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On Products of Cyclic and Non-Abelian Finite p-Groups

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Abstract

For an odd prime p we present some results concerning the structure of factorised finite p-groups of the form $G = AB$, where A is a cyclic subgroup and B is a non-abelian subgroup whose class does not exceed $\frac{p}{2}$ in most cases. Bounds for the derived length of such groups are also presented.

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1 Introduction

The present paper explores the structure of factorised finite p-groups of the form $G = AB$, where p is an odd prime and A and B are subgroups of G such that A is cyclic. It has been shown in [6] Theorem 6 that if B is abelian of exponent at most p^k, then $\Omega_k(A)B \trianglelefteq G$, where the characteristic subgroup $\Omega_k(W)$ of the finite p-group W is given by $\Omega_k(W) = \langle w \in W \mid w^{p^k} = 1\rangle$. Here we generalise this theorem in certain cases where B is non-abelian. To this end, we present in Section 2 a series of results leading to Theorem 2.9, which shows that if B has class less than $\frac{p}{2}$ and exponent at most p^k, then $\Omega_k(A)B \trianglelefteq G$. The example of Section 3 shows that the result of Theorem 2.9 does not always hold when the class of B exceeds $\frac{p}{2}$. As an application of Theorem 2.9, it is shown in Corollary 4.3 that if $p \geqslant 5$ and B has class two and exponent p, then G has derived length at most three. Section 4 further provides a generalisation of [4] Theorem 5. This is used in Theorem 4.6 to show that the derived length of G can also be at most three if $p = 3$ and B has class two and exponent 3. The latter bound is further shown to apply in the case where $p \geqslant 5$ and B has class two and exponent p^2.

We denote the nth term of the derived series of a group G by $G^{(n)}$. Thus $G^{(0)} = G$, $G^{(1)} = G'$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geqslant 1$. The derived length of a soluble group G is denoted by $d(G)$. The ith term of the lower (or descending) central series of G will be denoted by $K_i(G)$. Hence

$$K_1(G) = G, \quad K_2(G) = G' \quad \text{and} \quad K_{i+1}(G) = [K_i(G), G]$$

for $i \geqslant 2$. We denote the jth term of the upper (or ascending) central series of G by $Z_j(G)$. Thus $Z_0(G) = 1$, $Z_1(G) = Z(G)$ and

$$Z_{j+1}(G)/Z_j(G) = Z(G/Z_j(G))$$

for $j \geqslant 1$. If G is nilpotent then $c(G)$ will denote the class of G. U_G denotes the core of the subgroup U of a group G. Thus

$$U_G = \bigcap_{g \in G} U^g.$$

The normal closure of U in G is denoted by U^G, so that

$$U^G = \langle U^g \mid g \in G \rangle.$$

We finally denote the cyclic group of order p^n by C_{p^n}.

2 Structural results

In this section we make extensive use of the following theorem which is a consequence of two fundamental results concerning regular p-groups (see [3], III 10.2 Satz and 10.5 Hauptsatz).

Theorem 2.1 *Let G be a finite p-group such that $c(G) < p$. Then, for all k, $\Omega_k(G) = \{g \in G \mid g^{p^k} = 1\}$.*

Our first four results deal with special cases that will find application in the proofs of Theorems 2.6 and 2.9.

Lemma 2.2 *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic, $c(B) < \frac{p}{2}$ and $\exp(B) = p$. Then $\Omega_1(A)B \trianglelefteq G$.*

Proof — We use induction on $|G|$. We may assume that G is non-cyclic and that

$$G \neq \Omega_1(A)B.$$

In particular, we may assume that $\Omega_2(A) \simeq C_{p^2}$.

If $B_G \neq 1$, then there exists

$$1 \neq z \in B_G \cap Z(G)$$

such that $o(z) = p$. By induction, we have $\Omega_1(A\langle z\rangle/\langle z\rangle)B/\langle z\rangle \trianglelefteq G/\langle z\rangle$. If $A \cap \langle z\rangle = 1$, then

$$\Omega_1(A\langle z\rangle/\langle z\rangle) = \Omega_1(A)\langle z\rangle/\langle z\rangle,$$

so $\Omega_1(A)\langle z\rangle B = \Omega_1(A)B \trianglelefteq G$. We thus assume that $A \cap \langle z\rangle \neq 1$. Then

$$\langle z\rangle = \Omega_1(A),$$

so

$$\Omega_1(A\langle z\rangle/\langle z\rangle) = \Omega_1(A/\Omega_1(A)) = \Omega_2(A)/\Omega_1(A).$$

Hence $\Omega_2(A)B \trianglelefteq G$.

