*dedicated to Francesco de Giovanni on the occasion of his 60th birthday*

# Multiplicative ideal theory: Dedekind's and Kronecker's approaches compared

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*Napoli, Ottobre 2015*

# §1. The genesis

This story starts around 1850. More precisely,

- in 1847, Gabriel Lamé in Paris, submits to the *Académie des Sciences* a short note on a "new approach" to the problem of the solution of Fermat's Diophantine Equation :

$$\textbf{(FLT)} \qquad X^p + Y^p = Z^p \text{ where } p \text{ is a prime integer } p \geq 3\,.$$

At that time G. Lamé was a well-known mathematician: among other results, in 1839 he gave a positive solution to the Fermat's equation in case $p = 7$.

Note that

- ▶ The case of exponent 4 was solved by P. Fermat himself in 1637 (using the technique of *infinite descent*), reconducting his general conjecture to the case of prime exponents;
- ▶ $p = 3$ was solved by Leonhard Euler in 1753;
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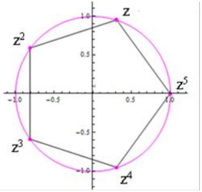
Lamé's "new idea" was based on considering the equation $X^p + Y^p = Z^p$ in a more general setting than that of the integer numbers $\mathbb{Z}$.

This new setting is $\mathbb{Z}[\zeta_p]$ *the ring of cyclotomic integers (of exponent p)*, where $\zeta_p$ is a *primitive p-th root of the unity*, i.e., it is a nontrivial solution of the polynomial equation:

$$X^p - 1 = (X-1)(X^{p-1} + X^{p-2} + \cdots + X + 1) = 0\,,$$

and so, for instance, $\zeta := \zeta_p := e^{2\pi i/p} = \cos(2\pi/p) + i\sin(2\pi/p)$.

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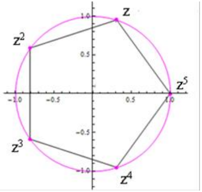


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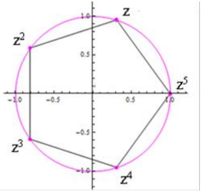


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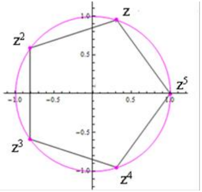


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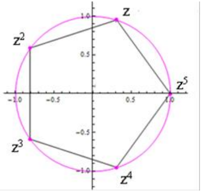


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in $\mathbb{Z}[\zeta_p]$, Fermat's polynomial "factorizes" as follows:

$$X^p + Y^p = (X+Y)(X+\zeta Y)(X+\zeta^2 Y)\cdots(X+\zeta^{p-2}Y)(X+\zeta^{p-1}Y) = Z^p\,.$$

Assuming that $\mathbb{Z}[\zeta_p]$ is an UFD, after proving that the factors of the previous factorization are coprime in $\mathbb{Z}[\zeta_p]$ (when evaluated in a non trivial solution of the **(FLT)**), one deduces by basic properties of the unique factorization that each of such factors is a $p$-th power of some element in $\mathbb{Z}[\zeta_p]$.

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Few months later, Liouville was informed by Dirichlet, a friend of Liouville from the times of his studies in Paris, that Ernst Kummer in Berlin proved in 1843 that $\mathbb{Z}[\zeta_{23}]$ was not a UFD.

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In order to show the previous fact, Kummer "pulled a very special element of $\mathbb{Z}[\zeta_{23}]$ out of a hat".

He considered the element:

$$\mu := 1 - \zeta + \zeta^{21} \in \mathbb{Z}[\zeta], \quad \text{where } \zeta := \zeta_{23}.$$

- $\mu$ is not a prime element of $\mathbb{Z}[\zeta_{23}]$,

since for its norm we have that $\mathtt{N}(\mu) = 47 \cdot 139$ and so $\mu$ is not a power of a prime integer (as each prime element in a ring of algebraic numbers must have).

- $\mu$ is an irreducible element of $\mathbb{Z}[\zeta_{23}]$.

He proved this second statement, by introducing a new important tool that he called *ideal-number factorization*.

First he observed that

$$\langle \mu \rangle = \mu \mathbb{Z}[\zeta_{23}] = P \cdot Q$$

where $P$ and $Q$ are what we call now prime ideals in $\mathbb{Z}[\zeta_{23}]$, with $\mathtt{N}(P) = 47$ and $\mathtt{N}(Q) = 139$. Since no element $\alpha \in \mathbb{Z}[\zeta_{23}]$ is such that $N(\alpha) = 47$, $P$ is not a principal ideal. Similarly for $Q$. Conclusion: $\mu$ is an irreducible element of $\mathbb{Z}[\zeta_{23}]$.

