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On the Supersoluble Residual of a Product of Supersoluble Subgroups

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To Leonid Kurdachenko on his 70th birthday

Abstract

Let $\mathbb{P}$ be the set of all primes. A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, if either $H = G$, or there exists a chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G,$$

with $|H_i : H_{i-1}| \in \mathbb{P}$ for all i. A group $G = AB$ with $\mathbb{P}$-subnormal supersoluble subgroups A and B is studied. The structure of its supersoluble residual is obtained. In particular, it coincides with the nilpotent residual of the derived subgroup of G. Besides, if the indices of the subgroups A and B are coprime, then the supersoluble residual coincides with the intersection of the metanilpotent residual of G and all normal subgroups of G such that all corresponding quotients are primary or biprimary. From here new signs of supersolubility are derived.

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. We use the standard notations and terminology of [7, 8].

The *supersoluble* (*nilpotent*) *residual* of a group G is the smallest normal subgroup K of G such that the quotient G/K is supersoluble (nilpotent, respectively). The notation $Y \leqslant X$ means that Y is a subgroup of a group X and $\mathbb{P}$ be the set of all primes.

It is well-known that factorizable group $G = AB$ with normal supersoluble subgroups A and B may be non-supersoluble [2, 6]. Sufficient conditions for supersolubility of such groups were established by Baer [2], Friesen [4], Vasil'ev and Vasil'eva [19]. We collect these results in

Theorem A. *Let* $G = AB$ *be the product of two normal supersoluble subgroups* A *and* B*. Then the following hold:*

(1) *if* G *has a nilpotent normal subgroup* W *such that all Sylow subgroups of* G/W *are abelian, then* G *is supersoluble* [19]*; in particular, if the derived subgroup* G' *is nilpotent, then* G *is supersoluble* [2]*;*

(2) *if the indices of* A *and* B *in* G *are coprime, then* G *is supersoluble* [4].

In every assertions of Theorem A, the normality of A and B can be replaced by subnormality. Indeed, we can replace the subgroups A and B by the normal subgroups

$$A^G = A(A^G \cap B) \quad \text{and} \quad B^G = (B^G \cap A)B,$$

which are supersoluble by induction, and then apply to $G = A^G B^G$ the corresponding assertion of Theorem A. Here $H^G = \langle H^g \mid g \in G\rangle$ is the smallest normal subgroup of G that includes H.

Not all sufficient conditions for supersolubility of a group $G = AB$ with normal supersoluble subgroups A and B can be generalized to groups with subnormal factors. For example, it is known that a group $G = AB$ with normal supersoluble subgroups A and B is supersoluble whenever $A \cap B$ is nilpotent [9, Corollary 5]. The example [2, p.186] demonstrates that the normality of any factor cannot be weakened to subnormality.

In [10] V.S. Monakhov and I.K. Chirik obtained that the supersoluble residual of a group $G = AB$ with subnormal supersoluble subgroups A and B coincides with nilpotent residual of mutual commutator of subgroups A and B. From this we can extract all three statements of Theorem A.

The normality of the factors A and B can be weakened to permutability of some subgroups of A and B. Asaad and Shaalan in [1] were the first who studied groups that factorized by mutually permutable subgroups, i.e. such subgroups A and B that satisfy the conditions: $UB = BU$ and $AV = VA$ for all $U \leqslant A$ and $V \leqslant B$. A detailed account on this topic can be found in the monograph [3].

Recall that a subgroup A is *seminormal* in G, if there exists a subgroup B such that $G = AB$ and AX is a subgroup of G for every subgroup X of B. Some results from [1] the authors of this article extended in [14] to groups with seminormal factors A and B. In particular, we proved the supersolubility of a group $G = AB$ with seminormal supersoluble subgroups A and B in the following cases: B is nilpotent, [14, Theorem 2.1]; the derived subgroup G' is nilpotent [14, Theorem 2.2]. Besides, we obtained that the supersoluble residual of a group $G = AB$ with seminormal supersoluble subgroups A and B coincides with the nilpotent residual of the derived subgroup of G. Moreover, if the indices of the subgroups A and B are coprime, then the supersoluble residual coincides with the metanilpotent residual of G [14, Theorem 2.3]. Also the supersolubility of $G = AB$ when all Sylow subgroups of A and of B are seminormal in G was proved [14, Theorem 2.4].

Another direction of research of a factorizable groups is related to the following concept of $\mathbb{P}$-subnormality. By Huppert's Theorem [7, VI.9.5], a group G is supersoluble if and only if for every proper subgroup H of G there exists a chain of subgroups

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G, \ |H_i : H_{i-1}| \in \mathbb{P}, \ \forall i. \qquad (1)$$

Thus the following definition naturally arises.

A subgroup H of a group G is called $\mathbb{P}$-*subnormal* in G, if either $H = G$, or there is a chain subgroups (1). We use the notation $H \,\mathbb{P}sn\, G$. This definition was proposed in [20] and besides, in this paper w-*supersoluble* groups (groups with $\mathbb{P}$-subnormal Sylow subgroups) were investigated.

By the Jordan-Hölder Theorem, in a soluble group every subnormal subgroup and every seminormal subgroup are $\mathbb{P}$-subnormal. But the converse statements do not hold in general. For example, in symmetric group S_4 a subgroup $\langle(1\,2)\rangle$ of order 2 is $\mathbb{P}$-subnormal, but not subnormal and seminormal.

