Groups in which every proper subgroup of Infinite Rank is Finite Rank-by-Hypercentral or Hypercentral-by-Finite Rank

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YRAC 2017

Richard Brauer " A tremendous effort has been made by mathematicians for more than a century to clear up the chaos in group theory. Still, we cannot answer some of the simplest questions."

Class of large groups

Following the context of **M. de Falco – F. de Giovanni – C. Musella, (2014)**, the notion of a "large group" can be formalized in the following way :

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Let $\mathcal{X}$ be a class of groups. Then $\mathcal{X}$ is said to be a class of large groups if it satisfies the following conditions :

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2. *If N is a normal subgroup of an $\mathcal{X}$-group G, then at least one of the groups N and G/N belongs to $\mathcal{X}$.*

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1. *If a group G contains an $\mathfrak{X}$-subgroup, then G belongs to $\mathfrak{X}$.*
2. *If N is a normal subgroup of an $\mathfrak{X}$-group G, then at least one of the groups N and G/N belongs to $\mathfrak{X}$.*
3. *No finite cyclic group lies in $\mathfrak{X}$.*

The rank of a group

Definition

- *A group is said to have finite rank r if every finitely generated subgroup can be generated by r elements, and r is the least positive integer with this property. If no such integer r exists then we say that the group has infinite rank.*

- Groups of infinite rank form a class of large groups.

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- C_{p^∞}, $\mathbb{Q}$ are locally cyclic groups, so they have rank 1.
- Every polycyclic group and every Černikov group has finite rank.
- A soluble minimax group has finite rank.
- Any free non-abelian has infinite rank.

Large class controls a subgroup property

Let $\mathcal{X}$ be a class of large groups, and let $\mathcal{P}$ be a property pertaining to subgroups of a group.

Definition

$\mathcal{X}$ controls $\mathcal{P}$ if and only if the following condition is satisfied :

- *G is any $\mathcal{X}$-group and all $\mathcal{X}$-subgroups of G have the property $\mathcal{P}$, then $\mathcal{P}$ holds for all subgroups of G.*

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- The class of cyclic groups controls periodicity.
- The class of finitely generated groups controls commutativity.

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  - $G \in L\mathcal{X}$ if every finite subset of G is contained in an $\mathcal{X}$-subgroup.
  - $G \in \acute{P}\mathcal{X}$ if G has an ascending series each of whose factors is a $\mathcal{X}$-group.

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  - $G \in L\mathcal{X}$ if every finite subset of G is contained in an $\mathcal{X}$-subgroup.
  - $G \in \acute{P}\mathcal{X}$ if G has an ascending series each of whose factors is a $\mathcal{X}$-group.
  - $G \in \grave{P}\mathcal{X}$ if G has a descending series each of whose factors is a $\mathcal{X}$-group.

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- So, we say a class $\mathcal{X}$ of groups is Λ-closed if and only if it is closed under the formation of local systems, subcartesien products and both ascending and descending normal series.
- In **1990**, **N. S. Černikov**, defined a class $\mathfrak{X}$ of groups by taking the Λ-closure of the class of periodic locally graded groups.
- The class $\mathfrak{X}$ is an extensive class of groups containing, in particular, the classes of locally (soluble-by- finite) groups, radical groups and residually finite groups.

Strongly locally graded groups

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Strongly locally graded groups

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- If G is locally graded then is $G \in \mathfrak{X}$?
- What is the structure of locally graded groups of finite rank ?
- The class $\mathfrak{X}$ is closed with respect to forming subgroups.
- All residually finite groups belong to $\mathfrak{X}$, and hence the consideration of any free non-abelian group shows that the class $\mathfrak{X}$ is not closed with respect to homomorphic images.

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Strongly locally graded groups

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- A group G is strongly locally graded if every section of G is a $\mathfrak{X}$-group. In particular, every strongly locally graded is an $\mathfrak{X}$-group.
- Strongly locally graded groups is a larger class of generalized soluble groups, which is closed with respect to subgroups and homomorphic images, and contains all locally (soluble-by-finite) groups.

