Coarse Structures on Infinite Groups

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B U O N   C O M P L E A N N O, F R A C E S C O !!!

joint work with Nicolò Zava

University of Udine

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- Coarse structures on groups – first steps
- The coarse classification of countable abelian-like groups
- Linear coarse structures and zero-dimensionality
- Some functorial coarse structures on groups
- Application to a problem of Banakh, Chervak, Lyaskovska
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Coarse space in the sense of Roe

According to Roe [Memoirs AMS, 2003], a coarse space is a pair $(X, \mathcal{E})$, where X is a set and $\mathcal{E} \subseteq \mathcal{P}(X \times X)$ a coarse structure on it, which means that

(D) $\Delta_X := \{(x,x) \mid x \in X\} \in \mathcal{E}$;

(I1) $\mathcal{E}$ is closed under passage to subsets;

(I2) $\mathcal{E}$ is closed under finite unions;

(U1) if $E \in \mathcal{E}$, then $E^{-1} := \{(y,x) \in X \times X \mid (x,y) \in E\} \in \mathcal{E}$;

(U2) if $E, F \in \mathcal{E}$, then

$$E \circ F := \{(x,y) \in X \times X \mid \exists z \in X \text{ s.t. } (x,z) \in E, (z,y) \in F\}.$$

(I1) and (I2) say that $\mathcal{E}$ is an ideal of $X \times X$. Replacing (I1) and (I2) with their dual properties (F1) and (F2) (saying that $\mathcal{E}$ is a filter of $X \times X$) the four properties (F1), (F2), (U1) and (U2) describe precisely a uniformity on X.

A uniformity can be equivalently described by means of uniform covers, i.e., the families $\{E[x] : x \in X\}$, $E \in \mathcal{E}$, where $E[x] = \{y \in Y : (x,y) \in E\}$ and $E[A] = \{y \in Y : x \in A, (x,y) \in E\}$ for $A \subseteq X$.

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Definition (Banakh, Protasov, Math. Studies, vol. 11, 2003)

A ball structure is a triple $\mathcal{B} = (X, P, B)$ where X and P are non-empty sets (called support and set of radii respectively) of the ball structure, and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X containing x, called ball of center x and radius α.

For a ball structure (X, P, B), $x \in X$, $\alpha \in P$ and $A \subseteq X$, we put

$$B^*(x, \alpha) := \{y \in X \mid x \in B(y, \alpha)\} \qquad B(A, \alpha) := \bigcup_{x \in A} B(x, \alpha).$$

A ball structure (X, P, B) is said to be

- upper symmetric if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$, $B(x, \alpha) \subseteq B^*(x, \alpha')$ and $B^*(x, \beta) \subseteq B(x, \beta')$;
- upper multiplicative if, for any $\alpha, \beta \in P$ there exists a $\gamma \in P$ such that for every $x \in X$, $B(B(x, \alpha), \beta) \subseteq B(x, \gamma)$.

A ballean is an upper symmetric and upper multiplicative ball structure.

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Balleans and coarse spaces are two faces of the same coin, since they are actually equivalent constructions:

Remark

Let X be a non-empty set.

- If $\mathcal{E}$ is a coarse structure on X, then $\mathcal{B}_{\mathcal{E}} = (X, \mathcal{E}, B_{\mathcal{E}})$, where $B_{\mathcal{E}}(x, E) := \{x\} \cup E[x]$ for every $x \in E$ and $E \in \mathcal{E}$, is a ballean.
- If (X, P, B) is a ballean, consider the family $\mathcal{E}_{\mathcal{B}}$ of all subsets E of $X \times X$ such that there exists an $\alpha \in P$ with $E \subseteq E_\alpha = \bigcup_{x \in X}\{(x, y) \mid y \in B(x, \alpha)\}$. Then $(X, \mathcal{E}_{\mathcal{B}})$ is a coarse structure.

This correspondence defines a bijection between coarse spaces ballean structures and ballean structures on X.

A ballean $\mathcal{B} = (X, P, B)$ is said to be connected if for all $x, y \in X$ there exists an $\alpha \in P$ such that $y \in B(x, \alpha)$, i.e., $X \times X = \bigcup_{E \in \mathcal{E}} E$.
A coarse structure $\mathcal{E}$ on a set X is said to be large scale connected if there is an entourage $E \in \mathcal{E}$ such that $X \times X = \bigcup_n \underbrace{E \circ \ldots \circ E}$.

