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Noetherian profinite groups

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Contents

Some notation

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- When we deal with subgroups of a profinite group, we will always mean *closed* subgroups.
- By compactness, each open subgroup has finite index.
- In general we can consider a pro-$\mathcal{C}$ group G, where $\mathcal{C}$ is any class of finite groups, requiring $G/N \in \mathcal{C}$ for any $N \triangleleft_o G$.

Profinite groups

Theorem

A profinite group G is (topologically) isomorphic to the subgroup of $\prod_{N \lhd_o G} G/N$ given by those elements $(gN)_{N \lhd_o G}$ such that $\pi_{N,M}(gN) = gM$ for any $N, M \lhd_o G$ such that $N \leqslant M$.

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- The *pro-*p *completion of the integers* $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is profinite.
- The group $\mathrm{GL}_n(\mathbb{Z}_p) = \varprojlim_k \mathrm{GL}_n\left(\mathbb{Z}/p^k\mathbb{Z}\right)$ is a profinite group.

Order of a profinite group and Sylow subgroups

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Let G be a profinite group, let $H \leqslant_c G$. The *index* of H in G is

$$\operatorname{lcm}\left\{[G : NH] \middle| N \lhd_o G\right\}$$

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Definition

A p-*Sylow subgroup* of a profinite group G is a maximal pro-p subgroup.

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- $\mathbb{Z}_p$ is noetherian.
- The *Nottingham group* over $\mathbb{F}_p$ is the group of formal power series $t + t^2\mathbb{F}_p[[t]]$. It is a pro-p group, but it is not noetherian.

Rank of a profinite group

Denote by $\mathrm{d}(G)$ the minimal number of generators for a group G.

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The *rank* of a profinite group G is one of the following equivalent values:

- $\sup\{\mathrm{d}(H) \mid H \leqslant_c G\}$
- $\sup\{\mathrm{d}(H) \mid H \leqslant_c G,\ d(H) < \infty\}$
- $\sup\{\mathrm{d}(H) \mid H \leqslant_o G\}$
- $\sup\{\mathrm{rk}(G/N) \mid N \lhd_o G\}$

An open problem

Question

Does every noetherian pro-p group have finite rank?

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The converse of this problem is true: in fact, a pro-p group is noetherian if and only if every (closed) subgroup is finitely generated.

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- The theorem follows, having H_i finite index.

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For it, consider the non-empty family of normal subgroups with infinite index: by noetherianity and Zorn's lemma, it admits a maximal element N_0.
So, G/N_0 is just-infinite.

The Dichotomy Theorem

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Branch groups are non-noetherian.

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Corollary

A just-infinite noetherian profinite group is a finite extension of a direct power of a hereditarily just-infinite group.

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Theorem

Let G be a just-infinite profinite group that is not virtually abelian. Then the following are equivalent:

- G is virtually pronilpotent;
- G is virtually pro-p;
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Thanks for your attention!