Autocommutators in infinite groups

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UNIVERSITÀ DEGLI STUDI DI SALERNO

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October 7th - 8th, 2015

Dedication

In honor of **Francesco de Giovanni**

on his 60th birthday

My dear teacher **Mario Curzio**

Bressanone, 11/7/1928 - Napoli, 7/9/2015

Remembering

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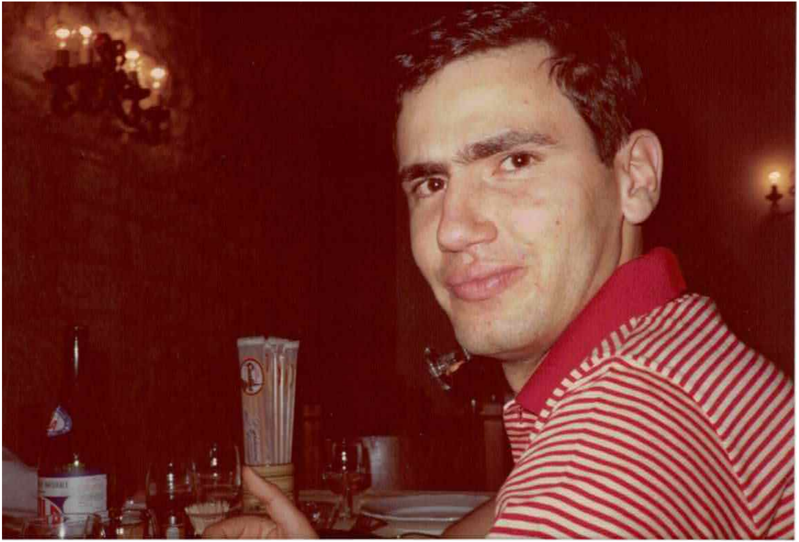

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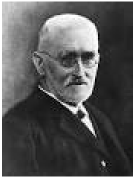
Let G be a group, $x, y \in G$.
The **commutator** of x and y is the element

$$[x, y] := x^{-1}y^{-1}xy = x^{-1}x^{y}.$$

1896



Julius Wilhelm Richard **Dedekind**
1831 - 1916

Ferdinand Georg **Frobenius**
1849 - 1917

Some history

Results proved by **Dedekind** in 1880

The conjugate of a commutator is again a commutator.

Therefore the **commutator subgroup** *generated by the commutators of a group is a normal subgroup of the group.*

Any normal subgroup with abelian quotient contains the commutator subgroup.

The commutator subgroup is trivial if and only if the group is abelian.

First published by **G.A. Miller** in 1896

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G.A. Miller, The regular substitution groups whose order is less than 48, *Quarterly Journal of Mathematics* **28** (1896), 232-284.

Dedekind had studied normal extensions of the rational field with all subfields normal. Some years later these investigations suggested to him the related problem:

Characterize those groups with the property that all subgroups are normal.

R. Dedekind, Über Gruppen, deren sämtliche Teiler Normalteiler sind, *Math. Ann.* **48** (1897), 548-561.

George Abram **Miller**
1863 - 1951

Heinrich Martin **Weber**
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In his 1896 paper G.A. Miller call the section about commutators:

"On the operation $sts^{-1}t^{-1}$"

The label commutator is used in

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(where the author expands the basic properties of the commutator subgroup and introduces the derived series of a group; he also shows that the derived series is finite and ends with 1 if and only if the group is solvable)
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The first textbook to introduce commutators and the commutator subgroup is **Weber**'s 1899 Lehrbuch der Algebra

Heinrich Martin **Weber**
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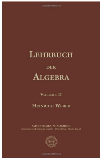

Lehrbuch der Algebra
1895 - 1896

the last important textbook on algebra published in the nineteenth century.

Some history

The first explicit statement of the

Question

Is the set of all commutators a subgroup?
i.e. Does the commutator subgroup consist entirely of commutators?

is found in Weber's 1899 textbook.

He states that

the set of commutators is not necessarily a subgroup.

In Miller's 1899 paper it is proved that the answer to the question is **yes** in the alternating group on n letters, $n \geq 5$, and in the holomorph of a finite cyclic group.

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The first example of a group in which the set of commutators in not equal to the commutator subgroup appears in

W.B. Fite, On metabelian groups, *Trans. Amer. Math. Soc.* **3** no. 3 (1902), 331-353.