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- If $\mathcal{E}$ is a coarse structure on X, then $\mathcal{B}_{\mathcal{E}} = (X, \mathcal{E}, B_{\mathcal{E}})$, where $B_{\mathcal{E}}(x, E) := \{x\} \cup E[x]$ for every $x \in E$ and $E \in \mathcal{E}$, is a ballean.
- If (X, P, B) is a ballean, consider the family $\mathcal{E}_{\mathcal{B}}$ of all subsets E of $X \times X$ such that there exists an $\alpha \in P$ with $E \subseteq E_\alpha = \bigcup_{x \in X} \{(x, y) \mid y \in B(x, \alpha)\}$. Then $(X, \mathcal{E}_{\mathcal{B}})$ is a coarse structure.

This correspondence defines a bijection between coarse spaces ballean structures and ballean structures on X.

A ballean $\mathcal{B} = (X, P, B)$ is said to be connected if for all $x, y \in X$ there exists an $\alpha \in P$ such that $y \in B(x, \alpha)$, i.e., $X \times X = \bigcup_{E \in \mathcal{E}} E$. A coarse structure $\mathcal{E}$ on a set X is said to be large scale connected if there is an entourage $E \in \mathcal{E}$ such that $X \times X = \bigcup_n \underbrace{E \circ \ldots \circ E}_{}$.

Example

1) Let X be a set and $\mathcal{J}$ be a base of an ideal on X (i.e., if $I_1, I_1 \in \mathcal{J}$ then $I_1 \cup I_2 \subseteq I$ for some $I \in \mathcal{J}$). If $x \in X$ and $I \in \mathcal{J}$, we define the ball $B_{\mathcal{J}}(x, I)$ of center x and radius I to be

$$B_{\mathcal{J}}(x, I) := \begin{cases} I & \text{if } x \in I, \\ \{x\} & \text{otherwise.} \end{cases}$$

$(X, \mathcal{J}, B_{\mathcal{J}})$ is a ballean.
The complements of the elements of $\mathcal{J}$ form a filter base φ on X. This is why the ballean $(X, \mathcal{J}, B_{\mathcal{J}})$ was defined as filter ballean and widely studied by the Ukraine School. It is connected iff the filter φ is not fixed.
2) Let (X, τ) be a topological space. The family $\mathcal{C}(X)$ of all compact subsets of X is a base of an ideal. The ballean $(X, \mathcal{C}(X), B_{\mathcal{C}(X)})$ is called compact ballean. It is always connected.
3) For a metric space (X, d) the metric ballean is defined by $(X, \mathbb{R}_+, B(x, r))$, where $B(x, r)$ is the usual d-ball for $r \in \mathbb{R}_+$.

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The word metric on a finitely generated group

For a group $G = \langle S \rangle$, with a finite set of generators $S = S^{-1}$, the Cayley graph $\Gamma = \Gamma(G, S)$ has as vertex sets the elements of a group G and as edges all pairs (g, gg_1), with $g \in G$ and $g_1 \in S$. Given an element g of G, its word norm $|g|$ with respect to the generating set S is the shortest length of a word over S whose evaluation is equal to g. The word metric with respect to S is defined by $d_S(g, h) = |g^{-1}h|$. This gives a left invariant metric d_S on Γ with finite balls (i.e., the metric d_S is proper)

$$B(g, R) = \{h \in G : d_S(g, h) \leq R\}, \; g \in G, \; R \in \mathbb{R}_+.$$

The growth of their size is one of the main topics in of the geometric group theory (Milnor's problems, Gromov's theorem).

Generally every graph $\Gamma = (V, E)$ has a metric defined by the shortest path between two vertexes $v_1, v_2 \in V$. Hence, every graph carries a natural ballean structure.

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Maps between coarse spaces

We say that two maps $f, g\colon S \to (X, \mathcal{E})$ from a set to a coarse space are close ($f \sim g$) if $\{(f(x), g(x)) \mid x \in S\} \in \mathcal{E}$.

