Sylow Permutability in Infinite Soluble Groups

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Permutability

(i) A subgroup H of a group G is *permutable* if $HK = KH$ for all subgroups K of G. In symbols

$$H \text{ per } G.$$

(ii) A subgroup H of a group G is *Sylow-permutable*, or *S-permutable*, if $HP = PH$ for all Sylow subgroups P of G. Write

$$H \ \mathrm{S}{-}\text{per } G.$$

The following result is fundamental.

Kegel's theorem

Theorem 1. (O. Kegel) *Let H be an S-permutable subgroup of a finite group G. Then H is subnormal in G.*

Proof. By induction on the group order, we can assume H is a maximal proper S-permutable subgroup and prove that $H \lhd G$. Assume this is false; then $H \neq H^P$ for some Sylow p-subgroup P. Now H^P is S-permutable in G, so $H^P = G$ by maximality of H. Hence $G = \langle H, P \rangle = HP$, so that $|G : H|$ is a power of p.

Next let $q \neq p$ be a prime and Q a Sylow q-subgroup of G; then $G_0 = HQ = QH$ and $|G_0 : H|$ is a power of q.

Kegel's theorem

But $|G_0 : H|$ divides $|G : H|$, which is a power of p; hence $Q \leq H$ and all p'-elements of G belong to H. Therefore H is subnormal in G. But $H^P = G$, so $H = G$. □

Permutability in infinite groups.

Theorem 2. (S. Stonehewer) *If H per G, then H is ascendant in G. If G is finitely generated, then H is subnormal.*

Open question. If H $\mathrm{S}-$per G where G is a locally finite group, is H serial in G?

Transitivity

Definitions

(i) A group G is a *PT-group* if H per K and K per G imply that H per G.

(ii) A group G is a *PST-group* if $H \ \mathrm{S}-$per K and $K \ \mathrm{S}-$per G imply that $H \ \mathrm{S}-$per G.

Also recall:

(iii) A group G is a T-group if $H \lhd K$ and $K \lhd G$ imply that $H \lhd G$.

Note the following consequences of Kegel's theorem.

Transitivity

Corollary 1. *Let G be a finite group. Then:*

(i) G is a PT-group if and only if every subnormal subgroup of G is permutable in G.

(ii) G is a PST-group if and only if every subnormal subgroup of G is S-permutable in

Hence

$$\mathrm{T} \Rightarrow \mathrm{PT} \Rightarrow \mathrm{PST}.$$

Recently there has been lots of interest in T-groups, PT-groups and PST-groups, especially in the finite soluble case.

Structure theorems for finite soluble groups

Theorem 3. (Gaschütz). *A finite group G is a soluble T-group if and only if it has an abelian Hall subgroup A of odd order such that G/A is a Dedekind group and elements of G induce power automorphisms in A.*

Theorem 4. (Zacher). *A finite group G is a soluble PT-group if and only if it has an abelian Hall subgroup A of odd order such that G/A is a nilpotent modular group and elements of G induce power automorphisms in A.*

Structure theorems for finite soluble groups

Theorem 5. (Agrawal). *A finite group G is a soluble PST-group if and only if it has an abelian Hall subgroup A of odd order such that G/A is nilpotent and elements of G induce power automorphisms in A.*

Notice that in all three theorems $A = [A, G]$, so that

$$A = \gamma_\infty(G)$$

is the limit of the lower central series of G.

Many other characterizations are known of T-groups, PT-groups and PST-groups.

Polycyclic groups

Recall the theorem of Mal'cev that if H is a subgroup of a polycyclic group G, then H is closed in the profinite topology, i.e.,

$$H = \bigcap_{|G/N|<\infty} HN.$$

The finite quotients of a polycyclic group can pass on information about the permutability of subgroups.

Polycyclic groups

Examples.

(i) *If every finite quotient of a polycylic group G is a T-group, then G is a T-group. (Hence G is finite or abelian).*

Proof. Let H be subnormal in G and let N be a normal subgroup of finite index. Then HN/N is subnormal in G/N, so $HN/N \lhd G/N$ and $HN \lhd G$. But $H = \bigcap_N HN$, so $H \lhd G$. Hence G is a T-group. □

Examples

(ii) *If every finite quotient of a polycylic group G is a PT-group, then G is a PT-group. (Hence G is finite or nilpotent modular).*

Proof. Here we need the result of Lennox and Wilson that for any subgroups H and K of a polycyclic group G,

$$HK = \bigcap_{|G/N|<\infty} (HKN).$$

Let H be a subnormal subgroup of G. Let N be a normal subgroup of finite index in G and let K be any subgroup. Since G/N is a PT-group, $HKN = KHN$. Hence $\bigcap_N (HKN) = \bigcap_N (KHN)$ and $HK = KH$. □

Question

Is there an analogue of (i) and (ii) for PST-groups? What can be said of a polycyclic group G if every finite quotient is a PST-group?

Comments

1. The answer cannot be just that G is a PST-group, for this says nothing if G is torsion-free.

2. Let G be an infinite dihedral group. Then every finite quotient of G, but not G itself, is a PST-group.

The key to the solution is a generalization of the infinite dihedral group.

Groups of infinite dihedral type

First recall that the *dihedral group on an abelian group A* is

$$\mathrm{Dih}(A) = \langle t \rangle \ltimes A,$$

where $a^t = a^{-1}$ for $a \in A$ and $t^2 = 1$.

Definition.
A group G is *of infinite dihedral type* if and only if the hypercentre $\bar{Z}(G)$ is a finite 2-group and $G/\bar{Z}(G) \simeq \mathrm{Dih}(A)$ where A is a finitely generated, infinite abelian group without involutions.