The factorizable groups with $\mathbb{P}$-subnormal factors were investigated in [12, 13, 21]. We state some results.

Theorem B. *Let* $G = AB$ *be the product of* $\mathbb{P}$-*subnormal supersoluble* *subgroups* A *and* B. *Then the following hold:*

(1) *if the derived subgroup* G' *is nilpotent, then* G *is supersoluble* (see Corollary 4.7.2 of [21]);

(2) *if* G *has a nilpotent normal subgroup* W *such that all Sylow subgroups of* G/W *are abelian, then* G *is* w*-supersoluble* (see Corollary 4.7.1 of [21]);

(3) *if the indices of* A *and* B *in* G *are coprime, then* G *is* w*-supersoluble* (see Corollary 4.7.1 of [21]).

Section 5 contains examples showing that in (2) and in (3) the group G may be non-supersoluble.

In the present work, further development of these directions is obtained. A group $G = AB$ with $\mathbb{P}$-subnormal supersoluble subgroups A and B is studied. In Section 3 the structure of its supersoluble residual is obtained. In particular, it coincides with the nilpotent residual of the derived subgroup of G. Besides, if the indices of the subgroups A and B are coprime, then the supersoluble residual coincides with the intersection of the metanilpotent residual of G and all normal subgroups of G such that all corresponding quotients are primary or biprimary. From here new signs of supersolubility are derived. In Section 4 p-analogs of some results of Section 3 are obtained. Section 5 provides examples illustrating the completeness of the results.

2 Preliminary results

In this section, we give some definitions and basic results which are essential in the sequel. A group whose chief factors have prime orders is called *supersoluble*. Recall that a p-*closed* group is a group with a normal Sylow p-subgroup and a p-*nilpotent* group is a group with a normal Hall p'-subgroup.

Denote by G', $Z(G)$, $F(G)$ and $\Phi(G)$ the derived subgroup, centre, Fitting and Frattini subgroups of G respectively; $O_p(G)$ and $O_{p'}(G)$ the greatest normal p- and p'-subgroups of G respectively. We use E_{p^t} to denote an elementary abelian group of order p^t and Z_m to denote a cyclic group of order m. The semidirect product of a normal subgroup A and a subgroup B is written as follows: $A \rtimes B$. Denote by $\pi(G)$ the set of all prime divisors of order of G. A group G is called *primary* if $|\pi(G)| = 1$, and *biprimary* if $|\pi(G)| = 2$.

The formations of all abelian, nilpotent and supersoluble groups are denoted by $\mathfrak{A}$,$\mathfrak{N}$ and $\mathfrak{U}$, respectively. Let $\mathfrak{F}$ be a formation. Then $G^{\mathfrak{F}}$ denotes the $\mathfrak{F}$-residual of G, that is the intersection of all those normal subgroups N of G for which $G/N \in \mathfrak{F}$. The subgroups $G^{\mathfrak{A}}$, $G^{\mathfrak{N}}$ and $G^{\mathfrak{U}}$ are called abelian, nilpotent and supersoluble residual of G, respectively. It is clear that the abelian residual of G coincides with the derived subgroup of G, i.e. $G^{\mathfrak{A}} = G'$. We define

$$\mathfrak{F} \circ \mathfrak{H} = \{G \in \mathfrak{E} \mid G^{\mathfrak{H}} \in \mathfrak{F}\}$$

and call $\mathfrak{F} \circ \mathfrak{H}$ the *formation product* of $\mathfrak{F}$ and $\mathfrak{H}$. Here $\mathfrak{E}$ is the class of all finite groups. As usually, $\mathfrak{F}^2 = \mathfrak{F} \circ \mathfrak{F}$.

Lemma 2.1 (see Lemma 6 of [10]) *Let* G *be a soluble group. Assume that* $G \notin \mathfrak{U}$, *but* $G/K \in \mathfrak{U}$ *for every non-trivial normal subgroup* K *of* G. *Then the following hold:*

(1) G *contains a unique minimal normal subgroup* N *and*

$$N = F(G) = O_p(G) = C_G(N)$$

for some $p \in \pi(G)$;

(2) $Z(G) = O_{p'}(G) = \Phi(G) = 1$;

(3) G *is primitive;* $G = N \rtimes M$, *where* M *is maximal in* G *with trivial core;*

(4) N *is an elementary abelian subgroup of order* p^n, $n > 1$;

(5) *if* V *is a subgroup* G *and* $G = VN$, *then* $V = M^x$ *for some* $x \in G$.

Lemma 2.2 (see Lemma 5.8 and Theorem 5.11 of [8]) *Let* $\mathfrak{F}$ *and* $\mathfrak{H}$ *be formations,* K *be normal in* G. *Then the following hold:*

(1) $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$;

(2) $G^{\mathfrak{F}\mathfrak{H}} = (G^{\mathfrak{H}})^{\mathfrak{F}}$;

(3) *if* $\mathfrak{H} \subseteq \mathfrak{F}$, *then* $G^{\mathfrak{F}} \leqslant G^{\mathfrak{H}}$.

