1 \leq j \leq m} J_{ij}^{01} = (1 - 1_B) R_i 1_B.$$

Deze deelruimten vormen de bouwstenen van $J(A)$.

Stelling 4.3.7. *[GJJ17] Zij A een eindig dimensionale S-gegradeerde-simple F-algebra. Zij B en $f_1, \ldots, f_m$, $f'_1, \ldots, f'_n$, respectievelijk, een gegradeerde deelalgebra en de orthogonale idempotenten uit stelling 4.3.2. Hiervoor geldt dat J_{*j}^{10} een linkse B-deelmoduul van $J(A)$ is en $J_{*j}^{10} = \bigoplus_{i=1}^n J_{ij}^{10}$. Verder is J_{i*}^{01} een rechtse B-deelmoduul van $J(A)$ en $J_{i*}^{01} = \bigoplus_{j=1}^m J_{ij}^{01}$. Bovendien zijn*

$$J(A) = \bigoplus_{i=1}^n J_{i*}^{01} \oplus \bigoplus_{j=1}^m J_{*j}^{10} \oplus J(A)^2 \quad \text{en} \quad J(A)^2 = \bigoplus_{i=1}^n \bigoplus_{j=1}^m J_{i*}^{01} J_{*j}^{10},$$

directe sommen van deelruimten.

Alsook bestaat er een F-lineaire afbeelding

$$\varphi \colon \bigoplus_{i=1}^n J_{i*}^{01} \oplus \bigoplus_{j=1}^m J_{*j}^{10} \to B$$

die zich 'heel goed' gedraagt. Verder is voor iedere $1 \leq i \leq n$*,* $1 \leq j \leq m$

$$\begin{aligned} A_{ij} = f_i' B f_j \oplus \left\{ \varphi(v) + v \mid v \in J_{ij}^{10} \oplus J_{ij}^{01} \right\} \\ \oplus \mathit{span}_F \left\{ \varphi(v)\varphi(w) + v\varphi(w) + \varphi(v)w + vw \mid v \in J_{i*}^{01},\ w \in J_{*j}^{10} \right\} \end{aligned} \quad (5.28)$$

een directe som van deelruimten.

Tenslotte is $B \cong M_k(D)$*, waarbij* D *een scheef lichaam is dat voldoet aan*

$$\dim_F \bigoplus_{i=1}^{n} J_{i*}^{01} \leq (n-1) \dim_F B = (n-1)k^2 \dim_F D, \quad (5.29)$$

$$\dim_F \bigoplus_{j=1}^{m} J_{*j}^{10} \leq (m-1) \dim_F B = (m-1)k^2 \dim_F D, \quad (5.30)$$

$$\dim_F J(A) \leq (nm-1) \dim_F B = (|S|-1) \dim_F B = (|S|-1)k^2 \dim_F D. \quad (5.31)$$

In de volledige versie van stelling 4.3.7 schrijven we bovendien wat we ermee bedoelen dat φ zich 'heel goed' gedraagt.

Vervolgens onderzoeken we in Hoofdstuk 5 gegradeerde codimensies en hun exponentiële groei voor een oneindige deelfamilie van de zojuist geklasseerde algebra's. Hierbij gebruiken we nagenoeg geen resultaten vanuit de classificatie. Hierdoor kunnen beide hoofdstukken onafhankelijk van elkaar gelezen worden. Daardoor verliest men echter wel wat intuïtie over de algebra's die in beschouwing worden genomen.

In dit hoofdstuk beschouwen we zowel associatieve, als niet-associatieve algebra's. Het associatieve deel van het hoofdstuk is verdeeld in twee gevallen die bepaald worden op basis van eigenschappen van de gradatie. Beide gevallen gedragen zich helemaal anders. In het eerste geval is het antwoord analoog met het groep-gegradeerde geval.

