Products of subsets in torsion-free groups

Mercede MAJ

UNIVERSITÀ DEGLI STUDI DI SALERNO

NAPLES 2015 CONFERENCE ON GROUP THEORY AND ITS APPLICATIONS

Società Nazionale di Scienze, Lettere e Arti in Napoli

October, 7th-8th, 2015

Dedication

In honor to

Francesco de Giovanni

on his 60th birthday

Francesco de Giovanni

has

- More than 180 papers on the best international journals
- 4 books
- (at least) 22 students
- He is editor of the Proceedings of 4 Conferences

Mathematics Genealogy Project

Francesco de Giovanni

Advisor 1: Mario Curzio
Students:

Name	School	Year
Maria Rosaria Celentani	Università degli Studi di Napoli Federico II	1992
Giovanni Cutolo	Università degli Studi di Napoli Federico II	1993
Ulderico Dardano	Università degli Studi di Napoli Federico II	1993
Silvana Rinauro	Università degli Studi di Napoli Federico II	1993
Alessio Russo	Università degli Studi di Napoli Federico II	1995
Giovanni Vincenzi	Università degli Studi di Napoli Federico II	1995
Maria Cristina Cirino Groccia	Università degli Studi di Napoli Federico II	1997
Tommaso Landolfi	Università degli Studi di Napoli Federico II	1999
Massimo Manfredino	Università degli Studi di Napoli Federico II	1999
Carmela Musella	Università degli Studi di Napoli Federico II	

Maria De Falco	Università degli Studi di Napoli Federico II	2001
Emanuela Massarotti	Università degli Studi di Napoli Federico II	2002
Isabella Buono	Università degli Studi di Napoli Federico II	2003
Paolo Santaniello	Università degli Studi di Napoli Federico II	2004
Fausto De Mari	Università degli Studi di Napoli Federico II	2006
Rocco De Luca	Università degli Studi di Napoli Federico II	2007
Susanna Di Termini	Università degli Studi di Napoli Federico II	2007
Francesco Russo	Università degli Studi di Napoli Federico II	2009
Natascia Tortora	Università degli Studi di Napoli Federico II	2010
Antonio Auletta	Università degli Studi di Napoli Federico II	2013
Maria Martusciello	Università degli Studi di Napoli Federico II	2014
Caterina Rainone	Università degli Studi di Napoli Federico II	2014

According to our current on-line database, Francesco de Giovanni has 22 students and 23 descendants.
ID.


Happy
Birthday

Basic definition

Definitions

If S, T are subsets of a group G, then we denote

$$ST := \{xy \mid x \in S, y \in T\}$$

$$S^2 = SS := \{xy \mid x \in S, y \in S\}$$

ST is also called the product set of S and T

If G is an additive group, then we denote

$$S + T = \{x + y \mid x \in S, y \in T\}$$

$$2S = S + S := \{x + y \mid x \in S, y \in S\}$$

$S + T$ is also called the (Minkowski) sumset of S and T, $2S$ the double of S

Basic Problem in Additive Combinatorics :

Problem

To estimate the cardinality of ST in terms of $|S|, |T|$, if S, T are finite non-empty subsets of G

Remark

$$max\{|S|, |T|\} \leq |ST| \leq |S||T|$$

Background

Basic Problem in Additive Combinatorics :

Problem

To estimate the cardinality of ST in terms of $|S|, |T|$, if S, T are finite non-empty subsets of G

Remark

$$max\{|S|, |T|\} \leq |ST| \leq |S||T|$$

Background

Remark

The bounds are sharp

Example

If $S = T$ is a subgroup of G, then $ST = S$

More generally, if $S = xH$, where H is a subgroup of G and $xH = Hx$, then $S^2 = xHxH = x^2H$ and $|S^2| = |S|$

Background: an inverse result

Proposition

Let S be a non-empty finite subset of a group G.

$|S^2| = |S|$ *if and only if* $S = xH$, *where* $H \leq G$ *and* $xH = Hx$

Proof. Assume $|S^2| = |S|$. Write $H = S^{-1}S$.
Obviously $1 \in H$ and $a^{-1} \in H$, for any $a \in H$.
Let $a, b \in S$. Then $a = s^{-1}t, b = x^{-1}y$, $s, t, x, y \in S$.
But $tS = S^2 = Sx = xS$, so $t^{-1}S = Sx^{-1} = x^{-1}S$,
hence $x^{-1}y = t^{-1}v$ for some $v \in S$ and
$ab = s^{-1}tx^{-1}y = s^{-1}tt^{-1}v = s^{-1}v \in H$.
Therefore H is a subgroup of G.
Moreover, $H = x^{-1}S = Sx^{-1}$, for any $x \in S$,
thus $S = xH = Hx$.
//

Background: an inverse result

More generally:

Proposition

Let S, T be non-empty finite subsets of an abelian group G.
$|ST| = |S|$ if and only if S is the union of cosets of some finite subgroup H of G and T is contained in a coset of H

Problem

What lower bound one can get on $|ST|$ if we assume G torsion-free?

