Growth and Polynomial Identities

Antonio Giambruno

Universitá di Palermo

Notation

F is a field of characteristic zero
A is an F-algebra
$F\langle X\rangle$ is the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over F

$Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ is the ideal of polynomial identities of A

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The sequence of non-negative integers

$$c_n(A) = \dim_F P_n(A), \;\; n \geq 1,$$

is called the sequence of codimensions of A

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More is true:

$$C_1 n^t exp(A)^n \leq c_n(A) \leq C_2 n^t exp(A)^n,$$

for some constants $C_1 > 0, C_2, t$, and $t \in \frac{1}{2}\mathbb{Z}$.

Property: The sequence $c_n(A)$, $n = 1, 2, \ldots$ is eventually non decreasing.

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where

$\alpha(A) =$ max k such that for $n \geq 1$, there exists

$$f(X_1, \ldots, X_n, Y) \notin Id(A),$$

where $|X_i| = k$, $i \geq 1$, and f is alternating on each set X_i.

$\beta(A) = \max l$ such that for $n \geq 1$, there exists

$$f(X_1, \dots, X_n, Y_1, \dots, Y_l, Z) \notin Id(A),$$

where $|X_i| = \alpha(A)$, $|Y_j| = \alpha(A) + 1$ and f alternating on each set X_i and Y_j.

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Remark

If A is a finitely generated PI-algebra, then $exp(A) = \alpha(A)$, where $(ind(A) = (\alpha(A), \beta(A))$.

Wedderburn-Malcev

If A is a finite dimensional algebra over $F = \bar{F}$, then $A = B + J$, where $B =$ max semisimple subalgebra and $J = J(A)$, the Jacobson radical of A. Also $B = B_1 \oplus \cdots \oplus B_t$, with the B_i's simple.

For A as above, let q be such that $J^q \neq 0$ and $J^{q+1} = 0$.

Remark. $ind(A) = (\alpha(A), \beta(A)) \leq (\dim B, q)$, in the lexicographic order.

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Definition

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A finite dimensional algebra is a direct sum of fundamental algebras.

Example.
Upper block triangular matrix algebras are fundamental.

$$A = \begin{pmatrix} B_1 & & & * \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_t \end{pmatrix},$$

$B_i = M_{d_i}(F)$,
$Id(B_1) = P_1, \ldots, Id(B_t) = P_t$, and
$exp(A) = exp(P_1) + \cdots + exp(P_t)$.

$ind(A) = (\sum_{i=1}^{t} d_i^2, \sum_{i=1}^{t} d_i - 1)$.

Example.

Consider the algebra A whose elements are the matrices of the type

$$\begin{pmatrix} a & & & * \\ & C & & \\ & & C & \\ 0 & & & a \end{pmatrix},$$

where $a \in F$, $C \in M_2(F)$.

$ind(A) = (5, 3)$.

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Question. Describe explicitly the fundamental algebras. Can one embed a fundamental algebra into upper block triangular matrices?

Question. Can one extend the Kemer index to arbitrary PI-algebras?

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Theorem (Kemer)

If A is a PI-algebra, $A \sim_{PI} G(B)$, where $B = B_0 \oplus B_1$ is a finite dimensional superalgebra, $G = G_0 \oplus G_1$ is the Grassmann algebra and $G(B) = A_0 \otimes G_0 \oplus A_1 \otimes G_1$.