Stelling 5.4.5. *[GJJ17] Zij* A *een eindig dimensionale* T*-gegradeerde-simpele algebra over een veld* F *van karakteristiek* 0 *voor een rechtse zero band* T*. Veronderstel* $A/J(A) \cong M_2(F)$*. Zij* $T_0, T_1 \subseteq T$ *en* $\sim$*, respectievelijk de deelverzamelingen en equivalentierelatie gedefinieerd in het begin van Sectie 5.4. Veronderstel verder dat (5.13) geldig is of* $T_0 = \varnothing$*. Dan bestaan er constanten* $C > 0$*,* $D \in \mathbb{R}$*, zodat*

$$Cn^D (\dim_F A)^n \leq c_n^{T\text{-gr}}(A) \leq (\dim_F A)^{n+1}.$$

In het bijzonder is $\mathrm{PIexp}^{T\text{-gr}}(A) = \dim_F A$*.*

In het tweede geval verkrijgen we echter een irrationaal gegradeerde PI-exponent. Ieder getal $1+\sqrt{m}+m$ kan bijvoorbeeld op deze wijze gerealiseerd worden.

Stelling 5.5.5. *[GJJ17] Zij A een eindig dimensionale T-gegradeerd-simpele algebra over een veld F van karakteristiek 0 voor een rechts zero band. Veronderstel $A/J(A) \cong M_2(F)$. Zij $T_0, T_1 \subseteq T$ en $\sim$, respectievelijk de deelverzamelingen en de equivalentie relatie gedefinieerd bij het begin van sectie 5.4. Indien $|\bar{t}_0| > \frac{|T_0|}{2}$ voor een $\bar{t}_0 \in T_0/\sim$. Dan,*

$$\exp^{T\text{-gr}}(A) = |T_0| + 2|T_1| + 2\sqrt{(|T_1| + |\bar{t}_0|)(|T_0| + |T_1| - |\bar{t}_0|)} < 2|T_0| + 4|T_1| = \dim A.$$

Om deze thesis te beëindigen bestuderen we niet-associatieve algebra's, meer bepaald Lie algebra's, en semigroep gradaties hierop. Meer in het bijzonder construeren we het eerste voorbeeld van een eindig dimensionale semigroep-gegradeerde Lie algebra met een niet-geheel gegradeerde PI-exponent. In één en dezelfde beweging bewijzen we onder andere een semigroep-gegradeerde versie van Ado's stelling die bevestigt dat een eindig dimenisonale Lie algebra een getrouwe eindig dimensionale representatie heeft. In tegenstelling tot het associatieve geval is ons voorbeeld niet gegradeerd-simpel.

Stelling 5.7.1. *Zij L de Lie algebra met $(\mathbb{Z}_2,\cdot)$-gradatie van het begin van sectie5.7. Dan $\exp^{\mathbb{Z}_2}(L) = \lim\limits_{n\to\infty} \sqrt[n]{c_n^{\mathbb{Z}_2}(L)} = 2+2\sqrt{2}$.*

Leidraad

In Hoofdstuk 1 geven we een kijk op de achtergrond die nodig is om het onderzoek dat tijdens dit project werd uitgevoerd, te begrijpen en om er intuïtie voor te krijgen. We beginnen door in sectie 1.1 alle basisdefinities te geven en we introduceren codimensies van een algebra gedefinieerd over elk hoofdideaaldomein. Als een algebra A gedefinieerd is over verschillende hoofdideaaldomeinen, kunnen verschillende codimensie rijen met A geassocieerd worden. Het verband daartussen wordt ook beschreven in dit hoofdstuk.

Om codimensies te berekenen, gebruiken we vervolgens de representatietheorie van de symmetrische groep. Dit is een zeer rijke theorie. De essentie wordt herhaald in sectie 1.2. Hierbij hebben we ervoor gekozen om de uiteenzetting zo onafhankelijk mogelijk van de grondring te houden, om beter de nadruk te leggen op waar het in de klassieke asymptotische theorie van polynoomidentiteiten nodig is dat A een F-algebra is met $\mathrm{char}(F) = 0$. Daarnaast is het ook van toepassing in Hoofdstuk 3, waar we alleen werken over $\mathbb{Z}$ en waarvan de codimensietheorie volgens ons meer overeenkomsten heeft met de $\text{char}(F) = p \neq 0$ context, dan het $\text{char}(F) = 0$ geval.

Met de S_n-representatietheorie ter beschikking, overzien we in sectie 1.3 de belangrijkste resultaten van de Regev conjectuur en bovendien leggen we uit hoe men de PI-exponent van een F-algebra met $\text{char}(F) = 0$ kan berekenen. Ten slotte, alvorens verder te gaan met onze reis door de wereld van invarianten, introduceren we in sectie 1.4 de belangrijkste aspecten van de Kemer theorie.

