Third homology of perfect central extension

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Homology of groups

A complex of left (or right) R-modules is a family

$$K_\bullet := \{K_n, \partial_n^K\}_{n\in\mathbb{Z}}$$

of left (or right) R-modules K_n and R-homomorphisms $\partial_n^K : K_n \to K_{n-1}$ such that for all $n \in \mathbb{Z}$,

$$\partial_n^K \circ \partial_{n+1}^K = 0.$$

Usually we show this complex as follow

$$K_\bullet : \qquad \cdots \longrightarrow K_{n+1} \xrightarrow{\partial_{n+1}} K_n \xrightarrow{\partial_n} K_{n-1} \longrightarrow \cdots$$

The n-th homology of this complex is defined as follow:

$$H_n(K_\bullet) := \ker(\partial_n^K)/\mathrm{im}(\partial_{n+1}^K)$$

We say $K_\bullet$ is an exact sequence if $H_n(K_\bullet) = 0$ for any n.

A projective resolution of a R-module M is an exact sequence

$$P_\bullet \xrightarrow{\varepsilon} M : \qquad \cdots \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where all P_i's are projetive.

If $P_\bullet \xrightarrow{\epsilon} M$ is a projective resolution and N is any R-module, we defined the Tor functor as follow

$$\mathrm{Tor}_n^R(M,N) := H_n(P_\bullet \otimes_R N).$$

FACTS: (1) The definition of $\mathrm{Tor}_n^R(M,N)$ is independent of a choice of projective resolution $P_\bullet \xrightarrow{\epsilon} M$, so it is well-defined. Moreover

$$\mathrm{Tor}_0^R(M,N) \simeq M \otimes_R N.$$

(2) If M and N are abelian groups ($\mathbb{Z}$-modules), then $\mathrm{Tor}_1^{\mathbb{Z}}(M,N)$ is a torsion group and $\mathrm{Tor}_n^{\mathbb{Z}}(M,N) = 0$ for all $n \geq 2$.

(3) If M or N is torsion free, then $\mathrm{Tor}_n^{\mathbb{Z}}(M,N) = 0$ for all $n > 0$.

Let G be a group and let $\mathbb{Z}G$ be its (integral) group ring.

The n-th homology of G with coefficients in a $\mathbb{Z}G$-module M is defined as follow

$$H_n(G, M) := \mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, M),$$

where $\mathbb{Z}$ is a trivial $\mathbb{Z}G$-module, i.e. $(\sum n_g g).m = \sum n_g m$.

EXAMPLES:
(1) $H_0(G, M) \simeq M_G := M/\langle gm - m \mid g \in G, m \in M\rangle$.
In particular, $H_0(G, \mathbb{Z}) \simeq \mathbb{Z}$.

(2) $H_1(G, \mathbb{Z}) \simeq G/[G, G]$. In particular, if G is abelian, then $H_1(G, \mathbb{Z}) \simeq G$.

(3) In $G \simeq \mathbb{Z}/l\mathbb{Z}$, then $H_n(G, M)$ is l-torsion.

(4) If G is abelian, then $H_2(G,\mathbb{Z}) \simeq \bigwedge_{\mathbb{Z}}^2 G$ and for any $n \geq 0$ we have an injective homomorphism

$$\bigwedge\nolimits_{\mathbb{Z}}^n G \to H_n(G,\mathbb{Z})$$

HOMOLOGY IS A FUNCTOR: Let M be a $\mathbb{Z}G$-module and N a $\mathbb{Z}H$-module. If $\alpha : G \to H$ and $f : M \to N$ are homomorphism such that

$$f(gm) = \alpha(g)f(m), \qquad g \in G, \quad m \in M,$$

then (α, f) induce a homomorphism of group homology

$$H_n(\alpha, f) : H_n(G, M) \longrightarrow H_n(H, N).$$

In particular, if $M = \mathbb{Z}$ is trivial $\mathbb{Z}G$ and $\mathbb{Z}H$-modules, we have the homomorphism

$$\alpha_* := H_n(\alpha, \mathrm{id}_{\mathbb{Z}}) : H_n(G,\mathbb{Z}) \longrightarrow H_n(H,\mathbb{Z}).$$

Third homology of perfect central extensions

An extension $A \overset{\beta}{\rightarrowtail} G \overset{\alpha}{\twoheadrightarrow} Q$ is called a perfect central extension if G is perfect, i.e. $G = [G, G]$, and $A \subseteq Z(G)$.