Lemma 2.3 (see Lemma 3 of [11]) *Let* H *be a subgroup of* G, *and* N *be a normal subgroup of* G. *Then the following hold:*

(1) *if* $N \leqslant H$ *and* $H/N \ \mathbb{P}sn \ G/N$, *then* $H \ \mathbb{P}sn \ G$;

(2) *if* $H \ \mathbb{P}sn \ G$, *then* $(H \cap N) \ \mathbb{P}sn \ N$, $HN/N \ \mathbb{P}sn \ G/N$ *and* $HN \ \mathbb{P}sn \ G$;

(3) *if* $H \leqslant K \leqslant G$, $H \ \mathbb{P}sn \ K$ *and* $K \ \mathbb{P}sn \ G$, *then* $H \ \mathbb{P}sn \ G$;

(4) *if* $H \ \mathbb{P}sn \ G$, *then* $H^g \ \mathbb{P}sn \ G$ *for any* $g \in G$.

Lemma 2.4 (see Lemma 4 of [11]) *Let* G *be a soluble group, and* H *be a subgroup of* G. *Then the following hold:*

(1) *if* $H \ \mathbb{P}sn \ G$ *and* $K \leqslant G$, *then* $(H \cap K) \ \mathbb{P}sn \ K$;

(2) *if* $H_i \ \mathbb{P}sn \ G$, $i = 1, 2$, *then* $(H_1 \cap H_2) \ \mathbb{P}sn \ G$.

Lemma 2.5 (see Lemma 5 of [11]) *If* H *is a subnormal subgroup of a soluble group* G, *then* H *is* $\mathbb{P}$*-subnormal in* G.

Lemma 2.6 (see Lemma 8 of [11]) *Let* p *be the greatest prime divisor of* $|G|$, *and* A *be a* p*-subgroup of* G. *If* A *is* $\mathbb{P}$*-subnormal in* G, *then* A *is subnormal in* G.

Lemma 2.7 (see Lemma 4.1 of [21]) *Let* A *and* B *be* $\mathbb{P}$*-subnormal subgroups of* G, *and* $G = AB$. *For the subgroup* A, *we fix a* $\mathbb{P}$*-subnormal chain*

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_{n-1} \leqslant A_n = G$$

such that $|A_i : A_{i-1}| \in \mathbb{P}$ *for all* i. *Then the intersection* $A_k \cap B$ *is* $\mathbb{P}$*-subnormal in* A_k *for all* k.

Lemma 2.8 (see Theorem 4.2 of [21]) *Let* $G = AB$ *be the product of soluble subgroups* A *and* B. *If* A *and* B *are* $\mathbb{P}$*-subnormal in* G, *then* G *is soluble.*

Lemma 2.9 (see Theorem 4.4 of [21]) *Let* A *and* B *be* $\mathbb{P}$*-subnormal subgroups of* G, *and* $G = AB$. *If* A *and* B *have an ordered Sylow tower of supersoluble type, then* G *has an ordered Sylow tower of supersoluble type.*

Recall that a group G is said to be *siding* if every subgroup of the derived subgroup G' is normal in G, see [16, Definition 2.1]. Metacyclic groups, t-groups (groups in which every subnormal subgroup is normal) are siding. The group $G = (Z_6 \times Z_2) \rtimes Z_2$ (IdGroup(G)=[24,8], [5]) is siding, but not metacyclic and a t-group.

Lemma 2.10 *Let* G *be siding. Then the following hold:*

(1) *if* N *is normal in* G, *then* G/N *is siding;*

(2) *if* H *is a subgroup of* G, *then* H *is siding;*

(3) G *is supersoluble.*

Proof — (1) By [8, Lemma 4.6], $(G/N)' = G'N/N$. Let A/N be an arbitrary subgroup of $(G/N)'$. Then

$$A \leqslant G'N,\ A = A \cap G'N = (A \cap G')N.$$

Since $A \cap G' \leqslant G'$, we have $A \cap G'$ is normal in G. Hence $(A \cap G')N/N$ is normal in G/N.

(2) Since $H \leqslant G$, it follows that $H' \leqslant G'$. Let A be an arbitrary subgroup of H'. Then $A \leqslant G'$ and A is normal in G. Therefore A is normal in H.

(3) We proceed by induction on the order of G. Let $N \leqslant G'$ and $|N| = p$, where p is prime. By the hypothesis, N is normal in G. By induction, G/N is supersoluble and G is supersoluble. □

Lemma 2.11 (see Theorem A of [13]) *Suppose that* G *has non-conjugate subgroups* H *and* K *of prime indices. If* H *is nilpotent and* K *is supersoluble, then* G *is supersoluble.*

3 Factorizable groups with $\mathbb{P}$-subnormal supersoluble subgroups

In what follows, we will need to study the structure of the supersoluble residual of a w-supersoluble group. For this we introduce the subgroup $\mathfrak{B}(G)$ as the intersection of all normal subgroups of G such that all corresponding quotients are primary or biprimary. More precisely, let p,q be primes and $\mathfrak{S}_{\{p,q\}}$ be the formation of all {p,q}-groups. Notice that $\mathfrak{N}_p \subseteq \mathfrak{S}_{\{p,q\}}$ and $\mathfrak{N}_q \subseteq \mathfrak{S}_{\{p,q\}}$. For a group G with $|\pi(G)| > 2$ we introduce the following notation:

$$\mathfrak{B}(G) = \bigcap_{\forall\{p,q\}\subseteq\pi(G)} G^{\mathfrak{S}_{\{p,q\}}}.$$

If $|\pi(G)| \leqslant 2$, we assume that $\mathfrak{B}(G) = 1$. Recall that $\mathfrak{N}^2$ is the class of all metanilpotent groups and $w\mathfrak{U}$ is the class of all w-supersoluble groups.