Background - direct results

Proposition

If S, T are non-empty finite subsets of the groups of the integers, then we have

$$|S+T| \geq |S|+|T|-1$$

Proof. We use the fact that the integers are totally ordered. Write $S=\{x_1, x_2, \cdots, x_k\}$, $T=\{y_1, y_2, \cdots, y_h\}$ and assume $x_1 < x_2 < \cdots < x_k$, $y_1 < y_2 < \cdots < y_h$. Clearly

$$x_1+y_1 < \ldots < x_1+y_h < x_2+y_h < \ldots < x_k+y_h$$

Hence $|S+T| \geq h+k-1$, as required. //

Background - direct results

More generally:

Theorem (J.H.B. Kemperman, Indag. Mat., 1956)

If S, T are non-empty finite subsets of a torsion-free group, then we have

$$|ST| \geq |S| + |T| - 1$$

Problem

Is this bound sharp?

An example

Definition

If $a, r \neq 1$ are elements of a group G, a geometric left (rigth) progression with ratio r and length n is the subset of G

$$\{a, ar, ar^2, \cdots, ar^{n-1}\}(\{a, ra, r^2a, \cdots, r^{n-1}a\})$$

If G is an additive group

$$\{a, a+r, a+2r, \cdots, a+(n-1)r\}$$

is called an arithmetic progression with difference r and length n

Example

If $S = \{a, ar, ar^2, \cdots, ar^{n-1}\}$ is a geometric progression in a torsion-free group and $ar = ra$, then $S^2 = \{a^2, a^2r, ar^2, \cdots, a^2r^{2n-2}\}$ has order

$$2|S| - 1$$

Background - inverse results

Proposition

If S, T are finite subsets of the groups of the integers, $|S|, |T| > 1$,

$$|S + T| = |S| + |T| - 1$$

if and only if

S and T are arithmetic progressions with the same difference

More generally:

Theorem (L.V. Brailovsky, G.A. Freiman, J. Algebra, 1990)

If S, T are non-empty finite subsets of a torsion-free group, $|S|, |T| > 1$,

$$|ST| = |S| + |T| - 1$$

if and only if

$S = \{a, aq, \cdots, aq^{k-1}\}, \ T = \{b, qb, \cdots, q^{n-1}b\}$, $a, b, q \in G, q \neq 1$

Direct results

In particular, if $|ST| = |S| + |T| - 1$, then S is contained in a left coset of a cyclic subgroup of G. In fact, we have:

Theorem (Y.O. Hamidoune, A.S. Lladó, O. Serra, Combinatorica, 1998)

If S, T are non-empty finite subsets of a torsion-free group, S not contained in the left coset of a cyclic subgroup, and $|T| \geq 4$, then

$$|ST| \geq |S| + |T| + 1$$

Theorem (K.J. Böröczky, P.P. Pálfy, O. Serra, Bull. London Math. Soc., 2012)

If S, T are non-empty finite subsets of a torsion-free group, S not contained in the left coset of a cyclic subgroup, and $|T| \geq c(k)$ for $k \geq 1$ and $c(k) = 32(k+3)^6$, then

$$|ST| \geq |S| + |T| + k$$

Doubling problems

Let G be a group and S a finite subset of G.
Let α, β real numbers

Problem

What is the structure of S if $|S^2|$ satisfies

$$|S^2| \leq \alpha|S| + \beta?$$

Problems of this kind are called **inverse problems of doubling** type in additive number theory. The coefficient α, or more precisely the ratio $\frac{|S^2|}{|S|}$ is called the **doubling coefficient of** S

Doubling problems

Inverse problems of doubling type have been first investigated by **G.A. Freiman**

During the last two decades, they became a central issue in Additive Combinatorics
E. Breuillard, Y. O. Hamidoune, B. Green, M. Kneser, A.S. Llad, A. Plagne, P.P. Pálfy. Z. Ruzsa, O. Serra, Y.V. Stanchescu, T. Tao...

There are two main types of questions one may ask

Doubling property

Problem

What is the general type of structure that S can have if

$$|S^2| \leq \alpha|S| + \beta?$$

How behaves this type of structure when α increases?

Studied recently by many authors

E.Breuillard, **B.Green**, **I.Ruzsa**, **T.Tao**...

