Permutable subgroups of groups

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Permutability
First definitions

Definitions

- If $A, B \leq G$, A permutes with B when $AB = BA$, that is, AB is a subgroup of G.
- A is permutable or quasinormal in G if H permutes with all subgroups of G (Ore, 1939).

Permutability
First definitions

Lemma

Let $\mathfrak{X}$ be a family of subgroups of a group G. If a subgroup A of G permutes with all subgroups $X \in \mathfrak{X}$, then it also permutes with their join $< \mathfrak{X} >$.

Therefore a subgroup A of a periodic group G is permutable in G if and only if A permutes with every p-subgroup of G for all $p \in \pi(G)$.

Permutability
First definitions

Let π a set of primes. A subgroup A of a periodic group G is called:

Definition

- π-permutable in G if A permutes with every q-subgroup of G for all $q \in \pi$.
- π-S-permutable or π-S-quasinormal in G if A permutes with every Sylow q-subgroup of G for all $q \in \pi$ (Kegel 1962).

Permutability
First definitions

If π is the set of all primes, then the π-permutable subgroups are just the permutable subgroups; the π-S-permutable subgroups are called S-permutable. In the case when $\pi = \pi(G) \setminus \pi(A)$, then A is called semipermutable (respectively, S-semipermutable) in G (Chen 1987).

Permutability
S-permutable embeddings

Theorem (Kegel (1962), Deskins (1963))

If A is an S-permutable subgroup of a finite group G, then A/A_G is contained in the Fitting subgroup of G/A_G. In particular, A is subnormal in G.

In particular, every permutable subgroup is subnormal (Ore, 1939).

Permutability
S-permutable embeddings

Theorem (Maier and Schmid, 1973)

If A is a permutable subgroup of a finite group G, then A/A_G is contained in the hypercentre of G/A_G.

- There are examples of permutable subgroups A of finite groups G such that A/A_G is not abelian (Thompson, 1967).
- There is no bound for the nilpotency class of a core-free permutable subgroup (Bradway, Gross and Scott, 1971).
- There is no bound for the derived length of a core-free permutable subgroup (Stonehewer, 1974).

Permutability
S-permutable embeddings

Definition (Carocca and Maier 1998)

A subgroup A of a group G is called hypercentrally embedded in G if A/A_G is contained in the hypercentre of G/A_G.

Every S-permutable subgroup of a finite group is hypercentrally embedded in G, but the converse is not true in general.

Permutability
S-permutable embeddings

Theorem (Schmid, 1998)

Let A be an S-permutable subgroup of a finite soluble group G. Then A hypercentrally embedded in G if and only if A permutes with some system normaliser of G.

Permutability
S-permutable embeddings

Theorem (Stonehewer, 1972)

If A is a permutable subgroup of a group G, then A is ascendant in G. If G is finitely generated, then A is subnormal in G.

Kargapolov (1961) showed that S-permutable subgroups of locally finite groups do not have to be ascendant.

Permutability
S-permutable embeddings

Theorem

Let A be an S-permutable subgroup of a periodic group. Then A is ascendant if:

- *G is locally finite with min-p for all p (Robinson, Ischia 2010).*
- *G is hyperfinite (B-B, Kurdachenko, Otal and Pedraza, 2010).*

Permutability
S-permutable embeddings

Maier and Schmid's theorem does not hold in the general case (Busetto and Napolitani, 1992). For locally finite groups, we have:

Theorem (Celentani, Leone and Robinson, 2006)

If A is a permutable subgroup of a locally finite group G, then A/A_G is locally nilpotent, and their Sylow subgroups are finite provided that G has min-p for all primes p.

Permutability
S-permutable embeddings

Theorem (Celentani, Leone and Robinson, 2006)

Let A be a permutable subgroup of a locally finite Kurdachenko group. Then A^G/A_G is finite and it is contained in a term of the upper central series of G/A_G of finite ordinal type.