Definition

Let $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ be two coarse spaces and $f\colon X \to Y$ a map. Then f is :
(a) bornological if $(f \times f)(E) \in \mathcal{E}_Y$ for all $E \in \mathcal{E}_X$;
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$L \subseteq (X, \mathcal{E})$ is large if $R[L] = X$ for some $R \in \mathcal{E}$.

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Maps between metric spaces (balleans)

A map $f\colon (X,P,B) \to (Y,P',B')$ between two balleans is bornological when for every $R \in P$ there exists $S \in P'$ such that $f(B(x,R)) \subseteq B(f(x),S)$.
In case of metric balleans, i.e., when (X,d) and (Y,d') are metric spaces, this means that there exists an increasing function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty} \rho(x) = \infty$, such that $d'(f(x),f(y)) \leq \rho(d(x,y))$ for all $x, y \in X$. In case $\rho(x) = Cx + A$ is linear, f is called generalized Lipschitzian.
Similarly, f is a coarse embedding if there exists also an increasing function $\rho_- : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{x\to\infty} \rho_-(x) = \infty$

$$\rho_-(d(x,y)) \leq d'(f(x),f(y))) \leq \rho(d(x,y)).$$

The map f is a quasi-isometry if ρ, ρ_- can be chosen to be linear.
The spaces (X,d) and (Y,d') are quasi-isometric, if there exists a quasi-isometry $f : X \to Y$ and $C > 0$ such that $Y = B(f(X),C)$.
Example. If $G = \langle S \rangle = \langle S_1 \rangle$ with S, S_1 finite and symmetric, then (G, d_S) and (G, d_{S_1}) are quasi-isometric.

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Maps between metric spaces (balleans)

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The coarse category

The objects of the coarse category **PreCoarse** are coarse spaces and its morphisms are the bornological maps. We consider also the quotient category **Coarse** $:=$ **PreCoarse**$/\sim$.

Theorem (epimorphisms in Coarse)

Let X and Y be two coarse spaces and $h\colon X\to Y$ a bornological map. Then the following are equivalent:
1) $f(X)$ is large in Y;
2) for every pair of bornological maps $f,g\colon X\to Y$, if $f\restriction_{f(X)}\sim g\restriction_{f(X)}$ then f,g are close.
3) h is a epimorphism in **Coarse**.

Theorem (monomorphism in Coarse)

Let X and Y be two coarse spaces and $h\colon X\to Y$ a bornological map. Then the following are equivalent:
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Definition

Let $\mathcal{B} = (X, P, B)$ be a ballean. Given an $n \in \mathbb{N}$, we put $\operatorname{asdim} \mathcal{B} \leq n$ if, for every $\alpha \in P$, there exists a uniformely bounded covering $\mathscr{U} = \mathscr{U}_0 \cup \cdots \cup \mathscr{U}_n$ of X such that $\mathscr{U}_i$ is α-disjoint for each $i = 0, \ldots, n$.
We denote by $\operatorname{asdim} \mathcal{B}$ the minimal n such that $\operatorname{asdim} \mathcal{B} \leq n$, if it exists, otherwise we put $\operatorname{asdim} \mathcal{B} = \infty$.

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Coarse structures on groups

Definition

Let G be a group. A coarse structure $\mathcal{E}$ on G is compatible, if $GE = \{(gx, gy) : (x, y) \in E\} \in \mathcal{E}$ for all $E \in \mathcal{E}$.

Definition (Banah, Protasov (2006), Nicas, Rosenthal (2012))

Let G be a group. A group ideal for G is a family $\mathcal{F} \subseteq \mathcal{P}(G)$ s.t.

i) there is an non-empty element $F \in \mathcal{F}$;

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iii) for every $F_1, F_2 \in \mathcal{F}$, $F_1F_2 := \{gh \in G \mid g \in F_1, h \in F_2\} \in \mathcal{F}$;

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If $\mathcal{F}$ is a group ideal, then it gives a compatible coarse structure on G by $\mathcal{E}_{\mathcal{F}} := \{E \subseteq G \times G \mid \exists F \in \mathcal{F} : E \subseteq G(F \times F)\}$.
If $\mathcal{E}$ is a compatible coarse structure on G, then $\mathcal{F} = \{\pi(E) : E \in \mathcal{E}\}$ is a group ideal (where $\pi(x, y) = y^{-1}x$ is the shear map).

