An elementary proof of Graham's theorem

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Definition

Let S be a semigroup. Then:

- An element $e \in S$ is said to be *idempotent* if $e^2 = e$. $\mathrm{E}(S) := \{e \in S \text{ idempotent}\}$.
- Let $e \in S$ idempotent. S_e denotes the maximal subgroup in S which has e as an identity element.
- For every subset $X \subseteq S$, $X^0 = X \cup \{0\}$ and $X^1 = X \cup \{1\}$.
- $x \in S$ is *regular* (in the sense of von Neumann), if $x \in xSx$. A semigroup S is said to be *regular* if every element $s \in S$ is regular.

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Green Relations

The following equivalence relations introduced by Green are fundamental:

$$\begin{aligned} s\,\mathcal{R}\,t &\text{ if and only if } sS^1 = tS^1 \\ s\,\mathcal{L}\,t &\text{ if and only if } S^1s = S^1t \\ s\,\mathcal{J}\,t &\text{ if and only if } S^1sS^1 = S^1tS^1 \end{aligned}$$

Given an element $u \in S$, we denote by R_u (L_u, J_u) the corresponding $\mathcal{R}$-class of u ($\mathcal{L}, \mathcal{J}$-class). Moreover, a $\mathcal{K}$-class ($\mathcal{K} = \mathcal{J}, \mathcal{L}, \mathcal{R}$), K, is said to be regular if it contains an idempotent, i.e. $K = K_e$ for $e \in \mathrm{E}(S)$.

Proposition

Let S be a semigroup (not necessarily finite). Then:

1. If S is regular, then $\langle \mathrm{E}(S) \rangle$ is a regular subsemigroup of S.
2. If T is a regular subsemigroup of S. Then, $\mathcal{K}_T = \mathcal{K}_S \cap (T \times T)$, where $\mathcal{K} = \mathcal{L}$ or $\mathcal{R}$.

W. Eberhart, C. Williams and L. Kinch, *Idempotent-generated regular semigroups*, J. Austral. Math. Soc. Ser. A, 15 (1973), 27–34.

D. Rees, *On semigroups*, Math. Proc. Cambridge Philos. Soc., 36 (1940), 387–400.

Definition

A semigroup S with zero is called 0-*simple* if $SxS = S$ for all $0 \neq x \in S$.

Note that every 0-simple semigroup S has a unique non-zero $\mathcal{J}$-class, $S \setminus \{0\}$, i.e. $s \, \mathcal{J} \, t$ for all non-zero $s \neq t \in S$.

Proposition

Let S be a 0-simple semigroup. Then

1. S is regular.
2. S_e is isomorphic to S_f for all $e, f \in \mathrm{E}(S) \setminus \{0\}$.
3. $S_e = (eSe) \setminus \{0\}$, for all $0 \neq e \in \mathrm{E}(S)$.

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Let A, B be non-empty sets and let G be a group. A *Rees matrix* C is a map $C : B \times A \longrightarrow G^0$. We say that the Rees matrix is *regular* if every row and every column has a non-zero entry.

Rees matrix semigroup

The *Rees matrix semigroup* with sandwich matrix C is the semigroup $\mathcal{M}^0(G, A, B, C)$ with underlying set $(A \times G \times B) \cup \{0\}$ and the operation: $0 \cdot (a, g, b) = (a, g, b) \cdot 0 = 0$, and

$$(a, g, b) \cdot (a', g', b') = \begin{cases} (a, gC(b, a')g', b') & \text{if } C(b, a') \in G, \\ 0, & \text{if } C(b, a') = 0, \end{cases}$$

for all $a, a' \in A$, $b, b' \in B$, $g \in G$.

The Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$ is regular if, and only if, the matric C is regular.

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Rees' Theorem [Rees40]

Every regular Rees matrix semigroup is a 0-simple semigroup. Conversely, every 0-simple semigroup S is isomorphic to a regular Rees matrix semigroup $\mathcal{M}^0(G, A, B, C)$, where G is isomorphic to the maximal subgroups S_e, for all $e \neq 0$.

Given a regular $\mathcal{J}$-class J of a semigroup S, set $J^0 = J \cup \{0\}$ and the operation:

$$\text{for all } a, b \in J \quad a \cdot b = \begin{cases} ab, & \text{if } ab \in J, \\ 0, & \text{if } ab \notin J \end{cases}, \quad a \cdot 0 = 0 \cdot a = 0$$

Then, J^0 forms a 0-simple semigroup and:

$$J^0 \cong \mathcal{M}^0(G, A, B, C)$$

where $G \cong S_e = (J^0)_e$, for all $e \in J$.

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Graham's Theorem

Let S be a 0-simple semigroup. Then there exists an isomorphism $\psi : S \longrightarrow \mathcal{M}^0(G, A, B, C)$ such that:

$$C = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} & \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$

where each matrix $C_i : B_i \times A_i \to G^0$ is regular and:

$$\langle \mathrm{E}(S) \rangle = \bigcup_{i=1}^{n} \mathcal{M}^0(G_i, A_i, B_i, C_i)$$

where G_i is the subgroup of G generated by all non-zero entries of C_i, for $i = 1, \ldots, n$.