Very powerful general results have been obtained (leading to a qualitatively complete structure theorem thanks to the concepts of nilprogressions and approximate groups)

Small doubling problems

But these results are not very precise quantitatively

Problem

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets S which satisfy

$$|S^2| \leq \alpha|S| + \beta,$$

with α and $|\beta|$ small

Problems of this kind are called **inverse problems of small doubling** type

Small doubling problems

We know that:

Remark

If G is torsion-free, S a finite subset of G, then

$$|S^2| \geq 2|S| - 1$$

Theorem (Freiman, Schein, Proc. Amer. Math. Soc. 1991)

If S is a finite subset of a torsion-free group, $|S| = k \geq 2$,

$$|S^2| = 2|S| - 1$$

if and only if

$$S = \{a, aq, \cdots, aq^{k-1}\}, \textit{and either } aq = qa \textit{ or } aqa^{-1} = q^{-1}$$

Small doubling problems

Theorem (Y.O. Hamidoune, A.S. Lladó, O. Serra, Combinatorica, 1998)

If S is a finite subset of a torsion-free group, $|S| = k \geq 4$, then

$$|S^2| = 2|S|$$

if and only if *there exist $a, q \in G, aq = qa$ such that*

$$S = \{a, aq, \cdots, aq^k\} \setminus \{c\}, \text{ with } c \in \{a, aq\}$$

Small doubling problems

Theorem (G. Freiman)

Let S be a finite set of integers with $k \geq 3$ elements and suppose that

$$|2S| \leq 3k - 4.$$

Then S is contained in an arithmetic progression of size $2k - 3$:

$$\{a, a+q, a+2q, \cdots, a+(2k-4)q\}$$

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$

Small doubling problems

Freiman studied also the case $|2S| = 3|S| - 3$ and $|2S| = 3|S| - 2$, S a subset of the integers. He proved that, with the exception of some cases with $|S|$ small, then either S is contained in an arithmetic progression or it is the union of two arithmetic progressions with same difference

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 11$, and

$$|S^2| \leq 3|S| - 2,$$

then S is contained in a geometric progression of length at most $2|S| + 1$ or it is the union of two geometric progressions with same ratio

Small doubling problems

Small doubling problems have been studied in abelian groups by many authors

Y. O. Hamidoune, B. Green, M. Kneser, A.S. Llad, A. Plagne, P.P. Pálfy. Z. Ruzsa, O. Serra, Y.V. Stanchescu...

In a series of papers with

Gregory Freiman

Marcel Herzog

Patrizia Longobardi

Yonutz Stanchescu

Alan Plagne

Derek Robinson

we study Freiman's Conjectures and more generally small doubling problems with doubling coefficient 3, in the class of orderable groups

Small doubling problems

Small doubling problems have been studied in abelian groups by many authors
Y. O. Hamidoune, B. Green, M. Kneser, A.S. Llad, A. Plagne, P.P. Pálfy. Z. Ruzsa, O. Serra, Y.V. Stanchescu...

In a series of papers with
Gregory Freiman
Marcel Herzog
Patrizia Longobardi
Yonutz Stanchescu
Alan Plagne
Derek Robinson
we study Freiman's Conjectures and more generally small doubling problems with doubling coefficient 3, in the class of orderable groups

Orderable groups

Definition

Let G be a group and suppose that a total order relation $\leq$ is defined on the set G.
We say that $(G, \leq)$ is an *ordered group* if for all $a, b, x, y \in G$,

$$\textit{the inequality } a \leq b \textit{ implies that } xay \leq xby.$$

Definition

A group G is *orderable* if there exists a total order relation $\leq$ on the set G, such that $(G, \leq)$ is an ordered group.

Orderable groups

Remark

Any ordered group is torsion-free.

Theorem (F.W. Levi)

An **abelian group** G *is orderable if and only if it is torsion-free.*

Theorem (K. Iwasawa - A.I. Mal'cev - B.H. Neumann)

The class of orderable groups contains the class of **torsion-free nilpotent groups**.

Orderable groups

Free groups are orderable. Pure braid groups are orderable.

The group

$$\langle x, c \mid c^x = c^{-1} \rangle$$

is not an orderable group.

More information concerning ordered groups may be found, for example, in

R. Botto Mura and A. Rhemtulla, *Orderable groups*,
Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, Inc., New York and Basel, 1977.

A.M.W. Glass, *Partially ordered groups*,
World Scientific Publishing Co., Series in Algebra, v. 7, 1999.