Hoofdstuk 2 is gewijd aan het bewijs van Giambruno's conjectuur die een waarde beschrijft voor het polynomiale deel van elementaire algebra's. Dit is gebaseerd op gezamenlijk werk met Aljadeff en Karasik [AJK17]. Het hoofdstuk maakt een intensief gebruik van de theorie besproken in sectie 1.4 en bestaat uit twee delen, de boven- en ondergrens.

Nadien gaan we in Hoofdstuk 3 over tot het onderzoek van $\mathbb{Z}$-algebra's, dat wil zeggen ringen. Onder andere bewijzen we een variant van de Amitsur en Regev conjectuur voor (unitale) torsievrije ringen en bespreken het bestaan van een $\mathbb{Z}S_n$-filtratie van $\frac{P_n(\mathbb{Z})}{P_n(\mathbb{Z})\cap \text{Id}(A)}$, die als vervanging kan dienen voor de directe som decompositie bij een veld F van karakteristiek 0. Het probleem van het bestaan wordt in sectie 3.2 gereduceerd tot *proper* polynoomidentiteiten. Het bestaan en de sterkte van een dergelijke filtratie worden vervolgens aangetoond in sectie 3.4 en sectie 3.5, respectievelijk in het geval van een algemene boventriangulaire matrixring en bij de Grassmann algebra. Dit hoofdstuk is gebaseerd op gezamenlijk werk met Gordienko [GJ13].

Hoofdstuk 4 verlaat voor de eerste keer de niet-gegeradeerde setting en focust op algebra's met een semigroep gradatie. Het hoofdstuk behandelt het probleem van de classificatie van alle eindige dimensionale T-gegradeerde simpele algebra's voor een willekeurige semigroep T. In eerste instantie reduceren we het probleem in sectie 4.1 tot drie soorten semigroepen. Daarna wordt voor de semigroepen $\mathcal{M}(\{e\}^0, n, m, P)$ het classificatiesprobleem opgelost in sectie 4.3 en sectie 4.4.

Tenslotte behandelen we in Hoofdstuk 5 de eerder geclassificeerde algebra's. Voor ieder van hen geven we in paragraaf 5.3 een bovengrens aan de exponentiële groei van hun gegradeerde codimensies. Vervolgens berekenen we de exacte waarde van de gegradeerde PI-exponent in de sectie 5.4 en sectie 5.5 voor een oneindige deelverzameling van de semigroep gegradeerd-simpele algebra's. Deze resultaten leveren willekeurig grote irrationele gegradeerde PI-exponenten op. Hoofdstuk 4 en de zojuist genoemde resultaten zijn gebaseerd op gezamenlijk werk met Gordienko en Jespers [GJJ17]. Tot slot concentreren we ons ook op Lie algebra's en produceren we in sectie 5.7 het eerste voorbeeld van een gegradeerde Lie algebra met niet-geheel gegradeerde PI-exponent.

A

Overview other work

In this appendix we give a survey of some of the research performed during the ph.d. on topics not explicitly connected to codimensions.

A.1 Exact linear independence of solvable groups

Let G be a finitely generated group with finite generating set S. The study of the growth of G is the study of the number of elements in the n-fold product $S^n = S \dots S \subseteq G$ as a function of n. This is the group-theoretical equivalent of the Gelfand-Kirillov dimension. A simple way to quantify the growth of G with respect to S is to introduce the exponential growth rate $\rho_{G,S} := \lim_{n\to\infty} |S^n|^{1/n}$. The group G is said to be of exponential growth if $\rho_{G,S} > 1$. While $\rho_{G,S}$ typically depends on S, the property that it is strictly larger than 1 is independent of the choice of the generating set S. Similarly one says that G has polynomial growth if there are constants $C, d > 0$ independent of n such that $|S^n| \leq Cn^d$ for all $n \geq 1$. By a celebrated theorem of Gromov [Gro81] this happens if and only if G has a nilpotent subgroup of finite index.

