Some new classes to tackle Enochs' conjecture

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This is base on joint work with Alberto Facchini appearing in the following papers:

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1) Equivalence of some homological conditions for ring epimorphisms, J. Pure Appl.
2) Covering classes, strongly flat modules, and completions, ArXiv:1808.02397.

Precover and Cover class in Mod-R

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The class $\mathcal{C}$ is called covering if every module has a cover from $\mathcal{C}$.

Direct set

Direct set: If I is a set with relation $\leq$ (reflexive and binary transitive), then $< I, \leq >$ is called a direct set if every pair of elements of I has an upper bound.
That is if $i, j \in I$, there exists $k \geq i, j$

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3) $f_{jk} f_{ij} = f_{ik}$, where $i \leq j \leq k$.

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AND if there exist a Y with $g_i : C_i \to Y$ with commutative diagram $\begin{tikzcd} C_i \arrow[r, "f_{ij}"] \arrow[dr, "g_i"'] & C_j \arrow[d, "g_j"] \\ & Y \end{tikzcd}$

Direct limit of direct system

Then there is unique map $h : X \to Y$ such that

$$\begin{tikzcd} C_i \arrow[r, "\phi_i"] \arrow[dr, "g_i"'] & X \arrow[d, "h"] \\ & Y \end{tikzcd}$$

is commutative.

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Flat module: A module is flat if it is a direct limit of a direct sysytem of projectives.

Right Perfect ring = Every Right module has projective cover.

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Since every direct limit of flats is flats, it was a conjecture for years if the class of flats is covering....

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Enochs conjecture: Is the converse of Fact (2) true? That means is a covering class closed under direct limit?

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For every flat module S, consider the class ${}^{\perp}(S^{\perp})$.
This class of modules lies between the class of projective modules and that of flat modules. It is called the class of S-strongly flat modules, denoted by $\mathcal{SF}$.

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3) If $\mathrm{Ext}^1(S, S^{(I)}) = 0$, for any index set I, the class S-strongly flat modules contains the modules which are direct summand of extensions of free modules by some copies of S.
4) The main question is this when this class is a covering?

The idea of such defination comes from?

If Q is the quotient field of domain R. In 2000, Jan Trlifaj called the class ${}^{\perp}(Q^{\perp})$ strongly flat. He left an open problem in his book: When is the class of strongly flat modules a covering class?

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Generalization to non-commutative ring

Assume that R is a ring (not necessary commutative) and $f : R \to S$ is ring homomorphism with the following properties:
(1) f is injective ($=$ one to one).
(2) epimorphism in category of rings, that is for ring homomorphisms $g, h : S \to T$, $gf = hf$ implies $g = h$.
(3) ${}_R S$ is flat.

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(3) ${}_RS$ is flat.
So with this S, we have the class of S-strongly flat modules. When this call is covering?

Using some homological properties, we can see that if the class strongly flat (here S is a left flat ring epimorphism of R) is covering,

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In commutative case the converse is true but in general not.

Uniserial modules

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a domain R which is uniserial as right R-module is called right chain domain. For such a domain I have classical quotient ring S such that elements of S are in form rs^{-1} where $r, s \in R$ and s is nonzero. ${}_RS$ is flat and the inclusion is a ring bimorphism.

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(1) R invariant, that is $Rx = xR$, for every $x \in R$.
(2) Flats are strongly flats.

Thanks!