Groups with P-subgroups of infinite rank

Many authors studied groups of infinite rank whose proper subgroups of infinite rank belong to the group theoretical property $\mathcal{P}$. In particular :

- **M. R. Dixon - M. J. Evans - H. Smith, (1999),** they proved that the class of groups of infinite rank controls the property “ nilpotent of class c $(c \geq 0)$” in the universe of strongly locally graded. Thus :

Theorem

- *If G is a strongly locally graded group of infinite rank such that each proper subgroup of infinite rank of G is is nilpotent of class c $(c \geq 0)$, then so is G.*

- In the same paper **M. R. Dixon - M. J. Evans - H. Smith,** proved also that the class of groups of infinite rank controls the property "locally nilpotent" in the universe of strongly locally graded groups. Thus :

Theorem

- *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are locally nilpotent. Then so is G .*

- **M. De Falco - F. De Giovanni - C. Musella - N. Trabelsi, (2014)**, they proved that the class of groups of infinite rank controls the property "has locally finite commutator subgroup" in the universe of strongly locally graded groups. Thus :

Theorem

- *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are (locally finite)-by-abelian. Then , then so is G.*

Groups with P-subgroups of infinite rank

- **De Giovanni, Trombetti, (2014)**, they proved that the class of groups of infinite rank controls the property "(locally finite)-by-(locally nilpotent)" in the universe of strongly locally graded. Thus :

Theorem

- *Let G be a strongly locally graded group of infinite rank whose proper subgroups of infinite rank are (locally finite)-by-(locally nilpotent). Then so is G.*

Groups with P-subgroups of infinite rank

- These results shows that the subgroups of infinite rank of a (generalized) soluble group of infinite rank have the power to influence the structure of the whole group and to force also the behaviour of the subgroups of finite rank of G.

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- In fact, it has been proved that, for some choices of the group theoretical properties $\mathcal{P}$, if G is a group in which all subgroups of infinite rank satisfy the property $\mathcal{P}$, then the samehappens also to the subgroups of finite rank.

Groups with P-subgroups of infinite rank

- The types of the group theoretical properties $\mathcal{P}$ that we are interested in here are : $\mathcal{N}\mathcal{X}$ (resp. $\mathcal{N}_c\mathcal{X}$, $\mathcal{Z}\mathcal{A}\mathcal{C}$), and their duals $\mathcal{X}\mathcal{N}$ (resp. $\mathcal{X}\mathcal{N}_c$, $\mathcal{C}\mathcal{Z}\mathcal{A}$), where $\mathcal{N}$, $\mathcal{N}_c$, $\mathcal{Z}\mathcal{A}$, $\mathcal{C}$, are respectively, the class of nilpotent, nilpotent of class c $(c \geq 1$ an integer$)$, hypercentral, Černikov, and $\mathcal{X}$ is any class contained in the class of finite rank groups.

Groups with P-subgroups of infinite rank

- Does the class of groups of infinite rankcontrols the group theoretical propert $\mathcal{P}$ in the universe of strongly locally graded groups ? In other words, if G is a group of infinite rank whose proper subgroups of infinite rank belong to $\mathcal{P}$. Then all proper subgroups of G in $\mathcal{P}$? Does G itself belong to $\mathcal{P}$?

Groups with P-subgroups of infinite rank

- Does the class of groups of infinite rankcontrols the group theoretical propert $\mathcal{P}$ in the universe of strongly locally graded groups ? In other words, if G is a group of infinite rank whose proper subgroups of infinite rank belong to $\mathcal{P}$. Then all proper subgroups of G in $\mathcal{P}$? Does G itself belong to $\mathcal{P}$?
- The description of minimal non-$\mathcal{P}$ groups for a certain group class $\mathcal{P}$ can be considered as the first step in the investigation of groups of infinite rank. We shall say that a group G is minimal non-$\mathcal{P}$ if it is not $\mathcal{P}$-group, but all its proper subgroups belong to $\mathcal{P}$.

Main results

We denote by $(\mathcal{X})^*$ the class of groups in which every proper non-$\mathcal{X}$ subgroup has finite rank.