In many classes of groups there is a growth gap, i.e. the growth can only be polynomial or exponential. For example this phenomenon occurs for linear groups, solvable groups and elementary amenable groups [Tit72, Mil68, Wol68, Cho80]. For all classes where such a gap phenomenon is known it was a consequence of the existence of subgroups which are free semigroups or free groups. Surprisingly these constructions can be made independent of the generating set, hence affording even uniform exponential growth (i.e. $\rho_G := \inf_{S\,:\,G=\langle S\rangle} \rho_{G,S} > 1$ where S varies among all finite generating subsets of G). In order to be more concrete following definition is useful.

Definition A.1.1. Two elements $a, b \in G$ are called linearly independent if they generate a free semigroup. The number

$$d_G^+(S) := \inf\{n \in \mathbb{N} \mid S^n \text{ contains two linearly independant elements }\}$$

is called the diameter of positive independence of S. Further the diameter of positive independence of the group G is defined by $d_G^+ = \sup_{S\,:\,G=\langle S\rangle} \{d_G^+(S)\}$.

Uniform exponential growth and uniform constructions of free semigroups are connected by the inequality

$$\rho_G \geq 2^{1/d_G^+}.$$

Using this method uniform exponential growth was proven for linear, solvable and elementary amenable groups [Osi03, Osi04, EMO05]. In the eighties, Gromov asked whether a group with exponential growth must have uniform exponential growth. The question was answered negatively by Wilson [Wil04] who constructed examples of subgroups G of the automorphism group of a rooted tree with $\rho_G = 1$, although they contain a free subgroup and hence have exponential growth.

In joint work with Doryan Temmerman [JT17], optimizing the bound from [Bre07], we determined the linear independence of solvable groups.

Theorem A.1.2. *Let G be a finitely generated solvable group which is not virtually nilpotent. Then G has a subgroup H_0 of finite index such that $d_{H_1}^+ = 1$ for any finite index subgroup of H_0.*

Moreover if G is also metabelian and non-polycyclic, then H_0 can be taken to be G.

This result is a consequence from a careful study of the linear independence of some suitable chosen epimorphic image of G. Recall that free semigroups lifts and therefore it is enough to prove this statement for any epimorphic image of G. Now, there exists a normal subgroup N such that G/N is not virtually nilpotent but any strict quotient is virtually nilpotent. The group G/N is called 'just not virtually nilpotent', abbreviated JNVN. Thus it is enough to prove Theorem A.1.2 for finitely generated JNVN solvable groups. However this groups turn out to be very elegant, see [Gro78] or [Bre07].

Theorem A.1.3 ([Gro78] or [Bre07])**.** *Let G be a finitely generated JNVN solvable group. Then G is virtually metabelian. Moreover if G is metabelian, then it embeds in the group of affine transformations of the K-line, for some field K.*

In other words, [JT17] and Theorem A.1.2 consists of constructing in a strongly uniform way free semigroups in finitely generated subgroups of

$$\mathbb{A}(K) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \mid a \in K^{\times}, b \in K \right\}.$$

A.2 On the abelianisation of $\mathcal{U}(\mathbb{Z}G)$ and $GL_2(\mathcal{O})$

Given a finite group G it is natural to ask which information is contained in its category of complex representations $\mathrm{Rep}_{\mathbb{C}}(G)$. If we take the full symmetric monoidal structure into account, then using a result by Deligne [Del90], one can recover the group up to isomorphism. On the other hand only the ring structure of the regular representation $\mathbb{C}G$ contains few information about G. Even when varying the ground field, by an example of Dade [Dad71], group algebras do not always determine the underlying group. However Dade's examples can be distinguished by their integral group rings.

Question (Integral Isomorphism Problem (ISO))**.** *Let G be a finite a group. Is G uniquely determined by $\mathbb{Z}G$? In other words, if G and H are finite groups with $\mathbb{Z}G \cong \mathbb{Z}G$, is then $G \cong H$?*

A major breakthrough was by Roggenkamp-Scott [RS87] who proved it for nilpotent groups and by Whitcomb for metabelian groups, see the book of Sehgal [Seh93] for a full survey. For many decades the answer was believed to be positive, however surprisingly a counterexample was constructed by Hertweck [Her01]. Nevertheless still many properties of G are encoded in $\mathbb{Z}G$. Unfortunately the (module-theoretical) methods used in [RS87] do not explain how to actually reconstruct G from $\mathbb{Z}G$. In case $\mathcal{U}(\mathbb{Z}G)$ has a 'very nice' structure, e.g. it is virtually free-by-free, more can however be said on the reconstruction problem. Joint with Bächle, Jespers, Kiefer and Temmerman we started in [BJJ+17] the investigation of following question.