Metabelian = Nilpotent of class ≤ 2

Fite constructs an example G of order 1024, attributed to Miller, then provides a homomorphic image H of order 256 of G which is again an example.

H is the subgroup of S_{16}:

$$H = < (1,3)(5,7)(9,11), (1,2)(3,4)(13,15),$$

$$(5,6)(7,8)(13,14)(15,16), (9,10)(11,12) >$$

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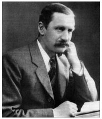
Some history

In

W. Burnside, On the arithmetical theorem connected with roots of unity and its application to group characteristics, *Proc. LMS* **1** (1903), 112-116

Burnside uses characters to obtain a criterion for when an element of the commutator subgroup is the product of two or more commutators.

William Benjamin **Fite**
1869 - 1932



William **Burnside**
1852 - 1927

Some history

The first occurence of the commutator notation probably is in

F.W. Levi, B.L. van der Waerden, Über eine besondere Klasse von Gruppen, *Abh. Math. Seminar der Universität Hamburg* **9** (1933), 154-158,

where the commutator of two group elements i, j is denoted by

$$(i, j) = iji^{-1}j^{-1}.$$

Friedrich Wilhelm **Levi**
1888 - 1966

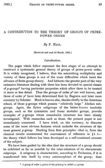
Bartel Leendert **van der Waerden**
1903 - 1996

Some history

Hans Julius **Zassenhaus**
1912 - 1991

Lehrbuch der Gruppentheorie
1937

Philip **Hall**
1904 - 1982



1932.] GROUPS OF PRIME-POWER ORDER. 29

A CONTRIBUTION TO THE THEORY OF GROUPS OF PRIME-POWER ORDER

By P. HALL.

[Received and read 10 March, 1933.]

Introduction.

The pages which follow represent the first stages of an attempt to construct a systematic general theory of groups of prime-power order. It is widely recognised, I believe, that the astonishing multiplicity and variety of these groups is one of the main difficulties which beset the advance of finite-group-theory. And hitherto the greater part of the very extensive literature dealing with them has been devoted to special classes of p-groups* having particular properties which allow them to be treated in more or less detail. Thus the groups of order p^4 are well known, and those of order p^5 have been determined first by Bagnera and later more concisely by Schreier. Much is known also, thanks chiefly to the American school, of those p-groups which possess "relatively large" Abelian subgroups. Again, the Sylow subgroups of the better-known insoluble groups, such as the symmetric and linear modular groups, provide examples of p-groups whose remarkable structure has been closely investigated. With researches such as these, the present paper is not immediately concerned. I have tried, on the contrary, to discover some of the more obvious features which underlie the structure of the most general p-group. Starting from first principles—that is, from the classical results enumerated for convenience of reference in § 1.1—the theory which follows has been developed as a self-contained account of the subject.

We have been guided by the idea that the structure of a group should be exhibited as far as possible by the inter-relations of its characteristic subgroups. A subgroup is characteristic, according to Frobenius, if it is transformed into itself by every automorphism of the group; thus

* A group of order a power of the prime p is called a p-group.

A contribution to the theory of groups of prime power order, 1934

Some history

Hans Julius **Zassenhaus**
1912 - 1991

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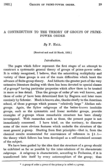
1937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Philip **Hall**
1904 - 1982

A contribution to the theory of groups of prime power order, 1934

Some Francesco' papers

- F. de Giovanni, D.J.S. Robinson, Groups with finitely many derived subgroups, *J. London Math. Soc.* **71** (2005), 658-668.
- F. De Mari, F. de Giovanni, Groups with finitely many derived subgroups of non-normal subgroups, *Arch. Math. (Basel)* **86** (2006), 310-316.
- M. De Falco, F. de Giovanni, C. Musella, Groups whose non-normal subgroups have small commutator subgroup, *Algebra Discrete Math.* **1** (2007), 46-58.
- M. De Falco, F. de Giovanni, C. Musella, Groups with finiteness conditions on commutators, *Algebra Colloq.* **19** (2012), 1197-1204.
- F. de Giovanni, M. Trombetti, Groups with minimax commutator subgroup, *International J. Group Theory* **3** (2014), 9-16.

Background

Let G be a group and put

$$K(G) := \{[g,h] \,|\, g,h \in G\}.$$

Then

$$G' = < K(G) >.$$

Question

Is $G' = K(G)$?

When is $G' = K(G)$?

Which is the minimal order of a counterexample?