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He introduced the notion of *class group for rings of cyclotomic integers*

$$\mathtt{Cl}(\mathbb{Z}[\zeta_p]) := \frac{\mathcal{F}(\mathbb{Z}[\zeta_p])}{\mathcal{P}(\mathbb{Z}[\zeta_p])},$$

proving that

*this group is always a finite group with order denoted by $h_p$, called the* class number of $\mathbb{Z}[\zeta_p]$.

This number $h_p$ misures the distance of $\mathbb{Z}[\zeta_p]$ from being UFD: more precisely,

*$h_p = 1$ if and only if $\mathbb{Z}[\zeta_p]$ is UFD.*

Kummer in 1851 proved that,

*if $p$ is a regular prime, i.e., if $p \nmid h_p$, the* **(FLT)** *holds for $X^p + Y^p = Z^p$.*

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$$\mathtt{Cl}(\mathbb{Z}[\zeta_p]) := \frac{\boldsymbol{\mathcal{F}}(\mathbb{Z}[\zeta_p])}{\boldsymbol{\mathcal{P}}(\mathbb{Z}[\zeta_p])},$$

proving that

*this group is always a finite group with order denoted by $h_p$, called the* class number of $\mathbb{Z}[\zeta_p]$.

This number $h_p$ misures the distance of $\mathbb{Z}[\zeta_p]$ from being UFD: more precisely,

$h_p = 1$ *if and only if* $\mathbb{Z}[\zeta_p]$ *is UFD.*

Kummer in 1851 proved that,

*if $p$ is a regular prime, i.e., if $p \nmid h_p$, the* **(FLT)** *holds for $X^p + Y^p = Z^p$.*

The importance of this result resides in the fact that

*all the prime integer* $p < 100$, *except three (i.e., 37, 59, and 67) are regular primes*.

In particular, *all the primes* $p < 23$ *have* $h_p = 1$ *(while* $h_{23} = 3$*)*.

Note also that $h_{37} = \underline{37}$, $h_{59} = 3 \cdot \underline{59} \cdot 233$, and $h_{67} = \underline{67} \cdot 12739$.

In 1993-95 **(FLT)** has became Theorem of A. Wiles and R. Taylor and so there is less interest in studying regular primes and their properties.

Note that one of the winner of the first edition (2015) of the (3-Million) Breakthrough Prize in Mathematics (founding sponsors Sergey Brin, Anne Wojcicki, Mark Zuckerberg and Yuri Milner) was R. Taylor (with Simon Donaldson, Maxim Kontsevich, Jacob Lurie and Terence Tao).

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Dedekind completely succeeded in his program in 1871.

What we call now Dedekind theory appeared as XI Supplement to *"Vorlesungen über Zhalentheorie"* ("Lectures in Number Theory") by Dirichlet (this volume appeared several years after Dirichlet's death, 1859).

One of the main result of Dedekind's theory can be stated as follows:

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*Let $D$ be an integral domain (not a field) and $I$ a proper ideal of $D$. The following are equivalent.*

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**(iv)** **(a)** *$D$ is Noetherian.*
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The properties in **(iv)** are called the *Noether's Axioms* and they define what are now called *Dedekind domains*.

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## A ring of integers of a quadratic number field which is a Dedekind domain but not a PID

Let $K := \mathbb{Q}(\sqrt{-5})$. In this case the ring of algebraic integers (i.e., the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{-5})$) is given by $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

- $\mathbb{Z}[\sqrt{-5}]$ is not a UFD (nor a PID)

  - It is not hard to show that the equalities

$$2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

provide two different factorizations of 6 into irreducible elements of $\mathbb{Z}[\sqrt{-5}]$ (since there is no element of $\mathbb{Z}[\sqrt{-5}]$ having norm equal to 2 or 3, after noting that $\mathtt{N}(2) = 4$, $\mathtt{N}(3) = 9$, $\mathtt{N}(1 + \sqrt{-5}) = \mathtt{N}(1 - \sqrt{-5}) = 6$).

Moreover,

- from the previous equalities, it follow also that the irreducible elements 2, 3, $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$ are *not primes*.

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- $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain (by Noeher's Axioms).

We provide an explicit example of the fact that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, by giving *a unique ideal-theoretic factorization of* $\langle 6 \rangle$.

Let

$$P := \langle 2, 1+\sqrt{-5}\rangle \ (= \langle 2, 1-\sqrt{-5}\rangle)$$
$$Q' := \langle 3, 1+\sqrt{-5}\rangle, \qquad Q'' := \langle 3, 1-\sqrt{-5}\rangle,$$

then it is not hard to see that $P, Q', Q'' \in \mathtt{Max}(\mathbb{Z}[\sqrt{-5}])$.