Theorem 3.1 *If* $G \in w\mathfrak{U}$, *then* $G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$.

Proof — Since $G/N \in w\mathfrak{U}$ for any normal subgroup N of G, it follows that

$$G/G^{\mathfrak{N}^2} \in w\mathfrak{U} \cap \mathfrak{N}^2$$

and by [20, Theorem 2.13], $G/G^{\mathfrak{N}^2}$ is supersoluble. Hence $G^{\mathfrak{U}} \leqslant G^{\mathfrak{N}^2}$. Because

$$G/G^{\mathfrak{S}_{\{p,q\}}} \in w\mathfrak{U} \cap \mathfrak{S}_{\{p,q\}},$$

by [20, Theorem 2.13] we have that $G/G^{\mathfrak{S}_{\{p,q\}}}$ is supersoluble and $G^{\mathfrak{U}} \leqslant G^{\mathfrak{S}_{\{p,q\}}}$. Since p and q are arbitrary,

$$G^{\mathfrak{U}} \leqslant \bigcap_{\forall\{p,q\}\subseteq\pi(G)} G^{\mathfrak{S}_{\{p,q\}}} = \mathfrak{B}(G).$$

Consequently $G^{\mathfrak{U}} \leqslant G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$. Check the converse inclusion. Since every supersoluble group is metanilpotent, it follows that $\mathfrak{U} \subseteq \mathfrak{N}^2$ and $G^{\mathfrak{N}^2} \leqslant G^{\mathfrak{U}}$ by Lemma 2.2 (3). Hence

$$G^{\mathfrak{N}^2} \cap \mathfrak{B}(G) \leqslant G^{\mathfrak{N}^2} \leqslant G^{\mathfrak{U}}.$$

The statement is proved. □

Corollary 3.2 *If* $G \in w\mathfrak{U} \setminus \mathfrak{U}$, *then* $|\pi(G^{\mathfrak{U}})| \leqslant |\pi(G)| - 2$.

Proof — Let $\pi(G) = \{p, q, \ldots\}$, $p < q < \ldots$. By [20, Proposition 2.8], every w-supersoluble group has an ordered Sylow tower of supersoluble type, hence G is $\{p, q\}$-nilpotent and $G^{\mathfrak{S}_{\{p,q\}}}$ is a $\{p, q\}'$-group. Since $\mathfrak{B}(G) \leqslant G^{\mathfrak{S}_{\{p,q\}}}$, it follows that $\mathfrak{B}(G)$ is a $\{p, q\}'$-group and

$$\pi(\mathfrak{B}(G)) \cap \{p, q\} = \emptyset.$$

By Theorem 3.1, $G^{\mathfrak{U}} \leqslant \mathfrak{B}(G)$, hence $|\pi(G^{\mathfrak{U}}| \leqslant |\pi(G)| - 2$. □

Theorem 3.3 *Let* A *and* B *be supersoluble* $\mathbb{P}$*-subnormal subgroups of* G, *and* $G = AB$. *Then the following hold:*

(1) $G^{\mathfrak{U}} = (G')^{\mathfrak{N}}$;

(2) *if* G *has a nilpotent normal subgroup* W *such that all Sylow subgroups of* G/W *are abelian, then* $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$;

(3) *if* $(|G : A|, |G : B|) = 1$, *then* $G^{\mathfrak{U}} = (G')^{\mathfrak{N}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$.

Proof — (1) If G is supersoluble, then $G^{\mathfrak{U}} = 1$ and G' is nilpotent. Consequently $(G')^{\mathfrak{N}} = 1 = G^{\mathfrak{U}}$ and the statement is true. Further, we assume that G is non-supersoluble. Since every supersoluble group has an ordered Sylow tower of supersoluble type, then by Lemma 2.9, G has an ordered Sylow tower of supersoluble type. Since $\mathfrak{U} \subseteq \mathfrak{N} \circ \mathfrak{A}$, we have

$$G^{(\mathfrak{N}\circ\mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{N}} = (G')^{\mathfrak{N}} \leqslant G^{\mathfrak{U}}$$

by Lemma 2.2 (2-3). Next we check the converse inclusion. For this we prove that $G/(G')^{\mathfrak{N}}$ is supersoluble. By Lemma 2.2 (1), the derived subgroup

$$(G/(G')^{\mathfrak{N}})' = G'(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} = G'/(G')^{\mathfrak{N}}$$

is nilpotent. Since

$$\begin{aligned} G/(G')^{\mathfrak{N}} &= (A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}})(B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}), \\ A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} &\simeq A/A \cap (G')^{\mathfrak{N}}, \\ B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} &\simeq B/B \cap (G')^{\mathfrak{N}}, \end{aligned}$$

the subgroups

$$A(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}} \quad \text{and} \quad B(G')^{\mathfrak{N}}/(G')^{\mathfrak{N}}$$

are supersoluble and by Lemma 2.3 (2), these subgroups are $\mathbb{P}$-subnormal in $G/(G')^{\mathfrak{N}}$. By Theorem B (1), $G/(G')^{\mathfrak{N}}$ is supersoluble.