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R.M. Guralnick, Expressing group elements as products of commutators, PhD Thesis, UCLA, 1977.

There are exactly two nonisomorphic groups G of order 96 such that $K(G) \neq G'$. In both cases G' is nonabelian of order 32 and $|K(G)| = 29$.

- $G = H \rtimes < y >$, where $H =< a > \times < b > \times < i, j >, a^2 = b^2 = y^3 = 1, < i, j > \simeq Q_8, a^y = b, b^y = ab, i^y = j, j^y = ij$;
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"On commutators in groups"

Groups St. Andrews 2005, Vol. 2, 531-558, London Math. Soc. Lecture Notes Ser., **340** , Cambridge University Press, 2007,

by

L-C. **Kappe**

R.F. **Morse**

Background

Many authors have considered subsets of a group G related to commutators asking if they are subgroups.

For instance, **W.P. Kappe** proved in 1961 that the set $R_2(G) = \{x \in G | [x, g, g] = 1, \forall g \in G\}$ of all right 2-Engel elements of a group G is always a subgroup.

W.P. Kappe, Die $\mathcal{A}$-Norm einer Gruppe, *Illinois J. Math.* **5** no. 2 (1961), 187-197.

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Basic definitions

Definition

Let G be a group, $g \in G$ and $\varphi \in Aut(G)$. The **autocommutator** of g and φ is the element

$$[g, \varphi] := g^{-1} g^{\varphi}.$$

We denote by

$$K^{\star}(G) := \{[g, \varphi] \mid g \in G, \varphi \in Aut(G)\}$$

the set of all autocommutators of G and, following P.V. Hegarty, we write

$$G^{\star} := \langle K^{\star}(G) \rangle.$$

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A new problem

Question

Is $G^\star = K^\star(G)$? Does it hold *if G is abelian?*

At "Groups in Galway 2003" Desmond MacHale brought this problem to the attention of L-C. Kappe. He added that there might be an abelian counterexample and that perhaps the two groups of order 96 given by Guralnick as the minimal counterexamples to the conjecture $G' = K(G)$ might also be counterexamples.

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Results in the finite abelian case

Theorem (D. Garrison, L-C. Kappe and D. Yull, 2006)

Let G be a finite abelian group. Then the set of autocommutators always forms a subgroup.

Furthermore there exists a finite nilpotent group of class 2 and of order 64 in which the set of all autocommutators does not form a subgroup. And this example is of minimal order.

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D. Garrison, L-C. Kappe, D. Yull, Autocommutators and the Autocommutator Subgroup, *Contemporary Mathematics* **421** (2006), 137-146.

Esempio

$$G =< a, b, c, d, e | a^2 = b^2 = c^2 = d^2 = e^4 = 1, [a, b] = [a, c] = [a, d] =$$
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Obviously G has order 64 and $< e^2 >= G' \subseteq Z(G) =< e >$. Hence G has nilpotency class 2. It is possible to show that e^{-1} is not an autocommutator. We have $(cd)(cde) = e^{-1}$ but there exist automorphims ρ and τ of G such that $[c, \rho] = cd$ and $[a, \tau] = cde$.

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The abelian case

Let $(G, +)$ be an abelian group, $g \in G$ and $\varphi \in Aut(G)$. Then the **autocommutator** of g and φ is the element

$$[g, \varphi] := -g + g^{\varphi}.$$

Proposition

Let G be an abelian torsion group without elements of even order. Then

$$K^{\star}(G) = G^{\star} = G.$$

Proof.

The mapping $\tau : g \in G \longmapsto 2g \in G$ is an automorphism of G and $[g, \tau] = -g + g^{\tau} = g$. □

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Theorem (D. Garrison, L-C. Kappe and D. Yull, 2006)

Let G be a finite abelian group. Write

$$G = B \oplus O,$$

where O is of odd order, B is a 2-group. Then we have:

- *If either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = G$.*
- *If $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G) = G_{2^{n-1}} = \{x \in G \mid 2^{n-1}x = 0\}$.*

In any case, $K^\star(G)$ is a subgroup of G.

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Paper

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First remarks

Remark (1)

In any abelian group G the map

$$\varphi_{-1} : x \in G \longmapsto -x \in G$$

is in $Aut(G)$, thus $[-x, \varphi_{-1}] = -(-x) + (-x)^{\varphi_{-1}} = 2x \in K^\star(G)$, for any $x \in G$, hence $2G \subseteq K^\star(G)$.

