Groups with Permutability Conditions on Subgroups of Infinite Rank

Anna Valentina De Luca
Università degli Studi della Campania Luigi Vanvitelli

Young Researchers Algebra Conference 2017

Napoli - May 23$^{\text{rd}}$, 2017

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- If G is a finite group and H is a permutable subgroup of $G \implies H$ is subnormal in G (Ore, 1937).
- If G is an arbitrary group and H is a permutable subgroup of $G \implies H$ is ascendant in G (Stonehewer, 1972).

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- If G is a quasihamiltonian group $\Longrightarrow$ G is locally nilpotent.
- If G is a periodic quasihamiltonian group $\Longrightarrow$ G is locally finite and it is the direct product of its primary components.

Theorem (Iwasawa, 1943)

Let p be a prime. A locally finite p-group G is quasihamiltonian if and only if

(i) *Every subgroup of G is normal, or*

(ii) *G contains an abelian normal subgroup A such that G/A is a finite cyclic group.*

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Theorem (Iwasawa, 1943)

Let G be a non-periodic quasihamiltonian group. Then:

(i) *The set T of all elements of finite order of G is an abelian subgroup of G and the factor group G/T is abelian.*

(ii) *Every subgroup of T is normal in G.*

(iii) *Either G is abelian or G/T is locally cyclic.*

Definition

A group G is said to have finite (Prüfer) rank r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. If such an r does not exist, we will say that the group G has infinite rank.

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- A group G has rank 1 if and only if it is locally cyclic.

Let G be a group of infinite rank and let Θ be a property concerning the subgroups of a group. If every subgroup of infinite rank of G satisfies Θ, can we say that Θ holds for all subgroups of G?

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- Finitely generated groups are "small".

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- If a group G contains a subgroup of infinite rank, then G has infinite rank.
- If N is a normal subgroup of a group G of infinite rank, then at least one of groups N and G/N has infinite rank.

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- It is true for different choices of Θ.

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Theorem (Dixon, Karatas, 2012)

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A group G is called generalized radical if it has an ascending series whose factors are either locally nilpotent or locally finite.

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$$|K:H| < \infty \qquad and \qquad KperG$$

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- A subgroup H of a group G is said to be almost normal if it has only finitely many conjugates in G or, equivalently, if H is normal in a subgroup of finite index of G.

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- If $G/Z(G)$ is finite $\Longrightarrow$ G' is finite (Schur, 1902).

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Theorem (De Falco, de Giovanni, Musella, Sysak, 2003)

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Theorem (A.V.D.L, Ialenti, 2017)

Let G be a locally finite group of infinite rank whose subgroups of infinite rank are almost permutable. Then every subgroup of G is almost permutable.

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1. G p-group.
2. G locally nilpotent.
3. G is (locally nilpotent)-by-finite.

Theorem (de Giovanni, Musella, Sysak, 2001)

Let G be a non-periodic group whose subgroups are almost permutable.

(i) *The set T of all elements of finite order of G is a normal subgroup of G and the factor group G/T is abelian.*

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(iii) *Either G is an FC-group or G/T is locally cyclic.*

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- If H is normal in K, we will say that X is finite-normal-finite.

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Corollary (A.V.D.L, Ialenti, 2017)

Let G be a non-periodic group whose subgroups are finite-permutable-finite and let T be the torsion subgroup of G. Then every subgroup of T is nearly normal in G.

Theorem (A.V.D.L, Ialenti, 2017)

Let G be a non-periodic generalized radical group of infinite rank whose subgroups of infinite rank are finite-permutable-finite. Then:

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Let G be a group as in the previous theorem whose subgroups of infinite rank are almost permutable (resp. nearly permutable).

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- (i) and (iii) hold.

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- (i) and (iii) hold.
- If the subgroups of infinite rank of G are almost permutable (resp. nearly permutable), every subgroup of T is almost normal (resp. nearly normal) in G.

$\mathcal{T}$hanks for $\mathcal{A}$ttention