(2–3) By Theorem B (2-3), G is w-supersoluble. Hence

$$G^{\mathfrak{U}} = G^{\mathfrak{N}^2} \cap \mathfrak{B}(G)$$

by Theorem 3.1. □

Theorem 3.4 *Let* G *be a group, and let* A *be a subgroup of* G *such that* $|G : A| = p^{\alpha}$, *where* $p \in \pi(G)$ *and* $\alpha \in \mathbb{N}$. *Suppose that* A *is supersoluble and* $\mathbb{P}$*-subnormal in* G. *If* G *is* p*-closed, then* G *is supersoluble.*

Proof — Let P be a Sylow p-subgroup of G. Since P is normal in G and $G = AP$, we have $G/P \simeq A/A \cap P \in \mathfrak{U}$, in particular, G is soluble.

We use induction on the order of G. Let N be a non-trivial normal subgroup of G. If $AN = G$, then

$$G/N = AN/N \simeq A/A \cap N \in \mathfrak{U}.$$

Let $AN < G$. Then AN/N is $\mathbb{P}$-subnormal in G/N by Lemma 2.3 (2) and supersoluble. Besides,

$$|G/N : AN/N| = |G : AN| = \frac{|G : A|}{|N : A \cap N|} = p^{\alpha_1}, \ \ 0 < \alpha_1 \leqslant \alpha.$$

Consequently G/N satisfies the hypothesis of the theorem and by induction, G/N is supersoluble. By Lemma 2.1, G contains a unique minimal normal subgroup

$$N = F(G) = O_p(G) = C_G(N)$$

such that $G = N \rtimes M$ and $O_{p'}(G) = 1$. Since P is normal in G, we have $N = P$ and M is a Hall p'-subgroup of G. Because $|G : A| = p^{\alpha}$, it follows that $M = A$. Hence $|P| = p$, a contradiction. □

Corollary 3.5 *Let A and B be supersoluble $\mathbb{P}$-subnormal subgroups of G, and $G = AB$. Suppose that $|G : A| = p^{\alpha}$, where $p \in \pi(G)$. Then G is p-supersoluble. If p is the greatest in $\pi(G)$, then G is supersoluble.*

Proof — Let p be the greatest in $\pi(G)$. Since every supersoluble group has an ordered Sylow tower of supersoluble type, then by Lemma 2.9, G has an ordered Sylow tower of supersoluble type. Hence G is p-closed. By Theorem 3.4, we have that G is supersoluble.

Let q be the greatest in $\pi(G)$, $q > p$ and Q be a Sylow q-subgroup of A. The subgroup Q is normal in A and $\mathbb{P}$-subnormal in G by Lemma 2.5 and Lemma 2.3 (3). By Lemma 2.6, Q is normal in G. The quotient A/Q is $\mathbb{P}$-subnormal in G/Q and $|G/Q : A/Q| = p^{\alpha}$. By induction, G/Q is p-supersoluble, hence G is p-supersoluble. □

In [15] we proved that *a group G is supersoluble if and only if, for every prime $p \in \pi(G)$, it has a supersoluble subgroup of index p*. A stronger result is obtained in Corollary 3.6.

Corollary 3.6 *Let G be a group, p be the greatest in $\pi(G)$, $p > q$ and $q \in \pi(G)$. If G has the supersoluble subgroups of indices p and q, then G is supersoluble.*

Proof — By Lemma 2.8, G is soluble. Let B be a supersoluble subgroup of index q and P be a Sylow p-subgroup of B. Then P is normal in B and hence P is $\mathbb{P}$-subnormal in G by Lemma 2.3 (3). By Lemma 2.6, P is normal in G. By Theorem 3.4, G is supersoluble. □

Theorem 3.7 *Let* A *be a supersoluble* $\mathbb{P}$*-subnormal subgroup of* G, *and* $G = AB$. *Then* G *is supersoluble in each of the following cases:*

(1) B *is nilpotent and normal in* G*;*

(2) B *is nilpotent and* $|G : B|$ *is prime;*

(3) B *is normal in* G *and is a siding group.*

Proof — We prove all three statements at the same time using induction on the order of G. Note that G is soluble in any case. By Lemma 2.5, B is $\mathbb{P}$-subnormal in G and G has an ordered Sylow tower of supersoluble type by Lemma 2.9. If N is a non-trivial normal subgroup of G, then AN/N is $\mathbb{P}$-subnormal in G/N by Lemma 2.3 (2) and

$$AN/N \simeq A/A \cap N$$

is supersoluble. The subgroup

$$BN/N \simeq B/B \cap N$$

is nilpotent or a siding group by Lemma 2.10 (1). Hence

$$G/N = (AN/N)(BN/N)$$

is supersoluble by induction. By Lemma 2.1, $F(G) = N = G_p$ is a unique minimal normal subgroup of G and $N = C_G(N)$, where p is the greatest in $\pi(G)$.

Since A is $\mathbb{P}$-subnormal in G, it follows that G has a subgroup M such that $A \leqslant M$ and $|G : M|$ is prime. By Dedekind's identity,

$$M = A(M \cap B).$$

The subgroup A is $\mathbb{P}$-subnormal in M. The subgroup $M \cap B$ satisfies the requirements (1)–(3). By induction, M is supersoluble.

(1) If B is nilpotent and normal in G, then $B = N$. Hence $G = AN$ and A is a maximal subgroup of G. Since A is $\mathbb{P}$-subnormal in G, we have $|G : A| = p = |N|$, a contradiction. So, in (1), the theorem is proved.