Question. *When does $\mathcal{U}(\mathbb{Z}G)$ have a non-trivial decomposition as amalgamated product? Which type of amalgamated decompositions imply (ISO)?*

Since $\mathbb{Z}G$ is a $\mathbb{Z}$-order in the finite dimensional semisimple $\mathbb{Q}$-algebra $\mathbb{Q}G$, its unit group $\mathcal{U}(\mathbb{Z}G)$ is a finitely presented (arithmetic) group [Bor62] and hence to investigate previous question it is natural use the rich tool box of geometric group theory. Herein a group presentation or decomposition as amalgamated product are obtained through actions on geometric objects. For instance, if for any tree T and any action of $\mathcal{U}(\mathbb{Z}G)$ on T there is an edge that is fixed by all elements of $\mathcal{U}(\mathbb{Z}G)$, then such non-trivial amalgamated decomposition do not exists. If $\mathcal{U}(\mathbb{Z}G)$ has such global fixed point for each action on a tree, one say that it satisfy Serre's property (FA). The latter has a group theoretical characterisation.

Theorem A.2.1 (Serre [Ser77])**.** *Let G be a finitely generated group. Then G has property (FA) if and only if G^{ab}, the abelianization of G, is finite and G is not a non-trivial amalgam.*

In [BJJ$^+$17] we describe, in function of G, when any subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ has property (FA), called property (HFA). We give now an overview of some of the results obtained. To start,

Proposition A.2.2. *Let G be a finite group and let $\mathbb{Q}G \cong \bigoplus_{i=1}^{q} M_{n_i}(D_i)$ with $\mathcal{O}_i$ an order in D_i. Suppose that $\mathcal{U}(\mathbb{Z}G)^{ab}$ is finite. Then G is a cut group (i.e. $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm\mathcal{Z}(G)$). Furthermore the following are equivalent:*

1. *$\mathcal{U}(\mathbb{Z}G)$ has (HFA),*
2. *$GL_{n_i}(\mathcal{O}_i)$ has (HFA) for all i,*
3. *G is a cut group and $SL_{n_i}(\mathcal{O}_i)$ has (HFA) for all i.*

Due to the previous proposition property (HFA) for $\mathcal{U}(\mathbb{Z}G)$ depend on lattices in the components of $\mathbb{Q}G$. Here the answer will depend on the rank of the Lie group $\mathcal{G}(K \otimes_{\mathbb{Q}} \mathbb{R})$ corresponding to the algebraic group $\mathcal{G} = SL_{n,D}$ associated with a central division algebra D with center K appearing in the decomposition of $\mathbb{Q}G$. Furthermore we can assume that the center of D is $\mathbb{Q}(\sqrt{-d})$ with $d \geq 0$ (these are by Dirichlet unit theorem the only algebraic number fields whose ring of integers only has a finite number of units). In case $n \geq 3$, we are in the so-called high rank setting and then strong fixpoint properties holds. More precisely, in this case some non-commutative version of the subgroup congruence problem holds. This allows to reduce the problem to the group $E_n(\mathcal{O})^{(m)}$ generated by the m-th powers of elementary matrices. For these groups we proved the following.

Theorem A.2.3. *Let $\mathcal{O}$ be an order in a finite dimensional rational division algebra D. Then, if $n \geq 3$, the group $E_n(\mathcal{O})^{(m)}$ satisfies property (FA) for each $m \in \mathbb{N}$. In particular, $SL_n(\mathcal{O})$ and $E_n(\mathcal{O})$ have property (HFA).*

In case of $SL_2(\mathcal{O})$ property (HFA) could not be retraced in the literature only in the case that D is a totally definite quaternion algebra over $\mathbb{Q}$ (in which case the rank of $\mathcal{G}(K \otimes_{\mathbb{Q}} \mathbb{R})$ is one). Interestingly in this case $SL_2(\mathcal{O})$ never satisfy property (HFA). This was obtained by investigations of the abelinisation of $E_2(\mathcal{O})$ and $GE_2(\mathcal{O}) = \langle E_2(\mathcal{O}), \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathcal{U}(\mathcal{O})\rangle$.