Moreover, the factorizations into maximal (or, prime) ideals of the following principal ideals are unique

$$\langle 2\rangle = P^2, \quad \langle 3\rangle = Q'\cdot Q'', \quad \langle 6\rangle = P^2\cdot Q'\cdot Q''$$

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• $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain (by Noeher's Axioms).

We provide an explicit example of the fact that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain, by giving *a unique ideal-theoretic factorization of* $\langle 6 \rangle$.

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If you are interested in recent developments of the ideal factorization, you might like to look at the following recent volume:

• M. Fontana, E. Houston, and T. Lucas, *Factoring Ideals in Integral Domains*, UMI Lecture Notes, Springer, Berlin 2013.

# §3. Kronecker's point of view: Divisors

Leopold Kronecker was a Kummer's student in high school (Gymnasium; at that time, Kummer was very young).
Later, he received his PhD in Berlin in 1845, having as advisor Dirichlet.

He also was interested, like Dedekind in Göttingen, to recover a "good" theory of divisibility, for integral domains for which the classical theory fails.

He solved, around 1859, a more general problem than Dedekind's problem, providing a solution in a more general setting than the setting of algebraic numbers. But, he did not published anything until 1880 (about 9 years after Dedekind's theory).

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Initially, immediately after the publication of his paper, which appeared in the Proceedings of a Conference in Kummer's honor (1880), Kronecker theory was not very successful essentially and roughly speaking for two reasons:

• The paper appeared many years after the very elegant Dedekind's theory, that was a very successful theory, widely adopted by Number Theorists.

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- W. Krull in 1936;
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Kronecker's "divisors", in a more modern approach might be represented by equivalent classes of polynomials.
More precisely, Kronecker functions rings –whose elements represent the Kronecker's divisors– can be introduced in the following general setting:

• Let $D_0$ be a PID (=Principal Ideal Domain) and let $K_0$ be its field of quotients (e.g., $D_0 = \mathbb{Z}$, $K_0 = \mathbb{Q}$).

• Let $K$ be a finite (algebraic) extension of $K_0$ and let $D := \overline{D_0}$ be the integral closure of $D_0$ in $K$ (i.e., $D = \{\alpha \in K \mid$ there exists a nonzero monic polynomial $f \in D_0[X]$ such that $f(\alpha) = 0\}$).

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• if $D_0 = \mathbb{Z}$ and $K = \mathbb{Q}(\sqrt{-5})$, then $D = \mathbb{Z}[\sqrt{-5}]$.

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With a modern terminology and notation, *the Kronecker function ring of a Dedekind domain D* is given by:

$$\mathtt{Kr}(D) := \left\{ \frac{f}{g} \mid f, g \in D[X] \ \textit{and} \ \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g) \right\} = \left\{ \frac{f'}{g'} \mid f', g' \in D[X] \ \textit{and} \ \boldsymbol{c}(g') = D \right\},$$

(where $\boldsymbol{c}(h)$ denotes *the content* of a polynomial $h \in D[X]$, i.e., the ideal of $D$ generated by the coefficients of $h$).

Note that the previous equality holds since we are assuming that $D$ is a Dedekind domain (since it is the integral closure of a PID $D_0$ in a finite field extension $K$ of the quotient field $K_0$ of $D_0$).

In this case, for each nonzero polynomial $g \in D[X]$, $\boldsymbol{c}(g)$ is an invertible ideal of $D$ and, by choosing a polynomial $u \in K[X]$ such that $\boldsymbol{c}(u) = (\boldsymbol{c}(g))^{-1}$, then we have $f/g = uf/ug = f'/g'$, with $f' := uf$, $g' := ug \in D[X]$ and, obviously, $\boldsymbol{c}(g') = D$.

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The fundamental properties of the Kronecker function ring are the following:

**Theorem (L. Kronecker, 1880)**

Let $D$ the integral closure of a PID in a finite extension of its field of fractions.

**(1)** *$\mathrm{Kr}(D)$ is a Bézout domain (i.e., each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and $D[X] \subseteq \mathrm{Kr}(D) \subseteq K(X)$ (in particular, the field of rational functions $K(X)$ is the quotient field of $\mathrm{Kr}(D)$).*

**(2)** *Let $a_0, a_1, \ldots, a_n \in D$ and set $f := a_0 + a_1X + \ldots + a_nX^n \in D[X]$, then:*

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Let $D$ the integral closure of a PID in a finite extension of its field of fractions.