If $B \trianglelefteq G$ then, since A is cyclic, we trivially have $\Omega_1(A)B \trianglelefteq G$. Now

$$\Omega_1(A) \leqslant B \quad \text{and} \quad \exp(B) = p,$$

so $|\Omega_2(A)B : B| = p$. Hence if $B \ntrianglelefteq G$, then for $g \in G \setminus N_G(B)$, we see, by comparison of orders, that $\Omega_2(A)B = B^gB$. In addition, B^g and B are normal in $\Omega_2(A)B$ and $c(B^g) = c(B) < \frac{p}{2}$. Thus

$$c(\Omega_2(A)B) \leqslant c(B^g) + c(B) < \frac{p}{2} + \frac{p}{2} = p.$$

Moreover, $\Omega_2(A)B$ is the product of two subgroups of exponent p and is thus generated by elements of order p. It follows by Theorem 2.1 that

$$\Omega_2(A)B = \Omega_1(\Omega_2(A)B) = \{g \in \Omega_2(A)B \mid g^p = 1\}.$$

But then

$$\exp(\Omega_2(A)B) = p,$$

which is a contradiction since $\Omega_2(A) \simeq C_{p^2}$. We thus conclude that

$$B = \Omega_1(A)B \trianglelefteq G.$$

If $B_G = 1$, then by a result of Morigi ([7], Lemma 1, or [1], Lemma 3.3.8), we have $A_G \neq 1$, so

$$1 \neq Z(G) \cap A \leqslant A.$$

By minimality, we then have $\Omega_1(A) \leqslant Z(G)$. We let $\widehat{B} = \Omega_1(A)B$ and have $\exp(\widehat{B}) = p$ and $c(\widehat{B}) < \frac{p}{2}$. Since $1 \neq \Omega_1(A) \leqslant \widehat{B}_G$, we can apply the above argument to see that $\Omega_1(A)B = \Omega_1(A)\widehat{B} \trianglelefteq G$. □

Lemma 2.3 *Let* p *be an odd prime and let* G *be a finite* p*-group such that* $c(G) < p$ *and* $\exp(G) = p^2$. *Suppose, in addition, that there exists* $z \in Z(G)$ *with* $o(z) = p$ *and such that* $\exp(G/\langle z\rangle) = p$. *Then* $|G : \Omega_1(G)| = p$.

Proof — We use induction on $|G|$. Since $\exp(G) = p^2$, there exists $x \in G$ such that $o(x) = p^2$. In addition, since $\exp(G/\langle z\rangle) = p$, we have $1 \neq x^p \in \langle z\rangle$, so $\langle z\rangle = \langle x^p\rangle$. We can thus assume that $x^p = z$. If $G = \langle x\rangle$, then $G \simeq C_{p^2}$ and $\Omega_1(G) \simeq C_p$, so $|G : \Omega_1(G)| = p$. We may therefore assume that $\langle x\rangle \neq G$. We let U be a maximal proper subgroup of G such that $x \in U$. Then $|G : U| = p$ and $\exp(U) = p^2$. In addition, $z = x^p \in U$. Thus $U/\langle z\rangle$ is a non-trivial subgroup of $G/\langle z\rangle$, so $\exp(U/\langle z\rangle) = p$. Hence, by induction, we have

$$|U : \Omega_1(U)| = p.$$

Now $\Omega_1(U)$ is characteristic in U, so $\Omega_1(U) \trianglelefteq G$. Since $o(z) = p$, we have $z \in \Omega_1(U)$, so $\exp(G/\Omega_1(U)) = p$. In addition,