Theorem A.2.4. *Let $\mathcal{O}$ be an order in a totally definite quaternion algebra or imaginary quadratic extension of $\mathbb{Q}$. Further let N be the two-sided ideal generated by the elements $u-1$ with $u \in \mathcal{U}(\mathcal{O})$. Then there exists following short exact sequence of groups*

$$1 \to (\mathcal{O}/N, +) \to GE_2(\mathcal{O})^{ab} \to \mathcal{U}(\mathcal{O})^{ab} \to 1.$$

Moreover $\mathcal{U}(\mathcal{O})^{ab} \cong GE_2(\mathcal{O})/E_2(\mathcal{O})$.

Note that, since $2 \in N$ and N is an ideal, $2b \in N$ for any $\mathbb{Z}$-basis element b of $(\mathcal{O}, +)$. In particular we have following corollary.

Corollary A.2.5. *Let $\mathcal{O}$ be an order in a totally definite quaternion algebra or imaginary quadratic extension of $\mathbb{Q}$. Then $GE_2(\mathcal{O})^{ab}$ is finite. In particular if $\mathcal{O}$ is a maximal order then $GE_2(\mathcal{O})^{ab}$ is finite.*

A natural approach to show that $GE_2(\mathcal{O})$ does not have property (HFA) is by proving that $E_2(\mathcal{O})$ has infinite abelianization. It turns out that this always works, except when D is equal to $(-1,-1)_{\mathbb{Q}}$ or $(-1,-3)_{\mathbb{Q}}$, due to following description of $E_2(\mathcal{O})^{ab}$ that we obtained.

Theorem A.2.6. *Let $\mathcal{O}$ be an order in a totally definite quaternion algebra or imaginary quadratic extension of $\mathbb{Q}$. Let M be the additive subgroup of $\mathcal{O}$ generated by the following set of elements:*

1. *$\alpha x \alpha - x$ with $x \in \mathcal{O}$ and $\alpha \in \mathcal{U}(\mathcal{O})$*
2. *$3(\alpha+1)(\beta+1)$ with $\alpha, \beta \in \mathcal{U}(\mathcal{O})$*
3. *the elements $4tr(x) + 6$ for each element $x \in \mathcal{O}$ with $|x|^2 = 2$*
4. *the elements $6tr(x)$ for each element $x \in \mathcal{O}$ with $|x|^2 = 3$.*

Then

$$E_2(\mathcal{O})^{ab} \cong (\mathcal{O}/M, +).$$

In the degenerate cases, other subgroups of finite index can be found, however in this overview we do go further into details. We only mention a last positive result.

Theorem A.2.7. *Let $D \in \{(-1,-1)_{\mathbb{Q}}, (-1,-3)_{\mathbb{Q}}\}$ and $\mathcal{O}$ its (unique) maximal order. Then $GL_2(\mathcal{O})$ has (FA) but not (HFA).*

All together we obtained that $\mathcal{U}(\mathbb{Z}G)$ has property (HFA) if and only if G is cut and without so-called exceptional components. Moreover we described, in terms of G, the existence of the latter.

A.3 Construction of amalgamated products in $\mathcal{U}(\mathbb{Z}G)$

As mentioned before, one main obstruction towards a positive solution to the integral isomorphism problem is the lack of information on how rigid G lies inside $\mathcal{U}(\mathbb{Z}G)$ and therefore one started the search for generic constructions of elements in $\mathcal{U}(\mathbb{Z}G)$ and one investigates the algebraic structure of the group generated by these units.

