Products of groups revisited

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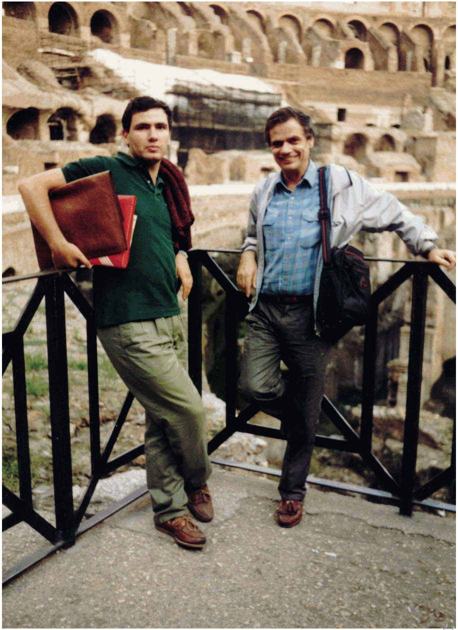
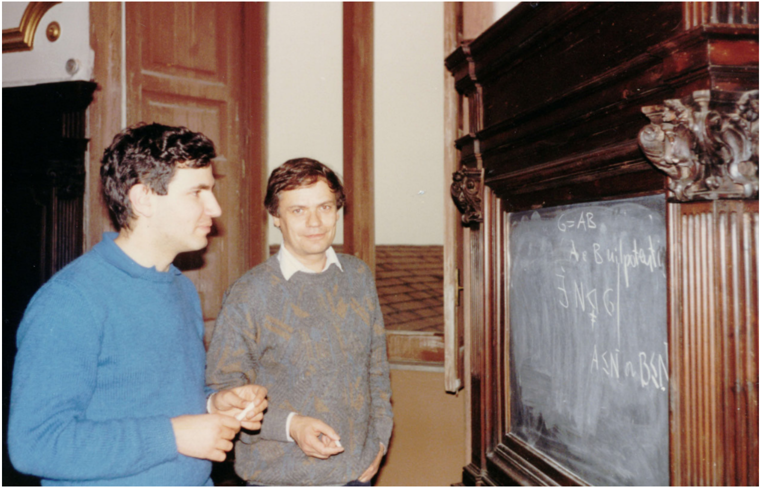
Some fotos

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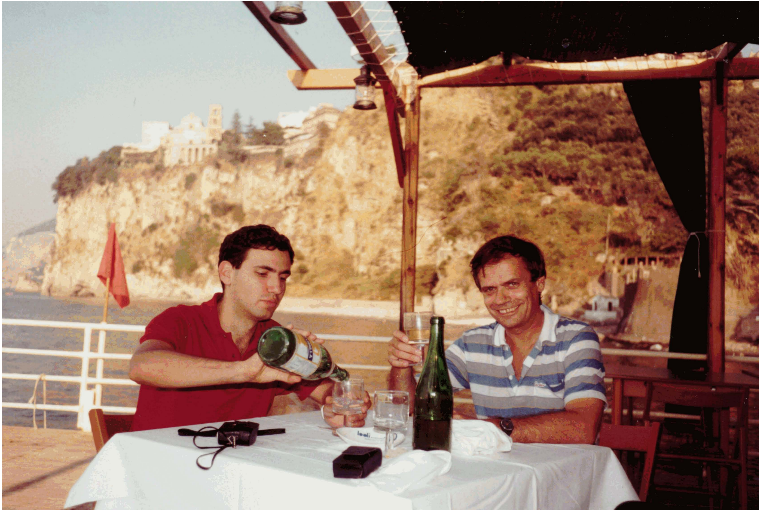

Products of groups

by B.A., Silvana Franciosi, Francesco de Giovanni [AFG]

Oxford Mathematical Monographs,
Clarendon Press, Oxford 1992

In the following we will discuss some relevant new results that have been obtained since that time and would have played a role if they were already known.

A. Products of generalized dihedral groups
B. Some results on finite factorized groups
C. Triply factorized groups and near-rings
D. Products of groups with finite rank

Main Problem

A group G is called **factorized**, if

$$G = AB = \{ab \mid a \in A,\ b \in B\}$$

is the product of two subgroups A and B of G.

What can be said about the structure of the factorized group G if the structures of its subgroups A and B are known?

If possible there should be no additional requirements on the group G.

A. Products of generalized dihedral groups

Theorem of N. Itô 1955 ([AFG], Theorem 2.1.1).

If the group $G = AB$ is the product of two abelian subgroups A and B, then G is metabelian.