Small doubling in orderable groups

Remark

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 2$. Then

$$t = |S^2| \geq 2|S| - 1.$$

If

$$t = |S^2| \leq 2|S| - 1,$$

then $\langle S \rangle$ is abelian. Moreover there exist $a, q \in G$, $g > 1$, such that

$$S = \{a, aq, aq^2, \cdots, aq^{k-1}\}.$$

Small doubling in orderable groups

Theorem (Freiman, Herzog, Longobardi, - , J. Austral. Math. Soc., 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G of size $k \geq 3$.

Assume that

$$t = |S^2| \leq 3|S| - 4.$$

Then $\langle S \rangle$ is abelian. Moreover, there exists $a, q \in G$, such that $qa = aq$ and S is a subset of

$$\{a, aq, aq^2, \cdots, aq^{t-k}\}.$$

Small doubling in orderable groups

Theorem (Freiman, Herzog, Longobardi, - , J. Austral. Math. Soc., 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G, $|S| \geq 3$. Assume that

$$|S^2| \leq 3|S| - 3.$$

Then $< S >$ is abelian

Theorem

Let G be an ordered group and let S be a finite subset of G, $S \geq 3$ If

$$|S^2| \leq 3|S| - 3,$$

then one of the following holds:

- (1) $|S| = 6$;
- (2) *S is a subset of a geometric progression of length at most $2|S| - 1$;*
- (3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$

Small doubling in orderable groups

Theorem (Freiman, Herzog, Longobardi, - , J. Austral. Math. Soc., 2014)

Let $(G, \leq)$ be an ordered group and let S be a finite subset of G, $|S| \geq 3$. Assume that

$$|S^2| \leq 3|S| - 3.$$

Then $< S >$ is abelian

Theorem

Let G be an ordered group and let S be a finite subset of G, $S \geq 3$
If

$$|S^2| \leq 3|S| - 3,$$

then one of the following holds:

- (1) $|S| = 6$;
- (2) *S is a subset of a geometric progression of length at most* $2|S| - 1$;
- (3) $S = \{ac^t \mid 0 \leq t \leq t_1 - 1\} \cup \{bc^t \mid 0 \leq t \leq t_2 - 1\}$

Remark

There exists an ordered group G with a subset S of order k (for any k) such that $\langle S \rangle$ is not abelian and $|S^2| = 3k - 2$

Example

Let

$$G = \langle a, b \mid a^b = a^2 \rangle,$$

the Baumslag-Solitar group $B(1,2)$ and

$$S = \{b, ba, ba^2, \cdots, ba^{k-1}\}$$

Then

$$S^2 = \{b^2, b^2a, b^2a^2, b^2a^3, \cdots, b^2a^{3k-3}\}$$

Thus $\langle S \rangle$ is non-abelian and $|S^2| = 3k - 2$

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Stanchescu, 2015)

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

- (1) $\langle S\rangle$ *is an abelian group, at most 4-generated;*
- (2) $\langle S\rangle = \langle a, b \,|[a, b] = c, [c, a] = [c, b] = 1\rangle$. *In particular $\langle S\rangle$ is a nilpotent group of class 2;*
- (3) $\langle S\rangle = \langle a, b \mid a^b = a^2\rangle$. *Therefore $\langle S\rangle$ is the Baumslag-Solitar group $B(1, 2)$;*
- (4) $\langle S\rangle = \langle a\rangle \times \langle b, c \mid c^b = c^2\rangle$;
- (5) $\langle S\rangle = \langle a, b \mid a^{b^2} = aa^b, aa^b = a^b a\rangle$.

The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

and $\langle S \rangle$ is nilpotent, then it is nilpotent of class at most 2

The structure of S if $|S^2| = 3|S| - 2$

If $\langle S \rangle$ is abelian and the structure of $|S^2| = 3|S| - 2$ can be obtained using some previous results by Freiman and Stanchescu

Theorem

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is abelian, then one of the following possibilities occurs:

- (1) $|S| \leq 11$;
- (2) *S is a subset of a geometric progression of length at most $2|S| + 1$;*
- (3) *S is contained in the union of two geometric progressions with the same ratio*

The structure of S if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Robinson, Stanchescu, J. Algebra, 2016)

Let G be an ordered group and let S be a subset of G of finite size $k > 2$. If

$$|S^2| = 3k - 2,$$

and $\langle S \rangle$ is non-abelian, then one of the following statements holds:

- (1) $|S| \leq 4$;
- (2) $S = \{x, xc, xc^2, \cdots, xc^{k-1}\}$, *where* $c^x = c^2$ *or* $(c^2)^x = c$;
- (3) $S = \{a, ac, ac^2, \cdots, ac^i, b, bc, bc^2, \cdots, bc^j\}$, *with* $1 + i + 1 + j = k$ *and* $ab = bac$ *or* $ba = abc$, $ac = ca$, $bc = cb$, $c > 1$

Conversely if S has the structure in (2) and (3), then $|S^2| = 3|S| - 2$.