Only few generic constructions are known. The most important are the so called Bass units and the bicylic units. Recalling that the bicyclic units in $\mathbb{Z}G$ are the elements of the type

$$b(g,\tilde{h}) = 1 + (1-h)g\tilde{h} \quad \text{and} \quad b(\tilde{h},g) = 1 + \tilde{h}g(1-h),$$

where $g,h \in G$. These units are non-trivial(i.e. they do not belong to G) if $g \notin N_G(\langle h \rangle)$, the normalizer of $\langle h \rangle$ in G. The Bass units are the units of the type

$$u_{k,m}(g) = (1 + g + \cdots g^{k-1})^m + \frac{1-k^m}{o(g)}\tilde{g},$$

where $g \in G$ and k,m are positive integers such that $k^m \equiv 1 \bmod o(g)$, and $1 \leq k < n$.

With these elements at hand, it is a natural problem to determine "how large" the group B generated by the Bass and bicyclic units is compared to $\mathcal{U}(\mathbb{Z}G)$. Furthermore, one would like to determine the relations between these units. Jespers and Leal [JL93] proved that for many finite groups G the group B is of finite index in $\mathcal{U}(\mathbb{Z}G)$; earlier results of this type were obtained by Ritter and Sehgal (see for example [RS91]). The groups G excluded are those that have a non-commutative fixed point free image and those for which the rational group algebra $\mathbb{Q}G$ does have an exceptional simple epimorphic image. The latter are by definition the non-commutative division algebras which are not a positive-definite quaternion algebra and matrix algebras $M_2(D)$ over a division algebra of the type $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-d})$ or a quaternion algebra $\left(\frac{a,b}{\mathbb{Q}}\right)$. Moreover, in [EKVG15], Eisele, Kiefer and Van Gelder reduced the number of exceptional cases by showing that the only cases that can occur as an epimorphic image of a rational groups algebra $\mathbb{Q}G$ are $d = 1, 2, 3$ and $(a, b) = (-1, -1), (-1, -3), (-2, -5)$. For a state-of-the-art we refer to [JdRo16a].

In recent years there have been a lot of investigations on determining whether there are any non-trivial relations between two given units that are Bass units or bicyclic units. It turns out that in many cases two such elements generate a non-cyclic free group. In this context, a result of Hartley and Pickel [HP80] states that $\mathcal{U}(\mathbb{Z}G)$ contains a non-cyclic free group except if G is abelian or an Hamiltonian 2-group. Actually, it turns that these cases correspond with $\mathcal{U}(\mathbb{Z}G)$ being abelian-by-finite, which on its turn is exactly the case when the unit group is solvable-by-finite (see [JdRo16a, Corollary 5.5.7]).

An explicit construction of a free subgroup of the unit group was given by Marciniak and Sehgal in [MS97]: it is shown that any non-trivial bicyclic unit together with its image under the classical involution (which also is a bicylic unit) generate a non-cyclic free group. Since then many more constructions of two bicyclic units, or two Bass units, or a Bass together with a bicyclic unit generating a free group have been discovered. For a survey we refer to [GDRo13, JdRo16b]. In [GP04], Gonçalves and Passman showed that $\mathcal{U}(\mathbb{Z}G)$ contains a free product $\mathbb{Z}_p \star \mathbb{Z}$ (with p a prime number) if and only if G contains a noncentral element of order p. Moreover, when this occurs, the $\mathbb{Z}_p$-part of the free product can be taken to be a suitable noncentral subgroup of G of order p. The proof of this result makes use of earlier work of Passman [Pas04] on the existence in $\mathrm{PSL}_n(R)$ (with R a commutative integral domain of characteristic zero) of a free product $G \star \mathbb{Z}$ when G is a finite subgroup of $\mathrm{PSL}_n(R)$. In the proofs of all these results, the element of infinite order is used in order to apply Tits alternative type techniques. We point out that this generator of the infinite cyclic part is only shown to exist, but no explicit constructions are obtained.