Theorem of N.S. Chernikov 1981 ([AFG], Theorem 2.2.5).

If the group $G = AB$ is the product of two central-by-finite subgroups A and B, then G is soluble-by-finite.

It is unknown whether G must be metabelian-by-finite in this case.

Products of abelian-by-finite groups

In view of Itô's theorem the following can be asked.

Question 3 in [AFG].

Let the group $G = AB$ be the product of two abelian-by-finite subgroups A and B. Is G always soluble-by-finite or perhaps even metabelian-by-finite?

Yes, if G is linear (Ya. Sysak) or residually finite (J. Wilson) (see [AFG], Theorem 2.3.4)

Question 3*.

Let $G = AB$. If A and B are abelian-by-(index 2), is G soluble-by-finite?

Generalized dihedral groups

Recall that a group is **dihedral** if it is generated by two involutions.

Definition. A group G is **generalized dihedral** if it is **of dihedral type**, i.e. G contains an abelian subgroup X of index 2 and an involution a which inverts every element in X.

Then $A = X \rtimes <a>$ is the semi-direct product of an abelian subgroup X and an involution a, so that $x^a = x^{-1}$ for each $x \in X$.

Clearly every (finite or infinite) dihedral group is also generalized dihedral. A periodic generalized dihedral group is locally finite and every finite subgroup is contained in a finite dihedral group.

Properties of generalized dihedral groups

Lemma. Let A be generalized dihedral. Then the following holds

1) every subgroup of X is normal in A;
2) if A is non-abelian, then every non-abelian normal subgroup of A contains the derived subgroup A' of A;
3) $A' = X^2$ and so the commutator factor group A/A' is an elementary abelian 2-group;
4) the center of A coincides with the set of all involutions of X;
5) the coset aX coincides with the set of all non-central involutions of A;
6) two involutions a and b in A are conjugate if and only if $ab^{-1} \in X^2$;
7) if A is non-abelian, then X is characteristic in A.

Products of generalized dihedral groups

Theorem (B.A., Ya.Sysak, J. Group Theory 16 (2013), 299-318).

(a) Let the group $G = AB$ be the product of two subgroups A and B, each of which is either abelian or generalized dihedral. Then G is soluble.

(b) If, in addition, one of the two subgroups, B say, is abelian, then the derived length of G does not exceed 5.

Products of two subgroups which are (locally cyclic)-by-(index 2)

Corollary. Let the group $G = AB$ be the product of two subgroups A and B.

(a) If both A and B contain torsionfree locally cyclic subgroups of index at most 2, then G is soluble and metabelian-by-finite.

(b) If A and B are cyclic-by-(index 2), then G is metacyclic-by-finite.

Some special cases

Let the group $G = AB$ be the product of two generalized dihedral subgroups A and B.

1. The second case of the corollary was first proved in
   B.A., Ya. Sysak, Arch. Math. 90 (2008), 101-111.

2. The special case of the theorem when A and B are periodic generalized dihedral was already treated in
   B.A., A. Fransman, L. Kazarin, J. Alg. 350 (2012), 308-317.

3. If A and B are Chernikov groups and (abelian)-by-(index 2), and one of the two is generalized dihedral, then G is a soluble Chernikov group. This was shown in
   B.A., L. Kazarin, Israel J. Math. 175 (2010), 363-389.

Remarks on the proof of the above Theorem

The proof of the above Theorem is elementary and almost only uses computations with involutions. Extensive use is made by the fact that every two involutions of a group generate a dihedral subgroup.

A main idea of the proof is to show that

the normalizer in G of a non-trivial normal subgroup of one of the factors A or B has a non-trivial intersection with the other factor.

If this is not the case we may find commuting involutions in A and B and produce a nontrivial abelian normal subgroup by other computations.

It is easy to see that we may assume that $|A \cap B| \leq 2$.

B. Some results on finite factorized groups

Products of finite groups

by A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad, Expositions in Mathematics 53, De Gruyter 2010

Theorem of H. Wielandt, O. Kegel ([AFG], Theorem 2.4.3).

If the finite group $G = AB$ is the product of two nilpotent subgroups A and B, then G is soluble.

Question 5 in [AFG].

Let the class of A be α and the class of B be β. Is the derived length $d(G)$ of G bounded by a function of α and β?

$d(G) \neq \alpha + \beta$ in general (J. Cossey and S. Stonehewer 1998).

Generalization of Kegel-Wielandt

Theorem of L. Kazarin 1979

If the finite group $G = AB$ is the product of two subgroups A and B, each of which possesses nilpotent subgroups of index at most 2, then G is soluble.

