Krull-Schmidt-Remak Theorem, direct-product decompositions and G-groups

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Dedicated to Francesco

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central automorphism of G = automorphism of G that induces the identity $G/\zeta(G) \to G/\zeta(G)$. Here $\zeta(G)$ denotes the center of G.

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"Sur les produits directs", Bull. Soc. Math. France 41 (1913), 161–164: a simplified proof of Remak's main results.

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Abelian operator groups with ascending and descending chain conditions (operator groups $=$ Ω-groups. Here Ω is a set and an Ω-*group* is a pair (H, φ), where H is a group and $\varphi\colon \Omega \to \mathrm{End}(H)$ is a mapping).

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Let R be a ring, M_i ($i \in I$) be a right R-module, $\mathrm{End}_R(M_i)$ a *local* ring, $M = \bigoplus_{i \in I} M_i$. Then any two direct sum decompositions of M into indecomposable direct summands are isomorphic.

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The endomorphism ring of a biuniform module has at most two maximal right (left) ideals:

Biuniform modules and their endomorphism rings

Theorem

[F., T.A.M.S. 1996] *Let U_R be a biuniform module over a ring R,*

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(a) *either* E *is a local ring with maximal ideal* $I \cup K$*, or*

(b) E/I *and* E/K *are division rings, and* $E/J(E) \cong E/I \times E/K$*.*

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Weak Krull-Schmidt Theorem

Theorem

[F., T.A.M.S. 1996] *Let* $U_1, \ldots, U_n, V_1, \ldots, V_t$ *be* $n+t$ *biuniform right modules over a ring* R*. Then the direct sums* $U_1 \oplus \cdots \oplus U_n$ *and* $V_1 \oplus \cdots \oplus V_t$ *are isomorphic* R*-modules if and only if* $n = t$ *and there exist two permutations* σ *and* τ *of* $\{1, 2, \ldots, n\}$ *such that* $[U_i]_m = [V_{\sigma(i)}]_m$ *and* $[U_i]_e = [V_{\tau(i)}]_e$ *for every* $i = 1, 2, \ldots, n$.

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Auslander-Bridger modules (F., Girardi).
Also for direct products (Alahmadi, F., J. Algebra 2015).

Other algebraic structures?

Other algebraic structures, not only modules, could have the same behavior.

Groups, Lie algebras,. . .

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For instance, the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are uniform, and $\mathbb{Z}/p^n\mathbb{Z}$, simple groups, the symmetric groups S_n and the Prüfer groups $\mathbb{Z}(p^\infty)$ are biuniform.

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In our study, a predominant role is played by the *normal endomorphisms* of the group G, that is, the endomorphisms that commute with all inner automorphisms of G ($\varphi \in \mathrm{End}(G)$ and $\alpha_g \varphi = \varphi \alpha_g$ for every $g \in G$), and their generalizations to *normal homomorphisms* between normal subgroups and homomorphic images of G.

Biuniform groups

Theorem

Let $G_1, \dots, G_n, H_1, \dots, H_m$ *be groups with* $H_1, \dots, H_m$ *biuniform,* $G_1, \dots, G_n$ *indecomposable and* $G_1 \times \cdots \times G_n \cong H_1 \times \cdots \times H_m$. *Then:*

(a) $n \leq m$.

(b) $n = m$ *if and only if all the groups* $G_1, \dots, G_n$ *are biuniform.*

(c) *If the groups* $G_1, \dots, G_n$ *satisfy the maximal condition on normal subgroups or have centers which are either divisible or not torsion-free, then* $G_1, \dots, G_n$ *are biuniform,* $n = m$ *and there is a permutation* σ *of* $\{1, 2, \dots, n\}$ *such that* $G_i \cong H_{\sigma(i)}$ *for every* $i = 1, 2, \dots, n$.

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Thus completely indecomposable groups are the groups for which the partial ring of all normal endomorphisms is a sort of *local* (partial) ring.

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A group G is completely indecomposable $\Leftrightarrow$ the partial ring $\mathsf{NEnd}(G)$ is a local partial ring.