Permutability
S-permutable embeddings

Definitions (Asaad and Heliel, 2003)

- We say that $\mathfrak{Z}$ is a complete set of Sylow subgroups of a periodic group G if for each prime $p \in \pi(G)$, G contains exactly a Sylow p-subgroup G_p of G.
- If $\mathfrak{Z}$ is a complete set of Sylow subgroups of a group G, we say that a subgroup A of a group G is $\mathfrak{Z}$-permutable if A permutes with all subgroups in $\mathfrak{Z}$.

Permutability
S-permutable embeddings

If G is a periodic group, a complete set of Sylow subgroups of G composed of pairwise permutable subgroups G_p, p prime, is called a Sylow basis. The existence of Sylow basis characterises solubility in the finite universe.

Permutability
S-permutable embeddings

Theorem (Almestady, B-B, Esteban-Romero and Heliel, 2015)

Let A be a subnormal $\mathfrak{Z}$-permutable subgroup of a finite group G. Then A/A_G is is soluble. If $\mathfrak{Z}$ is a Sylow basis, then A/A_G is nilpotent.

Permutability
S-permutable embeddings

Theorem (B-B, Camp-Mora and Kurdachenko, 2014)

Let A be an S-permutable subgroup of a locally finite group G. If A is ascendant in G, then A^G/A_G is locally nilpotent.

Permutability
S-permutable embeddings

Lemma

Let H and S be periodic subgroups of a group G. Suppose that H is an ascendant subgroup of G permuting with S. If π is a set of primes containing $\pi(S)$, then $\mathrm{O}^{\pi}(H) = \mathrm{O}^{\pi}(HS)$.

Permutability
Lattice properties

Theorem (see Doerk and Hawkes (1992), I, 4.29)

If $\mathfrak{Z}$ is a Sylow basis of the finite soluble group G, then the set of all $\mathfrak{Z}$-permutable subgroups of G is a lattice.

Permutability
Lattice properties

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Permutability
Lattice properties

Theorem (Almestady, B-B, Esteban-Romero and Heliel, 2015)

If $\mathfrak{Z}$ is a complete set of Sylow subgroups of a finite group G, then the set of all subnormal $\mathfrak{Z}$-permutable subgroups of G is a sublattice of the lattice of all subgroups of G.

Permutability
Lattice properties

Theorem

Let p be a prime and U and V subgroups of a finite group G. If U and V permute with a Sylow p-subgroup G_p of G and U is subnormal in G, then $U \cap V$ permutes with G_p.

Permutability
Lattice properties

Corollary (Kegel, 1962)

S-permutable subgroups of a finite group G form a sublattice of the subgroup lattice of G.

According to a result of Dixon, Kegel's lattice result also holds for radical locally finite groups with min-p for all p.
We do not know whether

- $\mathfrak{Z}$-permutable subgroups is a sublattice of the subgroup lattice of a finite G, even in finite soluble groups.
- Kegel's result holds for locally finite groups.

Permutability
π-permutability

Definition

A group G is called a T_0-group if the Frattini factor group $G/\Phi(G)$ is a T-group, that is, a group in which normality is a transitive relation.

Permutability
π-permutability

Theorem (B-B, Beidleman, Esteban-Romero and Ragland, 2014)

Let G be a group with nilpotent residual L, $\pi = \pi(L)$. Let θ_1 (respectively θ_2) denote the set of all primes p in $\theta = \pi'$ such that G has a non-cyclic (respectively cyclic) Sylow p-subgroup. Then every maximal subgroup of G is S-semipermutable if and only if G satisfies the following:

1. *G is a T_0-group.*
2. *L is a nilpotent Hall subgroup of G.*
3. *If $p \in \pi$ and P is a Sylow p-subgroup of G, then a maximal subgroup of P is normal in G.*

Permutability
π-permutability

5. Let p and q be distinct primes with $p \in \theta_1$ and $q \in \theta$. Further, let P be a Sylow p-subgroup of G and Q a Sylow q-subgroup of G. Then $[P, Q] = 1$.
6. Let p and q be distinct primes with $p \in \theta_2$ and $q \in \theta$. Further, let P be a Sylow p-subgroup of G, Q a Sylow q-subgroup of G and M the maximal subgroup of P. Then $QM = MQ$ is a nilpotent subgroup of G.