Note that if $p \neq 2$ then a group contains $\mathbb{Z}_p \star \mathbb{Z}$ if and only if it contains a group $\mathbb{Z}_p \star \mathbb{Z}_p$. Hence, a natural problem is to give explicit generic constructions of units $b_1, b_2 \in \mathcal{U}(\mathbb{Z}G)$ such that $\langle b_1, b_2 \rangle \cong \mathbb{Z}_p \star \mathbb{Z}_p$. As such, one also obtains a generic construction of a unit $b = b_2 b_1 b_2$ such that $\langle b_1, b \rangle \cong \mathbb{Z}_p \star \mathbb{Z}$, provided $p \neq 2$. Till our joint work with Jespers and Temmerman, [JJT17], such a result had not been obtained mainly because of the lack of generic constructions of non-trivial torsion units. In [JJT17] we used so called Bovdi units to give the first (generic) constructions of free products of two finite cyclic groups in $\mathcal{U}(\mathbb{Z}G)$ provided G is a finite nilpotent group. Recall that Bovdi units are the following suitable modifications of bicyclic units, turning them into torsion units:

$$b_k(g,\tilde{h}) = h^k + (1-h)g\widetilde{h} \quad \text{and} \quad b_k(\tilde{h},g) = h^k + \widetilde{h}g(1-h),$$

with $g,h \in G$ and k a positive integer.

We can now state the main result we obtained in [JJT17].

Theorem A.3.1. *Let G be a finite nilpotent group, $g,h \in G$ such that $g \notin N_G(\langle h \rangle)$. If $o(h) = p$ (a prime number) and $1 \leq k,l \leq p-1$ then*

$$\langle b_k(g,\tilde{h}), b_l(g,\tilde{h})^* \rangle \cong C_p \star C_p \cong \langle b_k(g,\tilde{h}), b_l(\tilde{h},g^{-1}) \rangle.$$

Conversely, if $\mathcal{U}(\mathbb{Z}G)$ contains a subgroup isomorphic with $C_p \star C_p$ then there exist $g,h \in G$ satisfying the assumptions of the first part of the statement.

In the case of groups nilpotency class 2 we proved a more general statement.

Theorem A.3.2. *Let G be a finite nilpotent group of class 2 and let $g,h \in G$. Assume $o(h) = p^n$, with p a prime number, and $g \notin N_G(\langle h^{p^i} \rangle)$ for all $0 \leq i < n$. Then, for any $1 \leq l,t \leq p^n$,*

$$\langle b_l(g,\widetilde{h}), b_t(\widetilde{h},g^{-1}) \rangle \cong C_{n_l} \star C_{n_t} \cong \langle b_l(g,\widetilde{h}), b_t(g,\widetilde{h})^\star \rangle,$$

a free product of cyclic groups, where $n_l = o(b_l(g,\widetilde{h}))$ and $n_t = o(b_t(\widetilde{h},g^{-1}))$.

Moreover, for an arbitrary finite group G, we also deal with the problem of producing infinite solvable subgroups in $\mathcal{U}(\mathbb{Z}G)$ (that are not virtually nilpotent) using non-trivial Bass and bicyclic units and construct generators of a non-cyclic free submonoid in these groups.

Theorem A.3.3. *Let H be a subgroup of a finite group G. Suppose $h \in H$ with $o(h) = n$ and assume $\alpha \in \mathbb{Z}G$ is such that $s = 1 + (1-h)\alpha\widetilde{H} \neq 1$. Let ζ be a primitive n-th root of unity. Suppose $b_1 = u_{k_1,m_1}(h)$ and $b_2 = u_{k_2,m_2}(h)$ are two non-trivial Bass units (so, $1 < k_1, k_2 < n-1$, $(k_1,n) = (k_2,n) = 1$, $k_1^{m_1} \equiv 1 \mod n$ and $k_2^{m_2} \equiv \mod n$). If, for $i \in \{1,2\}$,*

$$m_i \geq \log_{\left|\frac{\zeta^{k_i-1}}{\zeta-1}\right|} 3 \quad \textit{and} \quad \left(\frac{\zeta^{k_1}-1}{\zeta-1}\right)^{m_1} \neq \left(\frac{\zeta^{k_2}-1}{\zeta-1}\right)^{m_2},$$

then

$$\{b_1 + (1-h)\alpha\widetilde{H}, b_2 + (1-h)\alpha\widetilde{H}\} \qquad \textit{and} \qquad \{b_1 s, b_2 s\}$$

generate free monoids of rank 2 that are contained in a solvable group. In particular, $\langle b(u_{k_1,m_1}(h), g, \tilde{h}), b(u_{k_2,m_2}(h), g, \tilde{h})\rangle$ *is a free monoid of rank 2 if* $g \notin N_G(\langle h\rangle)$.

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