The aim of this talk is to study the homomorphisms

$$\beta_* : H_3(A, \mathbb{Z}) \to H_3(G, \mathbb{Z})$$

and

$$\alpha_* : H_3(G, \mathbb{Z}) \to H_3(Q, \mathbb{Z})$$

Clearly by the functoriality of the homology functor we have $\mathrm{im}(\beta_*) \subset \ker(\alpha_*)$.

Our first main theorem is as follow

Theorem

Let A be a central subgroup of G and let $A \subseteq G' = [G,G]$. Then the image of $H_3(A,\mathbb{Z})$ in $H_3(G,\mathbb{Z})$ is 2-torsion. More precisely

$$\mathrm{im}\big(H_3(A,\mathbb{Z}) \to H_3(G,\mathbb{Z})\big) = \mathrm{im}\big(H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)) \to H_3(G,\mathbb{Z})\big),$$

where ${}_{2^\infty}A := \{a \in A :$ there is $n \in \mathbb{N}$ such that $a^{2^n} = 1\}$ and $\Sigma_2 = \{1, \sigma\}$ is symmetric group which σ is induced by the involution $\iota : A \times A \to A \times A$, $(a,b) \mapsto (b,a)$.

Sketch of proof:

(1) By a result of Suslin we have the exact sequence

$$0 \to \textstyle\bigwedge_{\mathbb{Z}}^3 A \to H_3(A,\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A,A)^{\Sigma_2} \to 0,$$

where the homomorphism on the right side is obtained from the composition

$$H_3(A,\mathbb{Z}) \xrightarrow{\Delta_*} H_3(A\times A,\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A,A).$$

Here Δ is the diagonal map $A \to A\times A$, $a\mapsto (a,a)$.

2) Since $A\subseteq G'$, the map $A = H_1(A,\mathbb{Z}) \to H_1(G,\mathbb{Z}) = G/G'$ is trivial. From the commutative diagram

$$\begin{array}{ccc} A\times A & \xrightarrow{\mu} & A \\ \downarrow & & \downarrow \\ A\times G & \xrightarrow{\rho} & G, \end{array} \tag{0.1}$$

where μ and ρ are the usual multiplication maps, we obtain the commutative diagram

$$\begin{array}{ccc} H_2(A,\mathbb{Z})\otimes H_1(A,\mathbb{Z}) & \longrightarrow & H_3(A,\mathbb{Z}) \\ \downarrow{\scriptstyle =0} & & \downarrow \\ H_2(A,\mathbb{Z})\otimes H_1(G,\mathbb{Z}) & \longrightarrow & H_3(G,\mathbb{Z}). \end{array}$$

3) On the other hand, $\Delta \circ \mu = \mathrm{id}_{A\times A}.\iota : A \times A \to A \times A$ induces the map

$$\mathrm{id} + \sigma : \mathrm{Tor}_1^{\mathbb{Z}}(A, A) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, A),$$

and thus

$$\begin{array}{ccc} H_3(A \times A, \mathbb{Z}) & \overset{(\Delta\circ\mu)_*}{\longrightarrow} & H_3(A \times A, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_1^{\mathbb{Z}}(A, A) & \xrightarrow{\mathrm{id}+\sigma} & \mathrm{Tor}_1^{\mathbb{Z}}(A, A), \end{array}$$

is commutative.

This implies that the following diagram is commutative:

$$\begin{array}{ccc} H_3(A \times A, \mathbb{Z}) & \xrightarrow{\mu_*} & H_3(A, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_1^{\mathbb{Z}}(A, A) & \xrightarrow{\mathrm{id}+\sigma} & \mathrm{Tor}_1^{\mathbb{Z}}(A, A)^{\Sigma_2}. \end{array}$$

4) From the diagram (0.1) we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{Tor}_1^{\mathbb{Z}}(A,A) \arrow[r, "\mathrm{id}+\sigma"] \arrow[ddd, bend left, "=0"] & \mathrm{Tor}_1^{\mathbb{Z}}(A,A)^{\Sigma_2} \\
\tilde{H}_3(A\times A,\mathbb{Z})/\bigoplus_{i=1}^{2} H_i(A,\mathbb{Z})\otimes H_{3-i}(A,\mathbb{Z}) \arrow[u, "\alpha", "\cong"'] \arrow[r, "\mu_*"] \arrow[d, "\tilde{\mathrm{inc}}_*"] & H_3(A,\mathbb{Z})/\bigwedge_{\mathbb{Z}}^3 A \arrow[u, "\beta", "\cong"'] \arrow[d, "\mathrm{inc}_*"] \\
\tilde{H}_3(A\times G,\mathbb{Z})/\mathrm{im}(H_1(A,\mathbb{Z})\otimes H_2(A,\mathbb{Z})) \arrow[r, "\rho_*"] \arrow[d] & H_3(G,\mathbb{Z}) \\
\mathrm{Tor}_1^{\mathbb{Z}}(A,H_1(G,\mathbb{Z})) &
\end{tikzcd}$$

