classes of locally nilpotent groups

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in honour of Francesco de Giovanni

classes of locally nilpotent groups

- $\mathfrak{N}$ the class of nilpotent groups.
- $\mathfrak{N}_1$ the class of groups in which every subgroup is subnormal
- $\mathcal{F}$ the class of Fitting groups: $\langle x\rangle^G$ nilpotent for every $x \in G$
- $\mathcal{B}$ the class of Bear groups: $\langle x\rangle$ subnormal for every $x \in G$

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- We have the following chain of proper inclusions:

$$\mathfrak{N} \subset \mathfrak{N}_1 \subset \mathcal{F} \subset \mathcal{B}$$

$\mathfrak{N}_1$-groups

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work of W. Möhres, H. Smith, C. Brookes, C. and others:

- *$\mathfrak{N}_1$-groups are soluble*
- *Torsion-free $\mathfrak{N}_1$-groups are nilpotent*
- *$\mathfrak{N}_1$-groups are metanilpotent*
- *periodic hypercentral or residually nilpotent $\mathfrak{N}_1$-groups are nilpotent*

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Conjecture: Any $\mathfrak{N}_1$-group is nilpotent-by-(finite rank)

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Theorem

Let G be a $\mathfrak{N}_1$-group such that

- *G is periodic* (C.), *or*
- *$\pi(T(G))$ is finite* (H. Smith 2013)

then G is nilpotent-by-Cernikov.

Fitting groups

Theorem (Puglisi, C.)

Let G be a countable group, which is either torsion-free or a p-group. The following are equivalent

1. *G is a Fitting group;*
2. *there exists a vector space V of countable dimension and a series $\mathcal{L}$ of subspaces of V, such that G embeds in the Hirsch-Plotkin radical of the stabilizer of $\mathcal{L}$.*

try to understand, beyond $\mathfrak{N}_1$, some other - possibly restricted - classes of Baer (Fitting) groups.

relaxing subnormality

classical open questions: groups in which every subgroup is ascendant (*the normalizer condition*) or descendant.

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$H \leq G$ is $\mathfrak{f}$-subnormal if there is a chain
$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$
such that $H_i \trianglelefteq H_{i+1}$ or $|H_{i+1} : H_i| < \infty$.

Theorem (Mainardis, C.)

Let G be a group in which every subgroup is $\mathfrak{f}$-subnormal. Then

- *G is finite-by-soluble;*
- *if G is torsion-free then it is nilpotent;*
- *if G is periodic then it is finite-by-$\mathfrak{N}_1$.*

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- Groups with finitely many normalizers of non-subnormal subgroups. (2007, with F. de Mari)
- Groups with dense subnormal subgroups (1999, with A. Russo)
- Groups with finite conjugacy classes of non-subnormal subgroups. (1998, with L. Kurdachenko, S. Franciosi)
- Groups with restrictions on non-subnormal subgroups. (1997, with L. Kurdachenko, S. Franciosi)
- Groups satisfying the minimal condition on non-subnormal subgroups. (1994, with S. Franciosi)
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- On groups with many subnormal subgroups (1993, with S. Franciosi)

- Groups whose non-subnormal subgroups have a transitive normality relation. (2003, with A. Russo, G. Vincenzi)

weakining $\mathfrak{N}_1$: a sample

A group G is said to have *dense* subnormal subgroups if for every $H < K \leq G$ either H is maximal in G or there exists a subnormal subgroup S such that $H \leq S \leq K$.

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Theorem (F. de Giovanni, A. Russo)

An infinite group with dense subnormal subgroups is a $\mathfrak{N}_1$-group.

weakening $\mathfrak{N}_1$ and nilpotency

Theorem (H. Smith)

Let G be a $\mathfrak{M}$-group in which every non-nilpotent subgroup is subnormal.

- *G is soluble.*
- *If G is torsion-free, then G is nilpotent.*
- *If G is locally finite then G is $\mathfrak{N}_1$-by-finite; if, in addition, G is Baer then $G \in \mathfrak{N}_1$.*

Definition. G is a $\mathfrak{M}$-group if every non-nilpotent finitely generated subgroup of G has a finite non-nilpotent homomorphic image.

strengthening $\mathcal{B}$

Say that a group G is

- *Strongly Baer:* every nilpotent subgroup of G is subnormal
- *Strongly Fitting:* H^G is nilpotent for every nilpotent subgroup H of G

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fact: *every $\mathfrak{N}_1$-group is strongly Fitting*

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example:

H a p-group of Heineken-Mohamed: $A = H'$ elementary abelian and $H/A = C_{p^\infty}$.

$K = C_p wr C_{p^\infty} = BC_{p^\infty}$ (B the base group is infinite elementary abelian).

In $W = H \times K$ let and $G/(A \times B)$ be the diagonal subgroup in $W/(A \times B) \simeq C_{p^\infty} \times C_{p^\infty}$.

Then G is a strongly Fitting group, but $G/A \simeq K$ is not even strongly Baer.

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- P.Hall generalized wreath powers. Let Ω be a totally ordered set with $|\Omega| \geq 2$, H a non-trivial transitive permutation group on X. Then WrH^{Ω} is a strongly Baer group if and only if Ω, X, H are finite, and H is a p-group for some prime p.

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$\Rightarrow$ Dark's groups [Baer p-groups with no non-trivial normal abelian subgroup] are not strongly Baer.

finite exponent

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A periodic $\mathfrak{N}_1$-group which is hypercentral or has finite exponent is nilpotent.

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Lemma

Let the p-group G be the extension of an elementary abelian group by an elementary abelian group. If $G \in \mathfrak{N}_1$ then it is nilpotent.

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A periodic $\mathfrak{N}_1$-group which is hypercentral or has finite exponent is nilpotent.

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for strongly Fitting groups this totally fails:

Theorem (A. Martinelli)

Fixed a prime p, there exists a non-nilpotent metabelian p-group G of finite exponent such that:

- for all $d \geq 1$, nilpotent subgroups of nilpotency class at most d are subnormal of defect bounded by a function of d.

Such G is strongly Fitting, and may be constructed so that:

a) has trivial center; or

b) is an FC-group

construction (b)

- A, A' countably infinite elementary abelian p-groups;

$$a \mapsto a' \quad (a \in A)$$

  an isomorphism .
- In the restricted wreath product $W = A\,wr\,A'$ identify A with the 1-component in the base group, and A' with the complement; then take
- $$G = \langle aa' \in W \mid a \in A \rangle.$$

Theorem

There exist strongly Fitting groups of arbitrary finite nilpotent length.

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Theorem

A torsion-free hypercentral strongly Baer group is soluble.