Remark (2)

If $G = 2G$, i.e. G is 2-divisible, then the map

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The infinite abelian case

Example

Let $G = \langle a \rangle \oplus \langle c \rangle$, where $\langle a \rangle$ is infinite cyclic and $|c| = 2$. Then $K^\star(G)$ is not a subgroup of G.

Proof.

Let $\varphi \in Aut(G)$, then $\varphi(c) = c$, and $\varphi(a) = \gamma a + \delta c$, where $\gamma \in \{1, -1\}$ and $\delta \in \{0, 1\}$.
Therefore we have: $Aut(G) = \{1, \varphi_1, \varphi_2, \varphi_3\}$,
where $1 = id_G$, $\varphi_1(a) = -a$, $\varphi_1(c) = c$,
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We have, for any $g = \alpha a + \beta c \in G$,
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In particular, $2a \in K^\star(G)$, $2a + c \in K^\star(G)$, but $4a + c \neq (-2\gamma)a + \gamma c$, for any integer γ.

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Finitely generated infinite abelian groups

Theorem (L-C. Kappe, P.L., M. Maj)

Let G be a finitely generated infinite abelian group. Write

$$G = \langle a_1 \rangle \oplus \cdots \oplus \langle a_s \rangle \oplus B \oplus O,$$

where $a_1, \cdots, a_s$ are aperiodic, O is a finite group of odd order, B is a finite 2-group. Then we have:
(i) If $s > 1$, then $K^\star(G) = G^\star = G$.
(ii) If $s = 1$ and either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = 2(\langle a_1)\rangle \oplus B \oplus O$ is a subgroup of G.
(iii) If $s = 1$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G)$ is not a subgroup of G.

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(ii) If $s = 1$ and either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = 2(\langle a_1)\rangle \oplus B \oplus O$ is a subgroup of G.

(iii) If $s = 1$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G)$ is not a subgroup of G.

Finitely generated infinite abelian groups

Theorem (L-C. Kappe, P.L., M. Maj)

Let G be a finitely generated infinite abelian group. Write

$$G = \langle a_1 \rangle \oplus \cdots \oplus \langle a_s \rangle \oplus B \oplus O,$$

where $a_1, \cdots, a_s$ are aperiodic, O is a finite group of odd order, B is a finite 2-group. Then we have:
(i) If $s > 1$, then $K^\star(G) = G^\star = G$.
(ii) If $s = 1$ and either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = 2(\langle a_1 \rangle) \oplus B \oplus O$ is a subgroup of G.
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(iii) If $s = 1$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G)$ is not a subgroup of G.
(iv) If $s = 0$ and either $B = 1$ or $B = \langle b_1 \rangle \oplus \langle b_2 \rangle \oplus H$, with $|b_1| = |b_2| = 2^n$, $expH \leq 2^n$, then $K^\star(G) = G^\star = G$. If $s = 0$ and $B = \langle b_1 \rangle \oplus H$, with $|b_1| = 2^n$, $expH \leq 2^{n-1}$, then $K^\star(G) = G_{2^{n-1}} = \{x \in G \mid 2^{n-1}x = 0\}$. In any case, if $s = 0$, $K^\star(G)$ is a subgroup of G.

Periodic abelian groups

Theorem

Let G be a periodic abelian group.
Write $G = O \oplus D \oplus R$, where D is a divisible 2-group, R is a reduced 2-group and every element of O has odd order. Then

$$K^\star(G) = O \oplus D \oplus K^\star(R),$$

where $K^\star(R) = R$ if either R is of infinite exponent or R is of finite exponent 2^n, and $R = \langle a \rangle \oplus \langle b \rangle \oplus H$, with $|a| = |b| = 2^n$, and $K^\star(R) = R_{2^{n-1}}$ otherwise.

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$$K^\star(G) \supseteq O \oplus D \oplus K^\star(R).$$

For, let $a \in O, b \in D, c \in K^\star(R)$, and let $\varphi \in Aut(R)$ such that $c = -t + t^\varphi$, for some $t \in R$. Consider the automorphism τ of G defined by putting $x^\tau = 2x$ for any $x \in O \oplus D$, $r^\tau = r^\varphi$, for any $r \in R$.
Then $[a + b + t, \tau] = -a - b - t + (a + b + t)^\tau = a + b + c$.

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Lemma

Let R be a reduced abelian 2-group of infinite exponent. Then

$$K^\star(R) = R.$$

Proof - Sketch.