Theorem (B.A., L. Kazarin, J.Alg. 311 (2007), 69-75).

Let the finite group $G = AB$ be the product of a nilpotent subgroup A and a subgroup B, then the normal closure of the center Z of B is a soluble normal subgroup of G.
In particular, if $Z \neq 1$, then G contains a non-trivial abelian normal subgroup.

Remark. The proof of this and the following theorem uses the **Classification of the Finite Simple Groups (CFSG)**.

Finite products of soluble groups

Theorem (L. Kazarin, Comm. Alg. 14 (1986), 1001-1066).

Let $G = AB$ be a finite group, where A and B are soluble. Then the composition factors of G are among the following groups

$$L_2(q),\ L_3(q),\ L_4(2),\ M_{11},\ PSp_4(3),\ U_3(8).$$

Theorem (L. Kazarin, A. Martinez-Pastor, M. Pérez-Ramos 2008).

Let π be a set of primes. If the finite group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.

C. Triply factorized groups and near-rings

Many proofs concerning factorized groups reduce to the consideration of a **triply factorized group**

$$G = AK = BK = AB, \text{ where } A, B, K \text{ are subgroups of } G,$$

$$K \text{ is normal in } G, \; A \cap K = B \cap K = 1.$$

Thus $G = A \ltimes K = B \ltimes K = AB$.

Such groups can be constructed using radical rings (see [AFG], Chapter 6).

Radical rings

Let R be an associative ring, not necessarily with identity element. Then R forms a semi-group with identity element 0 under the **circle operation**

$$a \circ b = a + b + ab \quad \text{for all } a, b \in R.$$

The group of all invertible elements in this semi-group is called the **adjoint group** R° of R.

R is **radical** if $R = R^\circ$, i. e. R coincides with its Jacobson radical.

Let R be any ring embedded into a ring R_1 with identity in an arbitrary way. Then R is radical if and only if $R + 1$ is a subgroup of the group of units of R_1 and this group is isomorphic with R°.

Construction of triply factorized groups

Construction. Let R be a radical ring, P a right ideal of R and $M = R/P$ as a right R-module.

The adjoint group $A = R^\circ$ operates on M via the rule $m^a = m + ma$ for all $a \in A$ and $m \in M$.

Consider the semidirect product $G(M) = A \ltimes M$ and its subgroup

$$B = \{am \mid m = a + P,\ a \in A\}.$$

It is easy to see that

$$G = G(M) = A \ltimes M = B \ltimes M = AB$$

(see [AFG], Section 6.1). Note that here M is an abelian group.

Near-rings

Definition. A set $(R, +, \cdot)$ with two binary operations, addition and multiplication, is called a (left) **near-ring** if

1. $(R, +)$ is a (not necessarily abelian) group,
2. $(R, \cdot)$ is a semi-group,
3. $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in R$.

If R contains a multiplicative identity 1, then R is a **near-ring with identity**.

In this case of multiplicatively invertible elements of R is a group $R^{\times}$. The additive group of R is written as R^{+}.

Construction subgroups

Example. Let G be any group (written additively).

Let $M(G) = \{\alpha : G \longrightarrow G\}$ the set of all mappings from G into G. Then $M(G)$ is a near-ring under pointwise addition of mappings and multiplication by composition.

Definition. Let R be a near-ring with 1. If the subgroup U of R^+ satisfies $(U + 1) \leq R^\times$, then U is admissible or a **construction subgroup** of R. In this case $(U + 1)U \subseteq U$.

For instance, if R is a ring with 1, then the Jacobson radical is a construction subgroup of R.

Construction of triply factorized groups using near-rings

Theorem (P. Hubert, Comm. Alg. 32 (2004), 1229-1235).

Let R be a near-ring with identity and U be a construction subgroup of R. Let N^+ be a normal subgroup of U^+ with $(U+1)N \subseteq N$. The group $A = U + 1$ operates on $M = U^+/N$ via the rule

$$(u+N)^{(v+1)} = (v+1)^{-1}u + N,$$

for all $u, v \in U$. In the semidirect product $G = G(R, U) = A \ltimes M$, the subgroup

$$B = \{((l+1)^{-1}, l+U), l \in R\}$$

is a complement of M, such that $G = A \ltimes M = B \ltimes M = AB$. Thus G is triply factorized by A, B, and M. (Note that here M is not necessarily abelian.)

The Converse theorem

Theorem (P. Hubert, 2005). Let the group

$$G = A \ltimes M = B \ltimes M = AB$$

be triply factorized by two subgroups A and B and a normal subgroup M of G, such that $A \cap B = 1$.
Then there exists a near-ring R which contains a construction subgroup U, such that $G(R, U) \simeq G$.