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Example (of connected group ideals)

1) $\mathcal{F}_{fin}$ is the collection of all the finite subsets of G. It generates the group-finite coarse structure used by Dranishnikov and Smith in [Fundam. Math. 2006]
2) [Nicas-Rosenthal (2012)] If G is a topological group, $\mathcal{C}(G) := \{K \subseteq G \mid K \text{ is compact}\}$ generates $\mathcal{E}_{C(G)}$ the group-compact coarse structure.
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Every countable group admits a proper left-invariant metric and any two such metrics are coarsely equivalent and discrete.

A metric is proper if all closed bounded sets are compact.

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The main aims of geometric group theory is the classification of the groups up to quasi-isometry or coarse equivalence.

Theorem (Dranishnikov-Smith: Fundam. Math. 2006)

(a) If G is a countable group then
$\operatorname{asdim} G = \sup\{\operatorname{asdim} F : F \text{ a finitely generated subgroup of } G\}$.
(b) If N is a normal subgroup of G, then
$\operatorname{asdim} G \leq \operatorname{asdim} N + \operatorname{asdim} G/N$.
(c) For abelian groups, $\operatorname{asdim} G = r_0(G)$, *the free rank of G.*
(d) For soluble groups $\operatorname{asdim}(G) \leq h(G)$, *the Hirsch lenght of G. Equality holds for (virtually) polycylic groups.*

Every tree Γ has $\operatorname{asdim}\Gamma = 1$, so all non-abelian free groups have $\operatorname{asdim} F = 1$ (as $\Gamma(F)$ is a tree). More generally one has

Theorem (Bell, Dranishnikov, Keesling: Fundam. Math. 2004)

If A and B are finitely generated groups (one of them being non-trivial), such that $\operatorname{asdim}(A) = n$ *and* $\operatorname{asdim}(B) \leq n$, *then* $\operatorname{asdim}(A * B) = \max\{1, n\}$.

Asymptotic dimension of discrete groups

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Theorem (Banakh, Higes, Zarichnyi Trans. AMS 2010)

For two countable abelian groups G and H endowed with proper left-invariant metrics, the following three statements are equivalent:
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Consequently, an abelian group G with $r_0(G) = n < \infty$ is coarsely equivalent to $\mathbb{Z}^n$ or to $\mathbb{Z}^n \times (\mathbb{Q}/\mathbb{Z})$ depending on whether G is finitely generated or not.

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A finitely generated subgroup H of a finitely generated group G is undistorted, if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding with respect to any word metrics on H and G. A countable group G is said to be undistorted if G can be written as the union $G = \bigcup_{n\in\mathbb{N}} G_n$ of an increasing sequence (G_n) of finitely generated subgroups such that each group G_n is undistorted in G_{n+1}.

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From the above theorems one gets:

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Two countable locally finite-by-abelian groups G,H are coarsely equivalent if and only if $\mathrm{asdim}(G) = \mathrm{asdim}(H)$ and the groups G, H are either both finitely generated or both are inf. generated.

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Linear coarse structures

Now we consider some group ideals of purely algebraic nature.

Definition

A compatible coarse structure $\mathcal{E}$ on a group G is said to be linear if its group ideal has a base $\mathcal{B}$ composed of subgroups of G.

Clearly, a linear coarse structure on G is connected when $\mathcal{B}$ contains all finitely generated subgroups.

Theorem

Let $(G, \mathcal{F})$ be a group endowed with a linear coarse structure. Then $\operatorname{asdim}(G, \mathcal{F}) = 0$.

Proof. Let $\mathcal{B}$ be a base for $\mathcal{F}$ which is guaranteed by the definition. Then for every $K \in \mathcal{B}$ (it is enough to test these), the partition $\mathscr{U} := \{xK \mid x \in G\}$ shows that $\operatorname{asdim}(G, \mathcal{F}) = 0$.