Some methods

Remark

Any orderable group is an R-group.

A group G is an **R-group** if, with $a, b \in G$,

$$a^n = b^n, \; n \neq 0, \text{ implies } a = b.$$

Remark

If G is an orderable group, $a, b \in G$ and if $a^n b = ba^n$ for some positive integer n, then $ab = ba$.

Some methods

Remark

Any orderable group is an $R^\star$-group.

A group G is an $\mathbf{R}^\star-$ **group** if, with $a, b, g_1, \cdots, g_n \in G$,

$$a^{g_1} \cdots a^{g_n} = b^{g_1} \cdots b^{g_n} \text{ implies } a = b.$$

Remark

A metabelian $R^\star$-group is orderable.

The structure of $\langle S \rangle$ if $|S^2| = 3|S| - 2$

Theorem (Freiman, Herzog, Longobardi, - , Plagne, Stanchescu, 2015)

Let G be an ordered group and let S be a finite subset of G. If

$$|S^2| = 3|S| - 2$$

then one of the following holds:

- (1) $\langle S \rangle$ *is an abelian group, at most 4-generated;*
- (2) $\langle S \rangle = \langle a, b \,|[a, b] = c, [c, a] = [c, b] = 1\rangle$. *In particular $\langle S \rangle$ is a nilpotent group of class 2;*
- (3) $\langle S \rangle = \langle a, b \mid a^b = a^2\rangle$. *Therefore $\langle S \rangle$ is the Baumslag-Solitar group $B(1, 2)$;*
- (4) $\langle S \rangle = \langle a \rangle \times \langle b, c \,|c^b = c^2\rangle$;
- (5) $\langle S \rangle = \langle a, b\rangle$, *with* $a^{b^2} = aa^b, aa^b = a^b a$;

The groups $\mathcal{BS}(m,n)$

For integers m and n, $m, n \neq 0$, the Baumslag-Solitar group $\mathcal{BS}(m,n)$ is a group with two generators a, b and one defining relation $b^{-1}a^m b = a^n$:

$$\mathcal{BS}(m,n) := \langle a, b \mid \ a^m b = ba^n \rangle.$$

These groups were introduced by Gilbert Baumslag and Donald Solitar in 1962 in order to provide some simple examples of non-Hopfian groups.

("Some two generator one-relator non-Hopfian groups", *Bull. Amer. Math. Soc.*, **689** (1962), 199-201).

A group is called *Hopfian* (or nowadays *Hopf*) if every epimorphism from the group to itself is an isomorphism.

The name is derived from the topologist **Heinz Hopf** who asked whether a finitely generated non-Hopfian group exists.

The groups $\mathcal{BS}(m, n)$

In 1962, **G. Baumslag** and **D. Solitar** showed that the group

$$\mathcal{BS}(2,3) = \langle a, b \mid \; a^2 b = ba^3 \rangle$$

is non-Hopfian.

More generally:

$$\mathcal{BS}(m,n) = \langle a, b \mid \; a^m b = ba^n \rangle$$

is Hopfian if and only if :

(i) $|m| = |n|$ or

(ii) $|m| = 1$ or

(iii) $|n| = 1$ or

(iv) $\pi(m) = \pi(n)$ where $\pi(m)$ denotes the set of prime divisors of m.

Dilates

Subsets of $\mathbb{Z}$ of the form

$$r * A := \{rx : x \in A\},$$

where r is a **positive** integer and A is a **finite** subset of $\mathbb{Z}$, are called *r-dilates*.

Minkosky sums of dilates have been recently studied in different situations by **Bukh, Cilleruelo, Hamidoune, Plagne, Rué, Silva, Vinuesa** and others.

In particular, they examined sums of two dilates of the form

$$A + r * A = \{a + rb \mid \ a, b \in A\}$$

and solved various **direct** and **inverse** problems concerning their sizes.

Dilates

For example:

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa, J. Comb. Number Theory, 2010)

Let S be a finite set of integers,

$$|S + 2 * S| \geq 3|S| - 2$$

and

$$|S + 2 * S| = 3|S| - 2$$

if and only if

S is an arithmetic progression.

Dilates

Theorem (Y.O. Hamidoune, A. Plagne, 2002)

Let S be a finite set of integers. Then

$$|S + r * S| \geq 3|S| - 2$$

for any integer $r \geq 2$.

Theorem (M.B. Nathanson, 2008)

Let A be a finite set of integers, $|S| \geq 4$. Then

$$|S + r * S| \geq \frac{7}{2}|S| - 2$$

for any integer $r \geq 3$.