(2) Let B be nilpotent and $|G : B|$ be prime. Since $G = MB$, it follows that M and B are non-conjugate maximal subgroups of prime

indices, M is supersoluble and B is nilpotent. By Lemma 2.11, G is supersoluble, a contradiction. So, in (2), the theorem is proved.

(3) Let B is normal in G and is a siding group. If B is nilpotent, then G is supersoluble by (1). Hence $B' \neq 1$. Because B' is normal in G and nilpotent, we have $N = B'$. If N is not contained in M, then

$$G = N \rtimes M$$

and $|N|$ is prime, a contradiction. Let N be contained in M and N_1 be a subgroup of prime order of N such that N_1 is normal in M. Then N_1 is normal in B by definition of siding group. Hence N_1 is normal in G, a contradiction. So, in (3), the theorem is proved. □

4 Applications to p-soluble groups

A group is said to be p-*soluble (*p-*supersoluble)*, if the order of each of its chief factors is either a p-power (equal to p), or a coprime to p. We write $p\mathfrak{S}$ for the class of all p-soluble groups and $p\mathfrak{U}$ for the class of all p-supersoluble groups. The classes of all p-closed and p-nilpotent groups are equal to the products $\mathfrak{N}_p \circ \mathfrak{E}_{p'}$ and $\mathfrak{E}_{p'} \circ \mathfrak{N}_p$ respectively, where $\mathfrak{N}_p$ is the class of all p-groups and $\mathfrak{E}_{p'}$ is the class of all p'-groups. The classes $p\mathfrak{S}$, $\mathfrak{N}_p \circ \mathfrak{E}_{p'}$ and $\mathfrak{E}_{p'} \circ \mathfrak{N}_p$ are radical hereditary saturated formations and

$$\mathfrak{N}_p \circ \mathfrak{E}_{p'} \cup \mathfrak{E}_{p'} \circ \mathfrak{N}_p \subseteq p\mathfrak{S}.$$

Lemma 4.1 (see Lemma 11 of [9]) *Suppose that a* p*-soluble group* G *is not belong to* $p\mathfrak{U}$*, but* $G/K \in p\mathfrak{U}$ *for every non-trivial normal subgroup* K *of* G*. Then the following hold:*

(1) $Z(G) = O_{p'}(G) = \Phi(G) = 1$;

(2) G *has a unique minimal normal subgroup* N *and*

$$N = F(G) = O_p(G) = C_G(N);$$

(3) G *is a primitive and* $G = N \rtimes M$*, where* M *is a maximal subgroup of* G *with trivial core;*

(4) N *is an elementary abelian group of order* p^n*,* $n > 1$*;*

(5) *if* M *is abelian, then* M *is cyclic of order dividing* $p^n - 1$ *and* n *is the smallest positive integer such that* $p^n \equiv 1 \pmod{|M|}$.

Lemma 4.2 *Let* $p \in \pi(G)$ *and* $(|\,G\,|, p-1) = 1$. *Then* G *is* p*-supersoluble if and only if* G *is* p*-nilpotent.*

Proof — It is obvious that every p-nilpotent group is p-supersoluble. Check the converse. Let G be a group of smallest order such that G is p-supersoluble, but not p-nilpotent. Let H be an arbitrary proper subgroup of G. Then H is p-supersoluble and $(|\,H\,|, p-1) = 1$. Therefore due to the choice of G, the subgroup H is p-nilpotent and G is a minimal non-p-nilpotent group. By [17, Theorem 10.3.3], G is a Schmidt group and by [18], $G = P \rtimes Q$, where P is a normal Sylow p-subgroup and Q is a cyclic Sylow q-subgroup. Since G is p-supersoluble, it follows that the order of p modulo q is equal to 1, i.e. $m = 1$, see [18]. Hence q divides $p - 1$, a contradiction. □

Lemma 4.3 (see Theorem 1 (1) of [12]) *Let* $G = AB$ *and* $r \in \pi(G)$. *If* A *and* B *are* $\mathbb{P}$*-subnormal* r*-soluble subgroups of* G, *then* G *is* r*-soluble.*

Lemma 4.4 (see Theorem 1 of [9]) *Let* A *and* B *are normal* p*-supersoluble subgroups of* G, *and* $G = AB$. *If the derived subgroup* G' *is* p*-nilpotent, then* G *is* p*-supersoluble.*

Lemma 4.5 (see Lemma 1.4 of [14]) *Let* H *be a maximal subgroup of* G. *The subgroup* H *is seminormal in* G *if and only if* $|G : H|$ *is prime.*

Lemma 4.6 *Suppose that* A *and* B *are seminormal subgroups in a* p*-soluble group* G, *and* $G = AB$. *Then* G *is* p*-supersoluble in each of the following cases:*

(1) A *is* p*-nilpotent and* B *is* p*-supersoluble* (see Theorem 3.1 of [14]);

(2) A *and* B *are* p*-supersoluble, and the derived subgroup* G' *is* p*-nilpotent* (see Theorem 3.2 of [14]).