Completely indecomposable groups

Theorem

Let $G_1, \ldots, G_n$ be completely indecomposable groups.
(a) *If $G_1 \times \cdots \times G_n = H \times L$, then there is a partition $I_H \dot{\cup} I_L$ of the set $\{1, 2, \ldots, n\}$ such that that $H \cong \prod_{i \in I_H} G_i$ and $L \cong \prod_{i \in I_L} G_i$ (direct products).*
(b) *If $G_1 \times \cdots \times G_n \cong H_1 \times \cdots \times H_m$, where the H_j are indecomposable groups, then $n = m$ and there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $G_i \cong H_{\sigma(i)}$ for every $i = 1, 2, \ldots, t$.*

The correct categorical setting: G-groups

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Equivalently, a G-group is a group H endowed with a mapping $\cdot\colon G \times H \to H$, $(g, h) \mapsto gh$, called *left scalar multiplication*, such that
(a) $g(hh') = (gh)(gh')$
(b) $(gg')h = g(g'h)$
(c) $1_G h = h$
for every $g, g' \in G$ and every $h, h' \in H$.

The category G-**Grp**

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Strict analogy with left modules over a ring R:
Objects of $R\text{-}\mathrm{Mod}$: all pairs (H, φ), where H is any abelian group and $\varphi\colon R \to \mathrm{End}(H)$ is a ring homomorphism.

The category G-**Grp**

A special object of G-**Grp** is the *regular G-group* (G, α). Here $\alpha\colon G \to \mathrm{Aut}(G)$, $g \mapsto \alpha_g$, where $\alpha_g(x) = gxg^{-1}$ for every $g, x \in G$.

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The regular G-group (G, α) plays, in the category G-**Grp**, a role pretty similar to the role of the regular module ${}_R R$ in the category R-Mod.

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The category G-**Set** of G-sets is a Boolean topos (which does not satisfy the Axiom of Choice), and the category of G-groups is the category of groups of that topos (Janelidze).

Spectral category

Construction of the spectral category of a Grothendieck category, due to Gabriel and Oberst, and its dual.

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It is also possible for the category G-**Grp**, or, better, for the full subcategory $\mathcal{C}_G$ of G-**Grp** consisting of all objects (H, φ) of G-**Grp** for which the image of the group homomorphism $\varphi\colon G \to \mathrm{Aut}(H)$ contains the group $\mathrm{Inn}(H)$ of all inner automorphisms of H.

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We thus get two categories $\mathrm{Spec}(G\text{-}\mathbf{Grp})$ and $\mathcal{C}'_G$ and a canonical functor $\mathcal{C}_G \to \mathrm{Spec}(G\text{-}\mathbf{Grp}) \times \mathcal{C}'_G$.

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For further details, [Arroyo - F., Category of G-Groups and its Spectral Category, 2015].

Modules vs groups

module M_R, $E := \mathrm{End}(M_R)$	group H
idempotents in E $\updownarrow$ $\{\,(A,B) \mid A, B \leq M_R,\ M_R = A \oplus B\,\}$	idempotents in $\mathrm{End}(H)$ $\updownarrow$ $\{\,(A,B) \mid A, B \leq H,\ H = A \rtimes B\,\}$
	normal idempotents in $\mathrm{End}(H)$ $\updownarrow$ $\{\,(A,B) \mid A, B \leq H,\ H = A \times B\,\}$

Modules vs groups

E-Mod $\quad {}_E E$ regular module

E-Mod is the category
in which it is natural to study
direct-sum decompositions
of ${}_E E$
$=$ direct-sum decompositions
of M_R

$$\begin{array}{ccc} \Omega\text{-groups} & & G\text{-sets} \\ & \diagdown \quad \diagup & \\ & G\text{-groups} & \end{array}$$

${}_G G$ regular G-group
G-**Grp** is the category
in which it is natural to study
direct-product decompositions
of G

$\mathrm{End}_{G\text{-}\mathbf{Grp}}(G) =$
$=\{$*normal* endomorphisms of $G\}$
$\mathrm{Aut}_{G\text{-}\mathbf{Grp}}(G) =$
$=\{$*central* automorphisms of $G\}$