Dilates

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa, 2010)

For any finite set S of integers we have

$$|S + 3 * S| \geq 4|S| - 4.$$

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, Europ. J. Comb., 2014)

If $r \geq 3$, then $|S + r * S| \geq 4|S| - 4$.

Dilates - Inverse results

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

If $|S + 2 * S| = 3|S| - 2$, *then* S *must be an arithmetic progression.*

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European. J. Comb., 2014)

$$\textit{If} \quad |S + 2 * S| < 4|S| - 4 \;, |S| \geq 3,$$

then S *is a* **subset** *of an arithmetic progression of size* $\leq 2|S| - 3$.

Theorem (A. Balog, G. Shakan)

For any relatively prime integers $1 \leq p < q$ *and for any finite set* A *of integers, one has*

$$|p * A + q * A| \geq (p + q)|A| - (pq)^{(p+q-3)(p+q)+1}.$$

The groups $\mathcal{BS}(1,n) = \langle a, b \mid \; ab = ba^n \rangle$

Let S be a finite subset of $\mathcal{BS}(1,n)$ of size k contained in the coset $b^r < a >$ for some $\mathbf{r} \geq \mathbf{0}$.

Then

$$S = \{b^r a^{x_0}, b^r a^{x_1}, \ldots, b^r a^{x_{k-1}}\},$$

where $A := \{x_0, x_1, \ldots, x_{k-1}\}$ is a subset of $\mathbb{Z}$. We introduce now the notation

$$S = \{b^r a^x : x \in A\} =: b^r a^A.$$

Thus $|S| = |A|$.

The groups $\mathcal{BS}(1,n) = \langle a, b \mid ab = ba^n \rangle$

Theorem

Suppose that $S = b^r a^A \subseteq \mathcal{BS}(1,n)$, $T = b^s a^B \subseteq \mathcal{BS}(1,n)$, *where* $r, s \in \mathbb{Z}$, $r, s \geq 0$ *and* A, B *are finite subsets of* $\mathbb{Z}$. *Then*

$$ST = b^{r+s} a^{n^s * A + B}$$

and

$$|ST| = |n^s * A + B|.$$

In particular,

$$S^2 = b^{2r} a^{n^r * A + A}$$

and

$$|S^2| = |n^r * A + A| = |A + n^r * A|.$$

The group $\mathcal{BS}(1,2) = \langle a, b \mid \ ab = ba^2 \rangle$

Theorem (J. Cilleruelo, M. Silva, C. Vinuesa)

If A is a finite set of integers, then $|A + 2 * A| \geq 3|A| - 2$ and $|A + 2 * A| = 3|A| - 2$ if and only if A is an arithmetic progression.

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, Europ. J. Combin., 2014)

If $S = ba^A \subseteq \mathcal{BS}(1,2)$, where A is a finite subset of $\mathbb{Z}$, then

$$|S^2| \geq 3|S| - 2$$

and if $|S^2| = 3|S| - 2$, then A is an arithmetic progression and S is a geometric progression.

The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, Europ. J. Comb., 2014)

If A is a finite set of integers, $|A| \geq 3$ and $|A + 2 * A| < 4|A| - 4$, then A is a **subset** *of an arithmetic progression of size $\leq 2|A| - 3$.*

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, Europ. J. Comb., 2014)

If $S = ba^A \subseteq \mathcal{BS}(1,2)$, $|S| \geq 3$ and $|S^2| < 4|S| - 4$, then A is a **subset** *of an arithmetic progression of size $\leq 2|S| - 3$.*

The group $\mathcal{BS}(1,2) = \langle a, b \mid ab = ba^2 \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European J. Comb., 2014)

If A is a finite set of integers, $r \geq 3$, then $|A + r * A| \geq 4|A| - 4$.

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European J. Comb., 2014)

Let $S = b^m a^A \subseteq \mathcal{BS}(1,2)$, where A is a finite set of integers of size $k \geq 2$ and $m \geq 2$. Then

$$|S^2| \geq 4k - 4.$$

The group $\mathcal{BS}(1, n) = \langle a, b \mid ab = ba^n \rangle$

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European J. Comb., 2014)

If A is a finite set of integers, $r \geq 3$, then $|A + r * A| \geq 4|A| - 4$.

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European J. Comb., 2014)

Let $S \subseteq \mathcal{BS}(1, n)$ be a finite set of size $k = |S| \geq 2$ and suppose that $n \geq 3$ and

$$S = ba^A,$$

where $A \subseteq \mathbb{Z}$ is a finite set of integers.