Theorem 4.7 *Let* $G = AB$, *where* A *and* B *are* $\mathbb{P}$*-subnormal in* G, *and* $p \in \pi(G)$. *Then* G *is* p*-supersoluble in each of the following cases:*

(1) A *and* B *are* p*-supersoluble, and* $(|G|, p-1) = 1$;

(2) A *is* p*-supersoluble,* B *is* p*-nilpotent and normal in* G;

(3) A *is* p*-supersoluble,* B *is* p*-nilpotent and* $|G : B|$ *is prime;*

(4) A *and* B *are* p*-supersoluble, and the derived subgroup* G' *is* p*-nilpotent.*

Proof — (1) We use induction on the order of G. Assume that the claim is false and let G be a minimal counterexample. By hypothesis, the subgroups A and B have the chains of subgroups:

$$A = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_n = G, \quad |A_i : A_{i-1}| \in \mathbb{P}, \quad \forall i;$$
$$B = B_0 \leqslant B_1 \leqslant \ldots \leqslant B_m = G, \quad |B_j : B_{j-1}| \in \mathbb{P}, \quad \forall j.$$

By Dedekind's identity,

$$A_{n-1} = A(A_{n-1} \cap B)$$

and by Lemma 2.7, $A_{n-1} \cap B$ is $\mathbb{P}$-subnormal in A_{n-1}. Since

$$A_{n-1} = A(A_{n-1} \cap B)$$

and A is $\mathbb{P}$-subnormal in A_{n-1}, we have by induction, A_{n-1} is p-supersoluble and $|G : A_{n-1}|$ is prime. Similarly, B_{m-1} is p-supersoluble and $|G : B_{m-1}|$ is prime. It is clear that

$$G = A_{n-1}B_{m-1}.$$

Denote $H = A_{n-1}$ and $R = B_{m-1}$.

If N is a non-trivial normal subgroup of G, then the subgroups RN/N and HN/N are $\mathbb{P}$-subnormal in G/N by Lemma 2.3 (2) and p-supersoluble. Consequently G/N satisfies the hypothesis of the theorem and by induction G/N is p-supersoluble. By Lemma 4.1, G contains a unique minimal normal subgroup N such that

$$N = F(G) = O_p(G) = C_G(N)$$

and N is an elementary abelian subgroup of order p^n, $n > 1$.

Suppose that N is not contained in R. Then $G = N \rtimes R$ and $|N| = p$ is prime, a contradiction. Therefore we can assume that $N \leqslant R \cap H$. By Lemma 4.2, R and H are p-nilpotent. Then $R_{p'}$ is normal in R and

$$R_{p'} \leqslant C_G(N) = N,$$

a contradiction. Hence R and H are p-groups. Thus G is a p-group, and therefore G is p-supersoluble.

(2–3) We prove all two statements at the same time using induction on the order of G. By Lemma 4.3, G is p-soluble in any case. If N is a non-trivial normal subgroup of G, then AN/N is $\mathbb{P}$-subnormal in G/N by Lemma 2.3 (2) and

$$AN/N \simeq A/A \cap N$$

is p-supersoluble, $BN/N \simeq B/B \cap N$ is p-nilpotent. If B is normal in G, then BN/N is normal in G/N. If $|G : B|$ is prime, then either $BN = G$ and $G/N \simeq B/B \cap N$ is p-supersoluble, or $|G/N : B/N| = |G : B|$ is prime. The quotient

$$G/N = (AN/N)(BN/N)$$

is p-supersoluble by induction and therefore we apply Lemma 4.1. We save to G the notation of this Lemma, in particular,

$$F(G) = N = O_p(G)$$

is a unique minimal normal subgroup of G and $|N| = p^{\alpha}$, $\alpha > 1$.

Since A is $\mathbb{P}$-subnormal in G, it follows that G has a subgroup M such that $A \leqslant M$ and $|G : M|$ is prime. By Dedekind's identity,

$$M = A(M \cap B).$$

The subgroup A is $\mathbb{P}$-subnormal in M. The subgroup $M \cap B$ satisfies the requirements (1)–(2). By induction, M is p-supersoluble.

If B is p-nilpotent and normal in G, then B is nilpotent and $B = N$. Hence $G = AN$ and A is a maximal subgroup of G. Since A is $\mathbb{P}$-subnormal in G, we have $|G : A| = p = |N|$, a contradiction. So, in (2), G is p-supersoluble.

Let B be p-nilpotent and $|G : B|$ be prime. By Lemma 4.5, M and B are seminormal in G. Since $G = MB$, M is p-supersoluble and B is p-nilpotent, it follows that by Lemma 4.6 (1), G is p-supersoluble. So, in (3), G is p-supersoluble.

(4) By induction, we can assume that $G = HR$, where H and R are p-supersoluble maximal subgroups of prime indices of G. By Lemma 4.5, H and R are seminormal in G. Since $G = HR$, we have by Lemma 4.6 (2), G is p-supersoluble. □

By Lemma 4.2, the p-nilpotency of G with $(|G|, p-1) = 1$ is equivalent to its p-supersolubility. Hence for the smallest $p \in \pi(G)$ we have the following result.