**(1)** $\mathtt{Kr}(D)$ *is a Bézout domain (i.e., each finite set of elements has a GCD and the GCD can be expressed as linear combination of these elements) and* $D[X] \subseteq \mathtt{Kr}(D) \subseteq K(X)$ *(in particular, the field of rational functions* $K(X)$ *is the quotient field of* $\mathtt{Kr}(D)$*).*

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One of the difficulties encountered by Krull for extending Kronecker's theory is related to the problem of extending the Gauss' content formula:

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**Dedekind-Mertens Formula, 1892**

*Let $D$ be an integral domain, $f, g \in D[X]$, and let $m := \deg(g)$, then:*

$$\boldsymbol{c}(f)^m \boldsymbol{c}(f \cdot g) = \boldsymbol{c}(f)^{m+1} \cdot \boldsymbol{c}(g) \,.$$

Krull's idea is based to the possibility of applying a "cancellative closure operation" $*$, defined on finitely generated ideals, to the Dedekind-Mertens Formula, in order to obtain a "weaker" form of the Gauss' Content Formula:

$$(\ \boldsymbol{c}(f)^m \boldsymbol{c}(f \cdot g)\ )^* = (\ \boldsymbol{c}(f)^{m+1} \cdot \boldsymbol{c}(g)\ )^* \ \Rightarrow$$

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Krull introduced an important class of algebraic "cancellative closure operations", that he called *a.b.-operations* (where a.b. stands for *arithmetisch brauchbar*). An important example of this kind of operation is the *b-operation* defined as follows:

Let $D$ be an integral domain with quotient field $K$ and let $\overline{D}$ be the integral closure of $D$ in $K$.
For each nonzero fractional ideal $E$ of $\overline{D}$, set:

$$E^{b} := \bigcap\{EV \mid V \text{ is a valuation overring of } \overline{D}\}.$$

- Note that *the b-operation (or, completion) coincides with the integral closure of ideals* (when restricted to the integer ideals, i.e., fractional ideals inside $\overline{D}$).
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Further generalizations of the Kronecker function ring were introduced independently and using different methods, by Fontana-Loper in two papers published in 2001 and 2003 and by F. Halter-Koch in 2003.

In particular, Halter-Koch's generalization is based on an axiomatic approach.

Let $K$ be a field, $X$ an indeterminate over $K$, $R$ a subring of $K(X)$ and $D := R \cap K$. If

- $X \in \boldsymbol{U}(R)$ (i.e., $X$ is a unit in $R$);
- $f(0) \in f \cdot R$ for each $f \in K[X]$ ;

then $R$ is called *a $K$–function ring of $D$*.

It is not difficult to show that the following are examples of $K$–function rings of $D$, after Halter-Koch:

- The Gaussian extension of a valuation $D(X)$ $(= D[X]_{(X)})$, where $D$ is a valuation domain;
- $\mathtt{Kr}(D, b)$, when $D$ is integrally closed and, in particular,
- $\mathtt{Kr}(D)$, when $D$ is a Dedekind domain.

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In the recent years, the Kronecker function rings have been used for studying Riemann-Zariski Spaces of valuation domains.

For stating some of the results that show this surprising and effective application of the Kronecker function rings, I need to recall briefly the following notions:

- *spectral spaces* (after M. Hochster);

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Spectral spaces were characterized by Hochster in a purely topological way:

*a topological space $X$ is spectral if and only if*

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Examples of valuation domains,

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(see [Zariski, 1939], [Zariski, 1944] and [Zariski-Samuel, 1960]),

• from rigid algebraic geometry started by J. Tate [Tate, 1971]
(see [Fresnel-van der Put, 1981] and [Fujiwara-Kato, 2014]) and

• from real algebraic geometry
(see [Schwartz, 1990] and [Huber, 1993]).

For a deeper insight on this topic see the paper [Huber-Knebusch, 1994].

## NOTATION

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- Let

$$\text{Zar}(K|A) := \{V \mid V \text{ valuation domain with } A \subseteq V \subseteq K = \text{qf}(V)\}.$$

- In case $A$ is the prime subring of $K$, then $\text{Zar}(K|A)$ includes all valuation domains with $K$ as quotient field and we denote it by simply $\text{Zar}(K)$.

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The topological structure on $Z := \mathtt{Zar}(K|A)$ is defined by taking, as *a basis for the open sets*, the subsets $\mathcal{U}_F := \{V \in Z \mid V \supseteq F\}$ for $F$ varying in the finite subsets of $K$, i.e., if $F := \{x_1, x_2, \dots, x_n\}$, with $x_i \in K$, then

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- First we proved, using a purely topological approach that:
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This result was later re-proved by several authors with a variety of different techniques:

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**Theorem [Dobbs-Fontana, 1986]**

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*With all good wishes to Francesco*

*and . . . thanks for your attention!*

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