Then

$$|S^2| = |A + n * A| \geq 4k - 4.$$

The group $\mathcal{BS}(1,2) = \langle a, b \mid \ ab = ba^2 \rangle$

Problem

What is the structure of an arbitrary subset of $\mathcal{BS}(1,2)$, satisfying some small doubling condition?

Definition

Consider the submonoid

$$\mathcal{BS}^+(1,2) := \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$$

of $\mathcal{BS}(1,2)$.

$\mathcal{BS}^+(1,2) = \{b^m a^x \in \mathcal{BS}(1,2) \mid x, m \in \mathbb{Z}, m \geq 0\}$

Theorem (G.A. Freiman, M. Herzog, P. L., - , Y.V. Stanchescu, European J. Comb., 2014)

Let S be a finite non-abelian subset of $\mathcal{BS}^+(1,2)$ and suppose that

$$|S^2| < \frac{7}{2}|S| - 4.$$

Then

$$S = ba^A,$$

where A is a set of integers of size $|S|$, which is contained in an arithmetic progression of size less than $\frac{3}{2}|S| - 2$.

The structure of $\langle S\rangle$ if $|S^2| = 3|S| - 2$

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

then $\langle S\rangle$ is metabelian

Corollary

Let G be an ordered group and let S be a finite subset of G, $|S| \geq 4$. If

$$|S^2| \leq 3|S| - 2$$

and $\langle S\rangle$ is nilpotent, then it is nilpotent of class at most 2

Problem

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that

$$|S^2| = 3|S| - 1$$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

In fact we have:

Theorem (Freiman, Herzog, Longobardi,- , Plagne, Stanchescu, 2015)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

Problem

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that

$$|S^2| = 3|S| - 1$$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

In fact we have:

Theorem (Freiman, Herzog, Longobardi,- , Plagne, Stanchescu, 2015)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

Problem

Is there an orderable group with a finite subset S of order k (for any $k \geq 4$) such that

$$|S^2| = 3|S| - 1$$

and $\langle S \rangle$ is non-metabelian (non-soluble)?

NO

In fact we have:

Theorem (Freiman, Herzog, Longobardi,- , Plagne, Stanchescu, 2015)

Let G be an ordered group, $\beta \geq -2$ any integer and let k be an integer such that $k \geq 2^{\beta+4}$. If S is a subset of G of finite size k and if

$$|S^2| \leq 3k + \beta,$$

then $\langle S \rangle$ is metabelian, and it is nilpotent of class at most 2 if G is nilpotent.

An example

Example

For any $k \geq 3$, there exists an ordered group, with a subset S of finite size k, such that $\langle S \rangle$ is not soluble and

$$|S^2| = 4k - 5.$$

Let

$$G = \langle a \rangle \times \langle b, c \rangle,$$

where $\langle a \rangle$ is infinite cyclic and $\langle b, c \rangle$ is free of rank 2.
For any $k \geq 3$, define

$$S = \{a, ac, \cdots, ac^{k-2}, b\}.$$

Then

$$|S^2| = 4k - 5.$$

Problems

Conjecture (G. Freiman)

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 4,$$

then S is contained in a geometric progression of length at most $2|S| - 3$

Theorem (K.J. Böröczky, P.P. Pálfy, O. Serra, Bull. London Math. Soc., 2012)

The conjecture of Freiman holds if

$$|S^2| \leq 2|S| + \frac{1}{2}|S|^{\frac{1}{6}} - 3$$

Conjecture

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian and, if it is nilpotent, it is nilpotent of class 2

It follows from results of **E. Breuillard, B. Green and T. Tao** that

Remark

If $|S^2| \leq 3|S| - 4$, then there exists a nilpotent subgroup H of nilpotency class $c(3)$ and generated by at most $d(3)$ elements such that $S \subseteq ZH$, for some subset Z of the group such that $|Z| \leq s(3)$.

Conjecture

If G is any torsion-free group, S a finite subset of G, $|S| \geq 4$, and

$$|S^2| \leq 3|S| - 2,$$

then $\langle S \rangle$ is metabelian and, if it is nilpotent, it is nilpotent of class 2

It follows from results of **E. Breuillard, B. Green and T. Tao** that

Remark

If $|S^2| \leq 3|S| - 4$, then there exists a nilpotent subgroup H of nilpotency class $c(3)$ and generated by at most $d(3)$ elements such that $S \subseteq ZH$, for some subset Z of the group such that $|Z| \leq s(3)$.

Thank you for the attention !