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Proof. Let $\mathcal{B}$ be a base for $\mathcal{F}$ which is guaranteed by the definition. Then for every $K \in \mathcal{B}$ (it is enough to test these), the partition $\mathscr{U} := \{xK \mid x \in G\}$ shows that $\operatorname{asdim}(G, \mathcal{F}) = 0$.

Theorem (Petrenko-Protasov 2012)

A group G endowed with a coarse structure is linear iff $\operatorname{asdim} G = 0$.

Now we consider some group ideals of purely algebraic nature.

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A compatible coarse structure $\mathcal{E}$ on a group G is said to be linear if its group ideal has a base $\mathcal{B}$ composed of subgroups of G.

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If G is a topological group, then $\mathcal{D}_<(G, \mathcal{C}(G)) \subseteq \mathcal{S}(G, \mathcal{C}(G))$.

Problem (T. Banakh, O. Chervak, N. Lyaskovska (2013))

Detect coarse spaces X for which $\mathcal{S}(X) = \mathcal{D}_<(X)$.

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If G is an abelian group endowed with the linear coarse structure $\mathcal{F}_{i,\kappa}$ and $F \in \mathcal{B}_{i,\kappa}$, then $B(A, G(F \times F)) = A + F$ for each subset A of G.

Proposition

Let i be a cardinal invariant which is subadditive. For an abelian group G with $i(G) \geq \kappa$, where κ is an infinite cardianal, we have $[G]^{<\omega_0} \subseteq \mathcal{S}(G, \mathcal{F}_{i,\kappa})$. If i is bounded, then $[G]^{<\kappa} \subseteq \mathcal{S}(G, \mathcal{F}_{i,\kappa})$.

Proof of the finite case

Let S be a finite subset and A a large one in G. Let $F \in \mathcal{F}_{i,\kappa}$ s.t. $G = A + F$. Then there exists a point $a \in A \setminus S$; in fact otherwise $G = \langle S \rangle + F \in \mathcal{F}_{i,\kappa}$. Thus we can conclude $G = (A \setminus S) + (\langle a - S \rangle + F)$, where $a - S := \{a - s \mid s \in S\}$, and the second addend belongs to $\mathcal{F}_{i,\kappa}$. $\square$

Since $\operatorname{asdim}(G, \mathcal{F}_{i,\kappa}) = 0$, one has $\mathcal{D}_<(G, \mathcal{F}_{i,\kappa}) = \{\emptyset\}$. This produces a huge amount of examples of coarse spaces $(X, \mathcal{E})$ with $\mathcal{D}_<(X, \mathcal{E}) \subsetneq \mathcal{S}(X, \mathcal{E})$.

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Let i be a cardinal invariant which is subadditive. For an abelian group G with $i(G) \geq \kappa$, where κ is an infinite cardianal, we have $[G]^{<\omega_0} \subseteq \mathcal{S}(G, \mathcal{F}_{i,\kappa})$. If i is bounded, then $[G]^{<\kappa} \subseteq \mathcal{S}(G, \mathcal{F}_{i,\kappa})$.

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Asymptotic dimension of LCA groups

For a topological group (G, τ), we let asdim $G :=$ asdim$(G, \mathcal{C}(G))$. $\widehat{G}$ will denote the Pontryagin dual of G, namely the group of all continuous characters $G \to \mathbb{T}$ endowed with the compact open topology.

Theorem (A. Nicas, D. Rosenthal (2013))

Let G be a LCA group. Then asdim $G =$ dim $\widehat{G}$ *and* asdim $\widehat{G} =$ dim G.

The Bohr modification G^+ of a topological group G is the topological group G^+ with support G equipped with the initial topology of the family of all continuous homomorphisms $G \to K$, where K is a compact group. Following J. von Neumann's terminology, G is MAP (maximally almost periodic) if G^+ is Hausdroff, MinAP (minimally almost periodic), if G^+ is indiscrete.

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Since asdim is coarse invariant, we obtain:

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The class of Glicksberg groups is large enough. Glicksberg proved that every LCA group is Glicksberg (1962). Banaszczyk extended this result by showing that all nuclear groups are Glicksberg.

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For a MinAP abelian group G we have $\operatorname{asdim} G^+ = 0$, thus the inequality $\operatorname{asdim} G \geq \operatorname{asdim} G^+ = 0$ holds.

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