Right Engel elements in the first Grigorchuk group

Marialaura Noce

University of Salerno - University of the Basque Country

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Outline

1. Engel groups
2. Automorphisms of a d-adic tree
3. The Grigorchuk group
4. Engel elements in the Grigorchuk group
5. Generalization

Engel groups

- Let G be a group. We say that $g \in G$ is a *right Engel element* if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[g, {}_n x] = 1$, where

$$[g,x] = g^{-1}g^x \text{ and } [g, {}_n x] = [[g, x, \overset{n-1}{\ldots}, x], x] \text{ if } n > 1.$$

- Similarly g is left Engel if for any $x \in G$, $\exists n = n(g,x) \geq 1$ such that $[x, {}_n g] = 1$.
- The group G is said to be an Engel group if all its elements are right Engel or left Engel.
- We will denote by $L(G)$ and $R(G)$ the sets of left and right Engel elements of G.

Are $L(G)$ and $R(G)$ subgroups?

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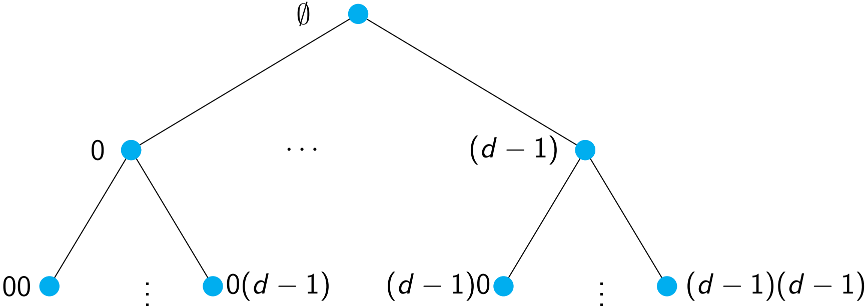
Automorphisms of a d-adic tree

A tree is a connected graph with no cycles.



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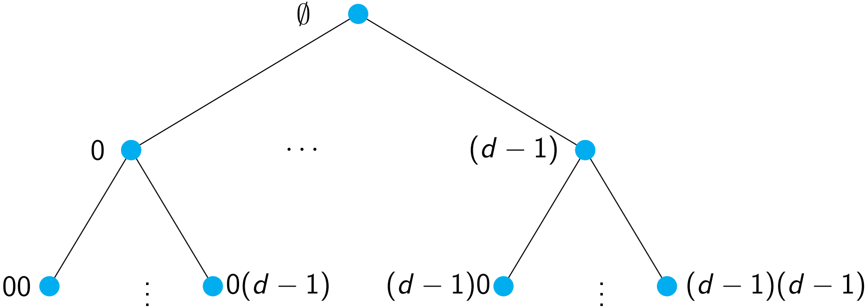

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Automorphisms of a d-adic tree

- A *vertex* is a word in X^*, i.e. the set of all words in the alphabet $X = \{0, \ldots, d-1\}$. Moreover X^n is the set of all words of length n.
- An automorphism is a bijective map from X^* to X^* which preserves incidence.
- The set of all of these automorphisms is a group

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If $g \in \operatorname{Aut} \mathcal{T}$, then:

(i) $g(\emptyset) = \emptyset$;

(ii) $g(X^n) = X^n$.

- The *label* of g at the vertex u is a permutation of S_d which describes how g acts on the descentants of u.
- The *portrait* of g is the set of all labels of g.

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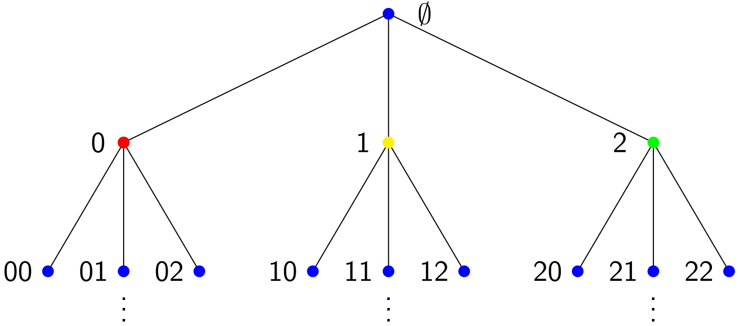

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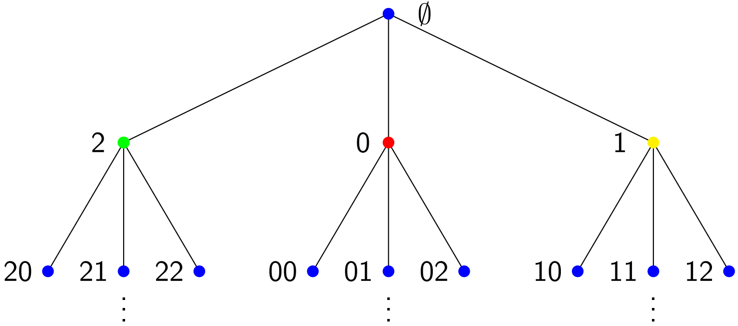

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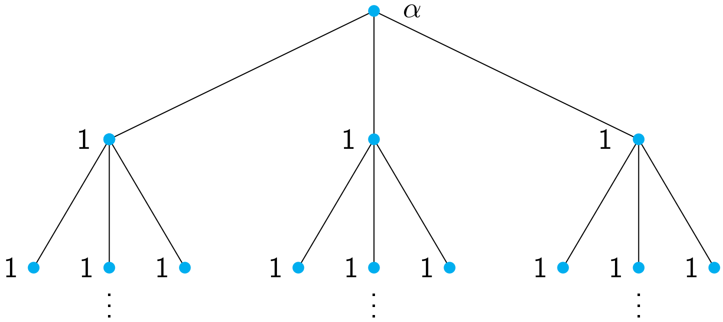
Let $\alpha = (012)$. The portrait of this automorphism is