M. Maj
Dipartimento di Matematica
Università di Salerno
via Giovanni Paolo II, 132, 84084 Fisciano (Salerno), Italy
E-mail address : mmaj@unisa.it

Bibliography

- A. Balog, G. Shakan, On the sum of dilatations of a set, *arXiv*:1311.0422, preprint, 1-10.
- O. Bogopolski, Abstract commensurators of solvable Baumslag-Solitar groups, *Communications of Algebra*, **40(7)** (2012), 2494-2502.
- E. Breuillard, B. Green, T. Tao, The structure of approximate groups, *arXiv*:1110.5008 (2011), preprint, 1–91.
- B. Bukh, Sums of dilates, *Combin. Probab. Comput.* **17** (2008), no. 5, 627–639.
- J. Cilleruelo, Y. O. Hamidoune, O. Serra, On sums of dilates, *Combin. Probab. Comput.* **18** (2009), no. 6, 871–880.

Bibliography

- J. Cilleruelo, M. Silva, C. Vinuesa, A sumset problem, *J. Comb. Number Theory* **2** (2010), no. 1, 79–89.
- G. A. Freiman, Foundations of a structural theory of set addition, Translations of mathematical monographs, v. 37. American Mathematical Society, Providence, Rhode Island, 1973.
- G. A. Freiman, Structure Theory of Set Addition, *Astérisque*, **258** (1999), 1-33.
- G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Small doubling in ordered groups, *J. Austral. Math. Soc.*, doi:10.1017/S1446788714000019.

Bibliography

G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Direct and inverse problems in Additive Number Theory and in non-abelian group theory, *European J. Combin.*, **40** (2014), 42–54.

G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, A small doubling structure theorem in a Baumslag-Solitar group, *European J. Combin.*, **44** (2015), 106-124.

G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Inverse problems in Additive Number Theory and in Non-Abelian Group Theory, *arXiv*:1303.3053 (2013), preprint, 1–31.

Bibliography

- G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Some inverse problems in group theory, Note di Matematica, 34 (2014), 89-104
- G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, D.J.S. Robinson, Y. V. Stanchescu, On the structure of subsets of an orderable group, with some small doubling properties, *J. Algebra*, (2016).
- G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, A. Plagne, Y. V. Stanchescu, Small doubling in ordered groups: generators and structures, submitted
- G. A. Freiman, M. Herzog, P. Longobardi, M. Maj, Y. V. Stanchescu, Small doubling in ordered nilpotent group of class 2, in preparation
- A. Geroldinger, I. Z. Ruzsa, *Combinatorial Number Theory and Additive Group Theory*, Birkäuser Verlag, Basel - Boston - Berlin, 2009

Bibliography

- B. Green, What is ... an approximate group ?, *Notices Amer. Math. Soc.* **59** (2012), no. 5, 655–656.
- Y. O. Hamidoune, A. Plagne, A generalization of Freiman's 3k–3 theorem, *Acta Arith.* **103** (2002), no. 2, 147–156.
- Y. O. Hamidoune, J. Rué, A lower bound for the size of a Minkowski sum of dilates, *Combin. Probab. Comput.* **20** (2011), no. 2, 249–256.
- M. I. Kargapolov, Completely orderable groups, *Algebra i Logica (2)* **1** (1962), 16-21.
- V. F. Lev, P. Y. Smeliansky, On addition of two distinct sets of integers, *Acta Arith.* **70** (1995), no. 1, 85–91.

Bibliography

- M. B. Nathanson, *Additive number theory - Inverse problems and geometry of sumsets*, Springer, New York, 1996.
- M. B. Nathanson, Inverse problems for linear forms over finite sets of integers, *J. Ramanujan Math. Soc.* **23** (2008), no. 2, 151–165.
- A. Plagne, S. Tringali, Sums of dilates in ordered groups, *arXiv*:1403.0934v2 (2014), preprint, 1–14.
- Z. Ruzsa, Generalized arithmetic progressions and sumsets, *Acta Math. Hungar.* **65(4)** (1994), 379-388.
- T. Sanders, The structure theory of set addition revisited, *Bull. Amer. Math. Soc.* **50(1)** (2013), 93-127.

Bibliography

- D. Shan-Shan, C. Hui-Qin, S. Zhi-Wei, On a sumset problem for integers, *arXiv*:1011.5438 (2010).
- Y. V. Stanchescu, On addition of two distinct sets of integers, *Acta Arith.* **75** (1996), no. 2, 191–194.
- Y. V. Stanchescu, On the structure of sets with small doubling property on the plane.I., *Acta Arith.* **83** (1998), no. 2, 127–141.
- Y. V. Stanchescu, The structure of d-dimensional sets with small sumset, *J. Number Theory* **130** (2010), no. 2, 289–303.
- T. C. Tao, Product set estimates for non-commutative groups, *Combinatorica* **28(5)** (2008), 547-594.


Happy Birthday