In the sequel, S will denote a 0-simple semigroup and $T := \langle \mathrm{E}(S) \rangle$. We use:

Key Lemma 1, [Rees40, Lemmas 2.61, 2.62, 2.63]

For each pair of non-zero idempotents e and f of S, eSf is non-zero. Moreover, if $0 \neq x \in eSf$ and $0 \neq y \in fSg$, then $0 \neq xy \in (eSf)(fSg) = eSg$.

Key Lemma 2, [Rees40, Lemma 2.7]

Let $e, f \in \mathrm{E}(S) \setminus \{0\}$. The sets eS and fS have either no non-zero elements in common or are identical. Similarly for the sets Se and Sf and, consequently, for the sets eSf and $e'Sf'$.

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From wich we prove:

Corollary 1

Let $0 \neq ef \in T$ with $e, f \in \mathrm{E}(S)$. Then $e \,\mathcal{R}_T\, ef \,\mathcal{L}_T\, f$. In particular, $e \,\mathcal{J}_T\, f \,\mathcal{J}_T (ef)$.

Corollary 2

Assume that $0 \neq e_1 \cdots e_r \in T$ for some $e_i \in \mathrm{E}(S)$, $1 \leq i \leq r$. Then $e_1 \,\mathcal{J}_T \ldots \mathcal{J}_T\, e_r \,\mathcal{J}_T (e_1 \cdots e_r)$.

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We split the proof into the following steps.

1. Let $(J_T)_{e_1}, \ldots, (J_T)_{e_n}$ be the non-zero $\mathcal{J}$-classes in T, with $e_1, \ldots, e_n \in \mathrm{E}(S)$. For each $k \in \{1, \ldots, n\}$, we write:

$$T^{(k)} = (J_T)^0_{e_k} \subseteq T$$

Corollary 1 ensures us that $T^{(k)}$ is a 0-simple subsemigroup of T.

2. Since S is regular and using Key Lemma 2, we can write:

$$S = \bigcup_{i=1, j=1}^{m_1, m_2} r_i S l_j \quad \text{where } r_i, l_j \in \mathrm{E}(S) \setminus \{0\},$$

with r_i, l_j such that $r_i S l_j \cap r_{i'} S l_{j'} = 0$ if $i \neq i'$ and $j \neq j'$. Let $r_1 = l_1 = e_1$.

3. Set $A := \{1, \ldots, m_1\}$ and $B := \{1, \ldots, m_2\}$. For each $1 \leq k \leq n$, we define: $A_k := \{i \in A : r_i \, \mathcal{J}_T \, e_k\}$, $B_k := \{j \in B : l_j \, \mathcal{J}_T \, e_k\}$.

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4. Let $k \in \{1, \dots, n\}$. Applying Key Lemma 1 we have:
   - There exist non-zero elements $x_{1k} \in e_1 S e_k$, $x_{k1} \in e_k S e_1$, such that for all $i \in A_k$, $j \in B_k$, we can take non-zero elements $\bar{p}_{ik} \in r_i T^{(k)} e_k$, $\bar{q}_{kj} \in e_k T^{(k)} l_j$, with

$$0 \neq x_{1k}\bar{q}_{kj} \in e_1 S l_j, \quad 0 \neq \bar{p}_{ik} x_{k1} \in r_i S e_1$$

5. Then, for all $i \in A$ and $j \in B$, we define:

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6. We have seen that the maximal subgroups S_e, for all $0 \neq e \in \mathrm{E}(S)$, are all isomorphic. Let $G^0 := e_1 S e_1 = (S_{e_1})^0$ and consider the Rees $(B \times A)$-matrix given by:

$$C(j,i) := \begin{cases} q_{1j} p_{i1} & \text{if } q_{1j} p_{i1} \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad j \in B, i \in A$$

7. The proof of Rees' Theorem in [Rees40] gives us an isomorphism

$$\psi : S \longrightarrow \mathcal{M}^0(G, A, B, C)$$

8. By Corollary 1, if $j \in B_k$ and $i \in A_{k'}$ and $k \neq k'$, it follows that $l_j r_i = 0$. Therefore $C(j,i) = 0$ and then:

$$C = \begin{array}{c} \\ B_1 \\ B_2 \\ \vdots \\ B_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$

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$$C = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} & \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$

6. We have seen that the maximal subgroups S_e, for all $0 \neq e \in \mathrm{E}(S)$, are all isomorphic. Let $G^0 := e_1 S e_1 = (S_{e_1})^0$ and consider the Rees $(B \times A)$-matrix given by:

$$C(j,i) := \begin{cases} q_{1j} p_{i1} & \text{if } q_{1j} p_{i1} \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad j \in B, i \in A$$

7. The proof of Rees' Theorem in [Rees40] gives us an isomorphism

$$\psi : S \longrightarrow \mathcal{M}^0(G, A, B, C)$$

8. By Corollary 1, if $j \in B_k$ and $i \in A_{k'}$ and $k \neq k'$, it follows that $l_j r_i = 0$. Therefore $C(j,i) = 0$ and then:

$$C = \begin{array}{c c} & \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ \vdots \\ B_n \end{array} & \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix} \end{array}$$