An identity having b-generalized skew derivations on multilinear polynomials

BALCHAND PRAJAPATI

अम्बेडकर विश्वविद्यालय दिल्ली
AMBEDKAR UNIVERSITY DELHI

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E. C. Posner, 1957

Thm 1 In a prime ring of characteristics not 2, if the iterate of two derivations is a derivation, then one of them is zero;

Thm 2 If d is a derivation of a prime ring such that, for all elements x of the ring, $xd(x) - d(x)x$ is central, then either the ring is commutative or d is zero.

Definitions:

A. Derivation d on a ring R is an additive mapping satisfying $d(xy) = d(x)y + xdy$ for all $x, y \in R$.

B. Skew derivation d associated with an automorphism α on a ring R is an additive mapping satisfying $d(xy) = d(x)y + \alpha(x)dy$ for all $x, y \in R$.

C. An additive mapping G from a ring R to R is said to be generalized derivation associated with a derivation d if $G(xy) = G(x)y + xd(y)$, for all $x, y \in R$.

D. An additive mapping G from a ring R to R is said to be generalized skew derivation associated with a skew derivation d and an automorphism α if $G(xy) = G(x)y + \alpha(x)d(y)$, for all $x, y \in R$.

Examples:

1. Ordinary Derivative on polynomial ring is a derivation.
2. The mapping $I_a(x) = [a, x]$ for all x, is a derivation, called inner derivation.
3. The mapping $G(x) = x + dx$, for all x, is a generalized derivation.
4. The mapping $G(x) = ax + \alpha(x)b$ for all x, is generalized skew derivation called generalized skew inner derivation.

b-generalized skew derivation:

For a prime ring R we have its maximal right ring of quotients which is called **Utumi's ring of quotients** U. The center of U, denoted by C, is said to be extended centroid of R.

E. Let $b \in U$. An additive mapping G from a ring R to R is said to be b-generalized skew derivation associated with a linear map $d : R \to R$ and an automorphism α of R if $G(xy) = G(x)y + b\alpha(x)d(y)$, for all $x, y \in R$.

- **Example:** The mapping $G : R \to R$ defined as $G(x) = ax + b\alpha(x)u$, for all $x \in R$ and for some $a, u \in R$ is a b-generalized skew derivation.

De Filippis, Vincenzo; Wei, Feng, 2017

- Let R be a prime ring, $\alpha \in Aut(R)$, $0 \neq b \in U$ and $G : R \to R$ be a b-generalized skew derivation associated with a linear map $d : R \to R$ then d becomes a skew derivation associated with automorphism α.
- Above b-generalized skew derivation G can be uniquely extended to U and assumes the form $G(x) = ax + bd(x)$, $a \in U$.

Multilinear Polynomial

- Let $\mathbb{Z}\langle X\rangle$ be the free algebra on the set $X = \{x_1, x_2, \dots\}$ over $\mathbb{Z}$. Let $f = f(x_1, \dots, x_n) \in \mathbb{Z}\langle X\rangle$ be a polynomial. Let R be a ring and $\phi \neq S \subset R$. We say that f is a polynomial identity on S if $f(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in S$. A polynomial $f = f(x_1, \dots, x_n) \in \mathbb{Z}\langle X\rangle$ is said to be multilinear if it is linear in every $x_i, 1 \leq i \leq n$.

Polynomial Identity, derivation and ring

1. I, R and U satisfy the same generalized polynomial identity with coefficients in U, [Chuang [2]].
2. I, R and U satisfy the same differential identity with coefficients in U, [Lee [3]].
3. Let R be a prime ring and $\alpha \in Aut(R)$ be an outer automorphism of R. If $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R then R also satisfies the non trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates, [Kharchenko [4]].

Polynomial Identity, derivation and ring

4. If $f(x_i, d(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring R, d is an outer skew derivation and α is an outer automorphism of R then R also satisfies the generalized polynomial identity $f(x_i, y_i, z_i)$, where x_i, y_i, z_i are distinct indeterminates, [Chuang and Lee [5]].

Main Theorem

Let R be a prime ring of char$\neq 2$ with center $Z(R)$ and F, G be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that $P \notin Z(R)$ s. t.

$$[P, [F(f(r)), f(r)]] = [G(f(r)), f(r)]$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) $\exists\ \lambda, \mu \in C$ s. t. $F(x) = \lambda x$, $G(x) = \mu x\ \forall\ x \in R$,

(2) $\exists\ a, b \in U$, $\lambda, \mu \in C$ s. t. $F(x) = ax + \lambda x + xa$, $G(x) = bx + \mu x + xb\ \forall\ x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

Corollary 1

Let R be a prime ring of char$\neq 2$ with center $Z(R)$ and F be b-generalized skew derivations on R. Let U be Utumi quotient ring of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R s. t.

$$[F(f(r)), f(r)] \in Z(R)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:

(1) $\exists\ \lambda \in C$ s. t. $F(x) = \lambda x\ \forall\ x \in R$,

(2) $\exists\ a \in U$, $\lambda \in C$ s. t. $F(x) = ax + \lambda x + xa\ \forall\ x \in R$ and $f(x_1, \ldots, x_n)^2$ is central valued on R.

Corollary 2

Let R be a prime ring of characteristic different from 2 and d be a skew derivation on R such that $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is a commutative ring.

Corollary 3

Let R be a prime ring of characteristic different from 2 and α be an automorphism on R such that $[\alpha(x), x] \in Z(R)$ for all $x \in R$, then either α is an identity automorphism or R is a commutative ring.

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Thank You