Permutability

π-permutability

Definition

Let p be a prime. A group G is *p-supersoluble* if there exists a series of normal subgroups of G

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

such that the factor group G_i/G_{i-1} is cyclic of order p or a p'-group for $1 \leq i \leq r$.

A group G is supersoluble if and only if G is p-supersoluble for all primes p.

Permutability
π-permutability

Theorem (Berkovich and Isaacs, 2014)

Let p be a prime and $e \geq 3$. Assume that a Sylow p-subgroup of a group G is non-cyclic with order exceeding p^e. If every non-cyclic subgroup of G of order p^e is S-semipermutable in G, then G is p-supersoluble.

Permutability
π-permutability

Theorem (Isaacs, 2014)

Let π be a set of primes. The normal closure A^G of an S-semipermutable π-subgroup A of a group G contains a nilpotent π-complement, and all π-complements are conjugate. Also, if π consists of a single prime, A^G is soluble. As a consequence, if A is a nilpotent Hall subgroup of G and A is S-semipermutable, then A^G is soluble.

Permutability
π-permutability

Theorem (B-B, Li, Su and Xie, 2014)

Let π and ρ be sets of primes. If A be a π-S-permutable ρ-subgroup of a group G, then $A^G/O_\rho(A^G)$ has nilpotent Hall π-subgroups.

Permutability
π-permutability

Lemma (Wielandt)

Let G be a group and let A and B be subgroups of G such that $AB^g = B^g A$ for all $g \in G$. Then $[A, B]$ is subnormal in G.

Permutability
π-permutability

Theorem (B-B, Li, Su and Xie, 2014)

If A is a nilpotent π-S-permutable subgroup of a group G, then $O^{\pi'}(A^G)$ is soluble.

Permutability
π-permutability

Corollary

If A is a nilpotent ρ-subgroup of a group G and A is S-semipermutable in G, then $O^{\rho}(A^G)$ is soluble. In particular, the normal closure of every nilpotent S-semipermutable subgroup of G of odd order is soluble.

Permutability
π-permutability

Theorem (Isaacs, 2014)

Let A be a subgroup of odd order of a finite group G. If A permutes with every 2-subgroup of G, then A^G is of odd order.

Permutability
π-permutability

Theorem (B-B, Li, Su and Xie, 2014)

Suppose that p is a prime such that $(p-1, |G|) = 1$. Let A be a p'-subgroup of a finite group G, and assume that A is permutable with every p-subgroup of G. Then A^G is a p'-group.

Permutability
π-permutability

Theorem (B-B, Li, Su and Xie, 2014)

Let p be a prime and let A be a p'-subgroup of a finite group G. Assume that A permutes with every p-subgroup of G. Then every chief factor of A^G whose order is divisible by p is simple and isomorphic to one of the following groups:

1. C_p,
2. A_p,
3. $\mathrm{PSL}(n, q)$, $n > 2$ prime, $p = \frac{q^n-1}{q-1}$, or $PSL(2, 11)$, $p = 11$,
4. M_{23}, $p = 23$, or M_{11}, $p = 11$.

If, moreover, A^G is p-soluble, then all p-chief factors are A^G-isomorphic when regarded as A^G-modules by conjugation.

Permutability
π-permutability

Corollary (B-B, Li, Su and Xie, 2014)

Let p be a prime and let A be a p'-subgroup of a finite G. Assume that A is permutable with every p-subgroup of G. If A^G is p-soluble, then $A^G/O_{p'}(A^G)$ is a soluble PST-group and either $A^G/O_{p'}(A^G)$ is nilpotent or the Sylow p-subgroups of A^G are abelian.