- We have shown that the class of infinite rank groups controls the property $\mathcal{X}\mathcal{N}$ (resp. $\mathcal{X}\mathcal{N}_c$) in the universe of strongly locally graded groups, where $\mathcal{X}$ is a class of group such that $\mathcal{X} \subseteq \mathcal{R}$ and $S\mathcal{X} = \mathcal{X}$ (i.e, $\mathcal{X}$ is closed under taking subgroups).Thus :

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Theorem

- *If G is a strongly locally graded group of infinite rank such that each proper subgroup of G is in the class $(\mathcal{X}\mathcal{N})^*$ (resp. $(\mathcal{X}\mathcal{N}_c)^*$), then each proper subgroup of G is an $\mathcal{X}\mathcal{N}$-group (resp. $\mathcal{X}\mathcal{N}_c$-group).*

- The preceding results admits the following consequences :

Let G be a strongly locally graded group of infinite rank.

- By combining the result of **N. Trabelsi - A. Zitouni** and the previous result, we have : if G belong to the class $(\mathcal{MN})^*$, then G is a $\mathcal{MN}$-group or $MN\mathcal{N}$ without maximal subgroups.
- By combining the result of **N. Trabelsi - A. Zitouni** and the previous result, we have : if G belong to the class $(\mathcal{MN}_c)^*$, then so is G.

Main results

- We have shown that the class of groups of infinite rank controls the property "$\mathcal{X}$" in the universe of strongly locally graded groups, where $\mathcal{X}$ satisfy $\mathcal{ZA} \subseteq \mathcal{X} \subseteq \mathcal{LN}$. Thus :

Theorem

If G is a strongly locally graded group of infinite rank such that $G \in (\mathcal{X})^*$. Then every proper subgroup of G is an $\mathcal{X}$-group.

As a consequences of the preceding result, we have the following :

- Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank of G are in the class Gruenberg (resp. normaliser condition), then all proper subgroups of G are Gruenberg (resp. normaliser condition).

Main results

- We have shown that the class of infinite rank groups controls the property "$\mathcal{CZA}$" in the universe of strongly locally graded groups. Thus :

Theorem

If G is a strongly locally graded group of infinite rank such that $G \in (\mathcal{CZA})^*$, then all proper subgroups of G are $\mathcal{CZA}$-groups.

- We proved that the class of infinite rankgroups controls the property "$\mathcal{ZAC}$" in the universe of strongly locally graded groups. Thus :

Theorem

If G is a strongly locally graded group of infinite rank such that $G \in (\mathcal{ZAC})^*$, then all proper subgroups of G are $\mathcal{ZAC}$-groups.

- We proved that the class of infinite rank groups controls the property $\mathcal{NC}$ (*resp.* $\mathcal{NPF}, \mathcal{NM}$) in the universe of strongly locally graded groups. Thus :

Theorem

If G is a strongly locally graded group of infinite rank such that G belong to $(\mathcal{NC})^*$(resp. $(\mathcal{NPF})^*, (\mathcal{NM})^*$), then all proper subgroups of G are $\mathcal{NC}$ (resp. $\mathcal{NPF}, \mathcal{NM}$)-groups.

As a consequences :
Let G be a strongly locally graded group of infinite rank.

- Using **A. O. Asar (2000)** result, we get : if G belong to the class $(\mathcal{NC})^*$, then so is G.
- Using **N. Trabelsi - A. Zitouni** result, we get : if G belong to the class $(\mathcal{NM})^*$, then so is G.
- **S. Franciosi - F. De Giovanni - Y. P. Sysac, (1999),** result, shows that a locally (polycyclic-by-finite) group whose proper subgroups belong to the class $\mathcal{NPF}$, then either G is $\mathcal{NPF}$ or locally finite countable group.
- Locally graded whose proper subgroups are $\mathcal{NPF}$. **(Future work)**.

- By using the result of **B. Bruno - F. Napolitani,(2004),** we proved the following :

Theorem

If G is a strongly locally graded group of infinite rank such that $G \in (\mathcal{N}_c\mathcal{C})^*$, then so is G.

Future works :

- The class of groups of infinite rank controls the property $\mathcal{N}_c\mathcal{PF}$ (resp. $\mathcal{N}_c\mathcal{M}$) in the universe of strongly locally graded groups.
- Locally graded group whose proper subgroups are $\mathcal{N}_c\mathcal{PF}$ (resp. $\mathcal{N}_c\mathcal{M}$).

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Thank you !