where

$$\tilde{H}_3(A\times A) := \ker(H_3(A\times A)\to H_3(A)\oplus H_3(A))$$

and

$$\tilde{H}_3(A\times G) := \ker(H_3(A\times G)\to H_3(A)\oplus H_3(G)).$$

5) Since $\mathrm{Tor}_1^{\mathbb{Z}}(A,A) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, H_1(G,\mathbb{Z}))$ is trivial, the map $\tilde{\mathrm{inc}}_* \circ \alpha^{-1}$ is trivial. This shows that $\mathrm{inc}_* \circ \beta^{-1} \circ (\mathrm{id} + \sigma)$ is trivial.

Therefore the image of $H_3(A,\mathbb{Z})$ in $H_3(G,\mathbb{Z})$ is equal to the image of

$$H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}(A,A)) = \mathrm{Tor}_1^{\mathbb{Z}}(A,A)^{\Sigma_2}/(\mathrm{id}+\sigma)(\mathrm{Tor}_1^{\mathbb{Z}}(A,A)).$$

6) Since $\mathrm{Tor}_1^{\mathbb{Z}}(A,A) = \mathrm{Tor}_1^{\mathbb{Z}}({}_{\mathrm{tor}}A, {}_{\mathrm{tor}}A)$, ${}_{\mathrm{tor}}A$ being the subgroup of torsion elements of A.
and
since for any torsion abelian group B, $B \simeq \bigoplus_{p \text{ prime}} {}_{p^\infty}B$, we have the isomorphism

$$H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}(A,A)) \simeq H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)). \qquad \diamond$$

Whitehead's quadratic functor:

In the study of the kernel of $\beta_* : H_3(G, \mathbb{Z}) \to H_3(Q, \mathbb{Z})$, Whitehead's quadratic functor plays a fundamental role.

We also will see that this functor is deeply related to the previous theorem.

A function $\psi : A \to B$ of (additive) abelian groups is called a **quadratic map** if
(a) for any $a \in A$, $\psi(a) = \psi(-a)$,
(b) the function $A \times A \to B$, with

$$(a, b) \mapsto \psi(a + b) - \psi(a) - \psi(b)$$

is bilinear.

FACT: For each abelian group A, there is a universal quadratic map

$$\gamma : A \to \Gamma(A)$$

such that if $\psi : A \to B$ is a quadratic map, there is a unique homomorphism $\Psi : \Gamma(A) \to B$ such that $\Psi \circ \gamma = \psi$.

Note that Γ is a functor from the category of abelian groups to itself.

The functions

$$\phi : A \to A/2, \;\; a \mapsto \bar{a}$$

and

$$\psi : A \to A \otimes_{\mathbb{Z}} A, \;\; a \mapsto a \otimes a$$

are quadratic maps.

Thus, by the universal property of Γ, we get the canonical homomorphisms

$$\Phi : \Gamma(A) \to A/2,\ \gamma(a) \mapsto \bar{a}$$

and

$$\Psi : \Gamma(A) \to A \otimes_{\mathbb{Z}} A,\ \gamma(a) \mapsto a \otimes a.$$

Clearly Φ is surjective and $\operatorname{coker}(\Psi) = H_2(A, \mathbb{Z})$.

Furthermore we have the bilinear pairing

$$[\ ,\ ] : A \otimes_{\mathbb{Z}} A \to \Gamma(A), \quad [a, b] = \gamma(a + b) - \gamma(a) - \gamma(b).$$

It is easy to see that for any $a, b, c \in A$,

$$[a, b] = [b, a], \qquad \Phi[a, b] = 0,$$

$$\Psi[a, b] = a \otimes b + b \otimes a, \quad [a + b, c] = [a, c] + [b, c].$$

Thus we get the exact sequences

$$\Gamma(A) \to A \otimes_{\mathbb{Z}} A \to H_2(A, \mathbb{Z}) \to 0,$$

$$A \otimes_{\mathbb{Z}} A \xrightarrow{[\ ,\ ]} \Gamma(A) \xrightarrow{\Phi} A/2 \to 0,$$

Our second theorem extends the first exact sequence to the left.