Groups of infinite dihedral type

There is an alternative definition which is often useful.

Lemma 1. *A group G is of infinite dihedral type if and only if it has a normal subgroup A such that:*

(i) *A is a finitely generated, infinite abelian group containing no involutions;*

(ii) *G/A is a finite 2-group and $|G : C_G(A)| = 2$;*

(iii) *elements in $G \setminus C_G(A)$ induce inversion in A.*

The key property of these groups appears next.

Groups of infinite dihedral type

Lemma 2. *Let G be a group of infinite dihedral type. Then G is not a PST-group, but all its finite quotients are PST-groups.*

Sketch of proof. By Lemma 1 there a finitely generated, infinite abelian normal subgroup A, without involutions, such that G/A is a finite 2-group, $|G : C_G(A)| = 2$ and $t \in G \backslash C_G(A)$ induces inversion in A. We show that G is not a PST-group. Write

$$A = F \times D$$

where $F \neq 1$ is free abelian and D has finite odd order.

Groups of infinite dihedral type

Choose $a \in F \backslash F^2$; then $(ta)^2 = t^2$, so t and ta are 2-elements of equal order. Let P be a Sylow 2-subgroup containing ta and put $|P| = 2^k$. Set

$$H = \langle t, A^{2^{k+1}} \rangle,$$

noting that H is subnormal in G.

Now suppose that G is is a PST-group. Since H is subnormal in G, transitivity shows that $H \ \mathrm{S}{-}\mathrm{per}\ G$ and $HP = PH = J$, say. Notice that $a \in J$ since $t \in H$ and $ta \in P$. Now

$$|J : H| = |P : H \cap P| \leq |P| = 2^k.$$

On the other hand,

$$|J:H| = |\langle t\rangle\langle a\rangle A^{2^{k+1}}P : \langle t\rangle A^{2^{k+1}}| \geq |\langle t\rangle\langle a\rangle A^{2^{k+1}} : \langle t\rangle A^{2^{k+1}}|.$$

Since $\langle a\rangle \cap (\langle t\rangle A^{2^{k+1}}) = \langle a\rangle \cap A^{2^{k+1}} = \langle a^{2^{k+1}}\rangle$,

$$|J:H| \geq |\langle a\rangle : \langle a\rangle \cap \langle t\rangle A^{2^{k+1}}| = |\langle a\rangle : \langle a^{2^{k+1}}\rangle| = 2^{k+1},$$

a contradiction which shows that G is not a PST-group.

□

The main theorem

Theorem. *Let G be a finitely generated soluble group. Every finite quotient of G is a PST-group if and only if G is one of the following:*

(i) *a finite PST-group;*

(ii) *a nilpotent group;*

(iii) *a group of infinite dihedral type.*

Since a group of infinite dihedral type contains involutions, there follows:

The main theorem

Corollary 2. *A finitely generated, torsion-free soluble group whose finite quotients are PST-groups is nilpotent.*

By Lemma 2 and the main theorem:

Corollary 3. *If G is a finitely generated soluble group such that G and all its finite quotients are PST-groups, then G is finite or nilpotent.*

Sketch of proof

The main task is to establish the theorem for polycyclic groups. Let G be a polycyclic group all of whose finite quotients are PST-groups. Argue by induction on the Hirsch number $h(G) > 0$.

Every finite quotient of G is a finite soluble PST-group and hence is supersoluble by Agrawal's theorem. According to a theorem of Baer, a polycyclic group whose finite quotients are supersoluble is itself supersoluble. Hence G is supersoluble and it has an infinite cyclic normal subgroup $N = \langle z \rangle$.

Sketch of proof

By induction hypothesis G/N is finite or nilpotent or a group of infinite dihedral type. The three cases must be discussed separately.

Consider the case where G/N *is infinite nilpotent*. We can assume that $N \not\leq Z(G)$, since otherwise G is nilpotent. Set $C = C_G(N)$; then $|G : C| = 2$, and $G = \langle t, C \rangle$ where t inverts in N. Note that C is nilpotent.

Let p be an odd prime and define $\bar{G} = G/C^p$ and $\bar{C} = C/C^p$; then $\bar{G}$ is finite and hence is a soluble PST-group.

Proof of the theorem – sketch

Suppose that $\gamma_\infty(\bar{G}) \neq 1$ for some $p > 2$. Now $1 < \gamma_\infty(\bar{G}) \leq \bar{C}$ and $\bar{C}$ is a p-group, so p divides $|\gamma_\infty(\bar{G})|$. Hence p does not divide $|\bar{G} : \gamma_\infty(\bar{G})|$ by Agrawal's theorem. Since G/N is nilpotent,

$$1 < \gamma_\infty(\bar{G}) \leq \bar{N} \leq \bar{C}$$

and $\bar{C}$ is a p-group, so it follows that $\bar{N} = \bar{C}$ and $C = C^p N$. But C/N is finitely generated and nilpotent, so it is finite, and therefore G/N is finite, a contradiction.

Proof of the theorem – sketch

It follows that $\gamma_\infty(\bar{G}) = 1$, i.e., $\bar{G}$ is nilpotent for all $p > 2$. Since t inverts in N and p is odd, it follows that $\bar{N} = 1$. i.e. $N \leq C^p$. Thus

$$N \leq \bigcap_{p>2} C^p = D.$$

Since C is finitely generated and nilpotent, D is finite and hence $N = 1$, a final contradiction.