A subgroup of Aut $\mathcal{T}$

If u is a vertex of $\mathcal{T}$, the *stabilizer* of u is:

$$\mathrm{st}(u) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u\}.$$

We can generalize and define stabilizers of levels:

$$\mathrm{st}(n) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u \;\forall u \in X^n\}.$$

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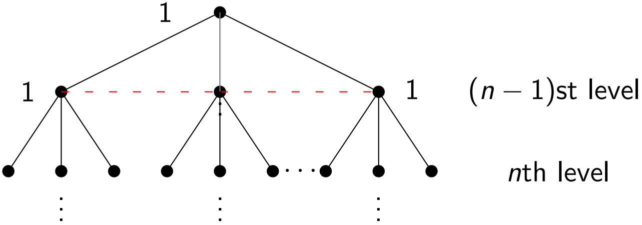

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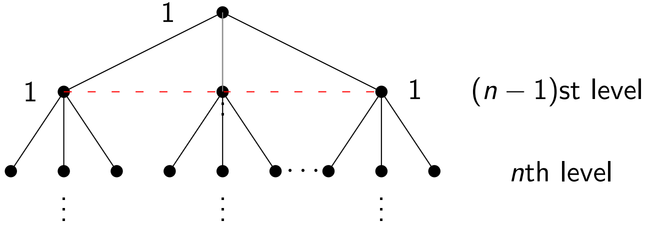

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Facts

If $n \in \mathbb{N}$, we can define a map

$$\psi_n : \mathrm{st}(n) \longrightarrow \operatorname{Aut} \mathcal{T} \times \overset{d^n}{\cdots} \times \operatorname{Aut} \mathcal{T}.$$

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$$\Gamma = \langle a, b, c, d \rangle$$

$$a = (01),\ \psi(b) = (a, c),\ \psi(c) = (a, d)\ \psi(d) = (1, b)$$

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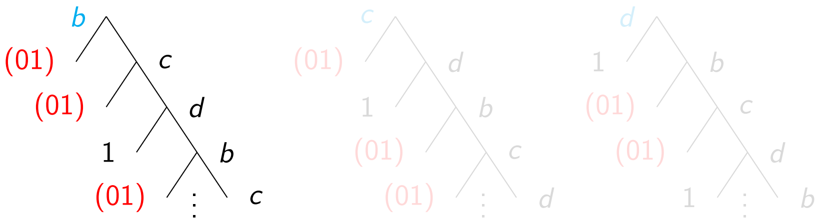

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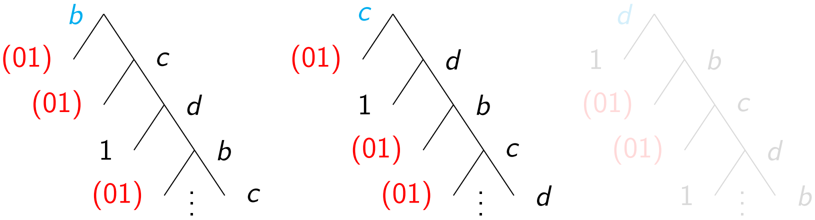

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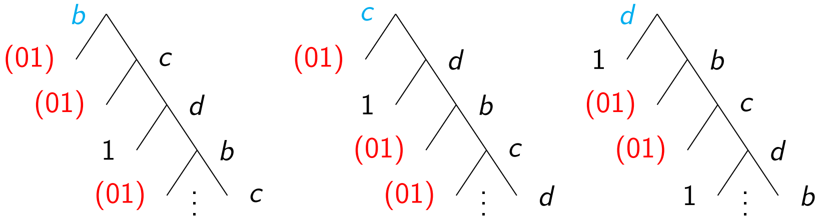

The Grigorchuk group

$$\Gamma = \langle a \rangle \ltimes \mathrm{st}_\Gamma(1)$$

Γ has the following properties:

- It is finitely generated
- It is a 2-group
- It is infinite
- $\psi : \mathrm{st}_\Gamma(1) \longrightarrow \Gamma \times \Gamma$
  It is regular branch over $K = \langle [a, b] \rangle^\Gamma$ (i.e. $\psi(K) \supseteq K \times K$)

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The set $L(G)$: Bludov's example

- Let $K = \langle g \rangle$, where $o(g) = 4$. We denote by $G = \Gamma \ wr \ K$. There exists
$$h = (1, ab, ca, d) \in (\Gamma \times \Gamma \times \Gamma \times \Gamma)$$
such that $[h,_n g] \neq 1$ for any $n \geq 1$.
- As a consequence, $H = \Gamma \ wr \ D_8$ is not an Engel group.
- Every involution in a 2-group is a left Engel element, then the set of left Engel element is not a subgroup.

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The set $L(\Gamma)$

Theorem (Bartholdi, 2015)

The Grigorchuk group is not Engel.

Right Engel elements in Γ

Theorem (Fernández-Alcober, N.)

The only right Engel element in the Grigorchuk group is the identity, i.e., $R(\Gamma) = \{1\}$.

Right Engel elements in Γ

Key facts used during the proof

- Γ is regular branch over K;
- K contains an element that is not left Engel.

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Let $G \leq \mathrm{Aut}\,\mathcal{T}$ be a group of automorphisms of a rooted p-adic tree. Then $R(G) = 1$ if:

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For example, one can prove that $R(\mathrm{Aut}\,\mathcal{T}) = 1$. Take $K = \mathrm{st}(1)$.

Open problem

Is there any left Engel element in K?

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