Corollary 4.8 (see Theorem 1 (2) of [12]) *Let* A *and* B *be* $\mathbb{P}$*-subnormal in* G*, and let* $G = AB$*. Suppose that* p *is the smallest prime divisor of the order of* G*. If* A *and* B *are* p*-nilpotent, then* G *is* p*-nilpotent.*

Theorem 4.9 *Let* A *and* B *be* $\mathbb{P}$*-subnormal* p*-supersoluble subgroups of* G*, and* $G = AB$*. Then* $G^{p\mathfrak{U}} = (G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$*.*

Proof — If G is p-supersoluble, then $G^{p\mathfrak{U}} = 1$ and the derived subgroup G' is p-nilpotent. Consequently $G^{p\mathfrak{U}} = 1 = (G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$ and the statement is true. Further, we assume that G is non-p-supersoluble. Since the derived subgroup of p-supersoluble group is p-nilpotent, it follows that $p\mathfrak{U} \subseteq \mathfrak{E}_{p'} \circ \mathfrak{N}_p \circ \mathfrak{A}$ and

$$G^{(\mathfrak{E}_{p'} \circ \mathfrak{N}_p \circ \mathfrak{A})} = (G^{\mathfrak{A}})^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} = (G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} \leqslant G^{p\mathfrak{U}}$$

by Lemma 2.2 (2-3).

Check the converse inclusion. For this we prove that $G/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$ is p-supersoluble. The derived subgroup

$$(G/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p})' = G'(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} = G'/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$$

is p-nilpotent. Since

$$G/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} = (A(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p})(B(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}),$$
$$A(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} \simeq A/A \cap (G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p},$$
$$B(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} \simeq B/B \cap (G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p},$$

the subgroups

$$A(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p} \quad \text{and} \quad B(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$$

are p-supersoluble and by Lemma 2.3 (2), this subgroups are $\mathbb{P}$-subnormal in $G/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$. By Theorem 4.7 (4), $G/(G')^{\mathfrak{E}_{p'} \circ \mathfrak{N}_p}$ is p-supersoluble. □

5 Examples

As can be seen in the following example, a group G factorized by $\mathbb{P}$-subnormal supersoluble subgroups and having coprime indices (Theorem B (3)) or containing a nilpotent normal subgroup W such that all Sylow subgroups of G/W are abelian (Theorem B (2)), can be non-supersoluble.

Example 5.1 *The minimal non-supersoluble group*

$$G = E_{7^2} \rtimes S_3$$

(IdGroup=[294,7]) is the product of subgroups

$$H = E_{7^2} \rtimes Z_3 \quad \textit{and} \quad K = E_{7^2} \rtimes Z_2$$

of indices 2 and 3. The subgroups H *and* K *are* $\mathbb{P}$*-subnormal in* G*.*

The following example shows that we cannot omit the condition «G is p-closed» in Theorem 3.4.

Example 5.2 *The group*

$$G = (S_3 \times S_3 \times S_3) \rtimes Z_3$$

(IdGroup=[648,705]) has a $\mathbb{P}$*-subnormal supersoluble subgroups*

$$A \simeq S_3 \times S_3 \times S_3.$$

Besides $|G : A| = 3$ *and* G *is not 3-supersoluble.*

The following example shows that we cannot omit the condition «$(|G : A|, |G : B|) = 1$» in Theorem 3.3 (3).

Example 5.3 *The group*

$$G = (S_3 \times S_3) \rtimes Z_2$$

(IdGroup=[72,40]) is metanilpotent and factorized by $\mathbb{P}$*-subnormal supersoluble subgroups* $A \simeq Z_3 \times S_3$ *and* $B = S_3 \times S_3$*. The supersoluble residual* $G^{\mathfrak{U}} \simeq Z_3 \times Z_3$*.*

The following example shows that in Theorem 4.7 (1) the normality of subgroup B cannot be weakened to $\mathbb{P}$-subnormality.

Example 5.4 *The group*

$$G = (Z_2 \times (E_{3^2} \rtimes Z_4)) \rtimes Z_2$$

(IdGroup=[144,115]) is non-supersoluble and factorized by subgroups

$$A = D_{12} \quad \textit{and} \quad B = Z_{12}.$$

The subgroup A *has the chain of subgroups*

$$A < S_3 \times S_3 < Z_2 \times S_3 \times S_3 < G$$

and B *has the chain of subgroups*

$$B < Z_3 \times (Z_3 \rtimes Z_4) < (Z_3 \times (Z_3 \rtimes Z_4)) \rtimes Z_2 < G.$$

Therefore A *and* B *are* $\mathbb{P}$*-subnormal in* G.

The following example shows that in Theorem 4.7 (2) it is impossible to weak the restrictions on the index of subgroup B.

Example 5.5 *The alternating group* $G = A_4$ *is non-supersoluble and factorized by subgroups* $A = E_{2^2}$ *and* $B = Z_3$*. It is clear that* A *is supersoluble and* $\mathbb{P}$*-subnormal in* G*, and* B *is nilpotent and* $|G : B| = 2^2$*. The group* $G = E_{5^2} \rtimes Z_3$ *is non-supersoluble and has a nilpotent subgroup* Z_3 *of index* 5^2*. Therefore even for the greatest* p *of* $\pi(G)$*, the index of* B *cannot be equal* p^α*,* $\alpha \geqslant 2$*.*

The following example shows that in Theorem 4.7 (3) the normality of subgroup B cannot be weakened to subnormality.

Example 5.6 *The group* $G = Z_3 \times ((S_3 \times S_3) \rtimes Z_2)$ *(IdGroup=[216,157]) is non-supersoluble and factorized by* $\mathbb{P}$*-subnormal supersoluble subgroup* $A \simeq S_3 \times S_3$ *and subnormal siding subgroup* $B \simeq Z_3 \times Z_3 \times S_3$*.*

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