Theorem

For any abelian group A, we have the exact sequence

$$0 \to H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)) \to \Gamma(A) \xrightarrow{\Psi} A \otimes_{\mathbb{Z}} A \to H_2(A) \to 0,$$

where $\sigma \in \Sigma_2$ is the natural involution on $\mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)$.

Corollary

For any abelian group A we have the exact sequence

$$H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)) \to A/2 \xrightarrow{\bar{\Psi}} (A \otimes_{\mathbb{Z}} A)_\sigma \to H_2(A, \mathbb{Z}) \to 0,$$

where $(A \otimes_{\mathbb{Z}} A)_\sigma := (A \otimes_{\mathbb{Z}} A)/\langle a \otimes b + b \otimes a : a, b \in A \rangle$ and $\bar{\Psi}(\bar{a}) = \overline{a \otimes a}$.

Eilenberg-Maclane in 1954 proved: For any abelian group A,

$$\Gamma(A) \simeq H_4(K(A, 2), \mathbb{Z}),$$

where $K(A, 2)$ is the Eilenberg-Maclane space.

Third homology of H-groups:

A perfect group Q is called an H-group if $K(Q,1)^+$ is an H-space, where $K(Q,1)^+$ is the plus construction of $BQ = K(Q,1)$ with respect to Q.

Our third theorem is as follow:

Theorem

Let $A \rightarrowtail G \twoheadrightarrow Q$ be a perfect central extension. If Q is an H-group, then we have the exact sequence

$$A/2 \to H_3(G,\mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G,\mathbb{Z})) \to H_3(Q,\mathbb{Z}) \to 0,$$

where $A/2$ satisfies in the exact sequence

$$H_1(\Sigma_2, \mathrm{Tor}_1^{\mathbb{Z}}({}_{2^\infty}A, {}_{2^\infty}A)) \to A/2 \xrightarrow{\bar{\Psi}} (A \otimes_{\mathbb{Z}} A)_\sigma \to H_2(A,\mathbb{Z}) \to 0.$$

Sketch of proof:
1) From the central extension and the fact that Q is perfect we obtain the fibration of Eilenberg Maclane spaces

$$K(A,1) \to K(G,1)^+ \to K(Q,1)^+$$

2) From this we obtain the fibration

$$K(G,1)^+ \to K(Q,1)^+ \to K(A,2)$$

Note that $K(A,2)$ is an H-space

3) We show that $K(Q,1)^+ \to K(A,2)$ is an H-map.

4) Since the plus construction does not change the homology of the space, from the Serre spectral sequence of the above fibration, we obtain the exact sequence

$$H_4(Q,\mathbb{Z}) \to H_4(K(A,2),\mathbb{Z}) \to H_3(G,\mathbb{Z})/\rho_*(A \otimes_{\mathbb{Z}} H_2(G,\mathbb{Z})) \to H_3(Q,\mathbb{Z}) \to 0.$$

5) From the commutative diagram, up to homotopy, of H-spaces and H-maps

$$\begin{array}{ccc} BQ^+ \times BQ^+ & \longrightarrow & BQ^+ \\ \downarrow & & \downarrow \\ K(A,2) \times K(A,2) & \longrightarrow & K(A,2), \end{array}$$

we obtain the commutative diagram

$$\begin{array}{ccc} H_2(Q,\mathbb{Z}) \otimes_{\mathbb{Z}} H_2(Q,\mathbb{Z}) & \longrightarrow & H_4(Q,\mathbb{Z}) \\ \downarrow & & \downarrow \\ A \otimes_{\mathbb{Z}} A & \longrightarrow & H_4(K(A,2),\mathbb{Z}). \end{array}$$

6) Since G is perfect,

$$H_2(Q, \mathbb{Z}) \to A$$

is surjective. Thus the diagram implies that the elements $[a, b] \in H_4(K(A, 2), \mathbb{Z})$ are in the image of $H_4(Q, \mathbb{Z})$.

This gives us the surjective map

$$A/2 \simeq H_4(K(A, 2), \mathbb{Z})/H \twoheadrightarrow H_4(K(A, 2), \mathbb{Z})/\mathrm{im}(H_4(Q, \mathbb{Z})),$$

where H is generated by the elements

$$[a, b] \in \Gamma(A) = H_4(K(A, 2), \mathbb{Z}).$$

This together with previous Corollary prove the theorem.$\diamond$

Thank You

Grazie