If A, B, M are abelian, then R may be chosen as a commutative radical ring R ([AFG], Proposition 6.1.4).

Local near-rings

Definition. Let R be a near-ring with identity element and let L_R be the set of elements of R which are not right invertible. If L_R is a subgroup of R^+, then R is called a **local near-ring**.

Remark. If R is a local ring, then L_R is the Jacobson radical $J(R)$. In every local near-ring R, the subgroup L_R is a construction subgroup.

Question. Is L_R in every local near-ring an ideal of R or at least a normal subgroup of R^+?

Examples of local near-rings

Lemma. Let N^+ be any p-group with finite exponent. Then there exists a local near-ring R such that L_R^+ contains N^+.

Theorem (B.A., P. Hubert, Ya. Sysak, J. Algebra 273, 2004, 700-717).

If R is a local near-ring, such that the group of units of R is a dihedral group, then R is finite. Moreover, R^+ is a p-group for $p = 2$ or $p = 3$ and $|R| \leq 16$.

D. Products of groups with finite rank

Definition. A group G has **finite Prüfer rank** if there exists a natural number r such that every finitely generated subgroup of G can be generated by r elements and r is minimal with this property.

Theorem (see [AFG], Theorems 4.6.12 and 4.3.5).
Let the soluble group $G = AB$ be the product of two subgroups A and B with finite Prüfer-rank $r(A)$ resp. $r(B)$.
Then G has finite Prüfer rank $r(G)$ which is bounded by a polynomial function of $r(A)$ und $r(B)$.

Question. Is there such a linear function?

This question reduces to a problem about finite p-groups.

Finite Prüfer rank

Question. Let $G = AB$ be a finite p-group. Is $r(G)$ bounded by a linear function $r(A)$ and $r(B)$?

B. Huppert has shown that for $p > 3$ every finite product of two cyclic subgroups is metacyclic.

Theorem (B. A., L. Kazarin, Comm. Alg. 27 (1999), 3895-3907).

Let $G = AB$ be a finite p-group

a) If A and B are metacyclic and $p > 3$, then $r(G) \leq 4$.

b) If $r(A), r(B) \leq r$ and $p > 2$, then

$$r(G) \leq 4r(\lceil \log_2 r \rceil + 2)^2.$$

Relation with Eggert's conjecture

Special Question. Let $G = AB$ be a finite p-group with A, B abelian. Is $r(G)$ bounded by a linear function of $r(A)$ and $r(B)$?

It turns out that this question has a positive answer if the following conjecture on commutative nilpotent p-algebras is true.

Eggert's conjecture

Let R be a finite dimensional commutative nilpotent p-algebra.
Let $R^{(p)} = \langle a^p \mid a \in R \rangle$, $R_{(p)} = \langle a \mid a^p = 0, a \in R \rangle$
Clearly $R/R_{(p)} \simeq R^{(p)}$ and $\dim R_{(p)} = r(R^\circ)$.

N. Eggert conjectured in 1979 that $\dim R \geq p \dim R^{(p)}$.
He proved this if $R = \langle x, y \rangle$ and $\dim R^{(p)} = 2$.

Some remarks on Eggert's conjecture

Eggert's conjecture has been verified in many special cases, for instance by R. Bautista (1976) and C. Stack (1996) for $\dim R^{(p)} \leq 3$ and B. A. and L. Kazarin (2003, 2008) for $\dim R^{(p)} \leq 4$.

Further results are for example by M. Korbelar (2010) for $R^{(p)} = \langle x, y \rangle$ and M. R. McLean (2004, 2006) for some graded algebras.

A detailed account on Eggert's conjecture can be found in:

G. M. Bergman, Thoughts on Eggert's conjecture, Contemporary Mathematics 609 (2014), 1-17.

Finite torsionfree rank

Definition. A group G has **finite torsion-free rank** if it has a series of finite length whose factors are either periodic or infinite cyclic. The number $r_0(G)$ of infinite cyclic factors in such a series is an invariant of G called its **torsion-free rank**.

Theorem (B.A., S. Franciosi, F. de Giovanni, 1991 [AFG], 4.1.8). If the group $G = AB$ with finite torsion-free rank is the product of two subgroups A and B, then

$$r_0(G) \leq r_0(A) + r_0(B) - r_0(A \cap B).$$

Question 8 in [AFG] Does the equality sign hold here?

This is true if G is soluble with finite abelian section rank (see [AFG], 4.1.10).