Let $g \in R$, and write $|g| = 2^n$.Then there exists $c \in R$ such that $|c| = 2^{n+1}$ and $R =< c > \oplus H$, for some subgroup H of R. It is possible to show that $R =< c + g > \oplus H$. Therefore there exixts an automorphism φ of R such that $c^\varphi = c + g, y^\varphi = y$ for any $y \in H$. Then $[c, \varphi] = -c + c^\varphi = g$, and $g \in K^\star(R)$, as required. □

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Generalizing the previous example it is easy to construct examples of mixed abelian groups G in which $K^\star(G)$ is not a subgroup. In fact, we have:

Example

Let T be a periodic abelian group with $K^\star(T) \subset T$ and consider the group $G = T \oplus \langle a \rangle$, where $\langle a \rangle$ is an infinite cyclic group. Then $K^\star(G)$ is not a subgroup.

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In the group G of the example the torsion subgroup $T(G) = T$ is contained in $K^\star(G)$, but $K^\star(T) \subset T$. Thus it is not true that $T \cap K^\star(G) \subseteq K^\star(T)$. Surprising, the reverse inclusion holds, in fact we have:

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Torsion-free abelian groups

Now consider torsion-free abelian groups.
Torsion-free abelian groups with a finite automorphim group have been studied by de Vries and de Miranda in 1958 and by Hallett and Hirsch in 1965 and 1970.

Theorem (J.T. Hallett, K.A. Hirsch)

If the finite group Γ is the automorphism group of a torsion-free abelian group A, then Γ is isomorphic to a subgroup of a finite direct product of groups of the following types:

(a) *cyclic groups of orders 2, 4, or 6;*

(b) *the quaternion group Q_8 of order 8;*

(c) *the dicyclic group $DC_{12} = < a, b | a^3 = b^2 = (ab)^2 >$ of order 12;*

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Proposition

Let G be a group with cyclic automorphism group. Then $K^\star(G)$ is a subgroup of G.

Proof.

From $G/Z(G) \simeq Inn(G) \leq Aut(G)$, we get that G is abelian. If $|G| \neq 1$ the map $x \in G \longmapsto -x \in G \in Aut(G)$ has order 2, therefore $Aut(G)$ is also finite. Write $Aut(G) = \langle \varphi \rangle$ and put $|Aut(G)| = n$. The map $\theta : x \in G \longmapsto -x + x^\varphi \in G$ is a homomorphism of G. Therefore $Im\theta$ is a subgroup of G. Obviously $Im\theta \subseteq K^\star(G)$. We show that $K^\star(G) = Im\theta$ and then it is a subgroup of G. Let $s \in K^\star(G)$. Then $s = -x + x^{\varphi^i}$, for some $i \in \{1, \cdots, n-1\}$ and some $x \in G$. We have $-x + x^\varphi, -x^\varphi + x^{\varphi^2}, \cdots, -x^{\varphi^{i-1}} + x^{\varphi^i} \in Im\theta$, thus $-x + x^\varphi - x^\varphi + x^{\varphi^2} - \cdots - x^{\varphi^{i-1}} + x^{\varphi^i} = -x + x^{\varphi^i} = s \in Im\theta$. Therefore $K^\star(G) = Im\theta$, as required. □

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There exist torsion-free abelian groups G of any rank with $Aut(G)$ of order 2, for them $K^{\star}(G) = 2G$ is a subgroup of G.

If G is a torsion-free abelian group of rank 1, then $G/2G$ has order at most 2, thus for any $x \in G$ and $\varphi \in Aut(G)$ we have $x^{\varphi} + 2G = x + 2G$, therefore $-x + x^{\varphi} \in 2G$ and $K^{\star}(G) = 2G$ is a subgroup of G.

de Vries and de Miranda and Hallett and Hirsch constructed many examples of abelian groups G, indecomposable or not, of rank ≥ 2, with $Aut(G) \simeq V_4$. In their examples $K^{\star}(G) = 2G$ is a subgroup of G.

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Proposition

There exists a torsion-free abelian group of rank 2 such that $Aut(G) \simeq V_4$ and $K^\star(G)$ is not a subgroup of G.

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Let G be a torsion-free abelian group such that $Aut(G) \simeq Q_8$. If $G/2G$ has rank at most 4, then $K^\star(G)$ is a subgroup of G.

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Thank you for the attention !

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