$$|G/\Omega_1(U)| = p^2.$$

Hence $G/\Omega_1(U)$ is elementary abelian of rank 2. In particular,

$$\Phi(G) \leqslant \Omega_1(U).$$

We let $y \in G \setminus U$ and have

$$|\langle y\rangle\Omega_1(U)| = p|\Omega_1(U)| = |U|.$$

Since $c(G) < p$, we see by Theorem 2.1 that $\Omega_1(G) = \{g \in G \mid g^p = 1\}$. Since $o(x) = p^2$ we have $G \neq \Omega_1(G)$, so $|G : \Omega_1(G)| \geqslant p$. Now if $o(y) = p$, then we have $\langle y\rangle\Omega_1(U) \leqslant \Omega_1(G)$ and see, by comparison of orders, that $\Omega_1(G) = \langle y\rangle\Omega_1(U)$ and hence $|G : \Omega_1(G)| = p$.

If $o(y) = p^2$, then $\langle y^p\rangle = \langle z\rangle$, and we may assume that $y^p = z^{-1}$. Applying the Hall-Petrescu Identity ([3], III 9.4 Satz), we see that there exist $c_2, \ldots, c_p$ with $c_2 \in K_2(G), \ldots, c_{p-1} \in K_{p-1}(G)$ and $c_p \in K_p(G)$ such that

$$x^p y^p = (xy)^p c_2^{\binom{p}{2}} \ldots c_{p-1}^{\binom{p}{p-1}} c_p.$$

Now, $c(G) < p$, so $c_p = 1$. In addition, p is a divisor of each of $\binom{p}{2}, \ldots, \binom{p}{p-1}$. Moreover, $\langle c_2, \ldots, c_{p-1}\rangle \leqslant G' \leqslant \Phi(G) \leqslant \Omega_1(G)$, so $c_2^p = \ldots = c_{p-1}^p = 1$. It follows that $1 = zz^{-1} = x^p y^p = (xy)^p$. But $xy \notin U$, as otherwise $U = G$. Hence $o(xy) = p$. We can now argue as above to see that

$$\Omega_1(G) = \langle xy\rangle\Omega_1(U),$$

and that $|G : \Omega_1(G)| = |G : \langle xy\rangle\Omega_1(U)| = p$. □

We note that the wreath product $G = C_p \operatorname{wr} C_p$ is a finite p-group that satisfies $Z(G) \simeq C_p$, $\exp(G) = p^2$ and $\exp(G/Z(G)) = p$. However, in this case we have $c(G) = p$ and $G = \Omega_1(G)$. This shows that the condition $c(G) < p$ in the statement of Lemma 2.3 is not redundant.

Lemma 2.4 *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic and $c(B) < \frac{p}{2}$. If $A \cap B = 1$, then $A\Omega_1(B) \leqslant G$.*

Proof — We may assume that A and B are both non-trivial subgroups of G and use induction on $|G|$. We also assume that $B \neq \Omega_1(B)$, as otherwise the result is trivial. By [7] Lemma 1, either $B_G \neq 1$

or $A_G \neq 1$. If $B_G = 1$ then we see, in particular, that

$$1 \neq \Omega_1(A) \leqslant Z(G) \cap A_G.$$

Then

$$A/\Omega_1(A) \cap B\Omega_1(A)/\Omega_1(A) = (A \cap B)\Omega_1(A)/\Omega_1(A) = 1_{G/\Omega_1(A)}.$$

By induction we have

$$A/\Omega_1(A)\Omega_1(B\Omega_1(A)/\Omega_1(A)) \leqslant G/\Omega_1(A).$$

Since $A \cap B = 1$, we see that

$$\Omega_1(B\Omega_1(A)/\Omega_1(A)) = \Omega_1(B)\Omega_1(A)/\Omega_1(A).$$

It follows that

$$A/\Omega_1(A)\Omega_1(B\Omega_1(A)/\Omega_1(A)) = A/\Omega_1(A)(\Omega_1(B)\Omega_1(A)/\Omega_1(A)),$$

and so we have $A\Omega_1(B)\Omega_1(A) = A\Omega_1(B) \leqslant G$.

We now assume that $B_G \neq 1$. Then $B_G \cap Z(G) \neq 1$, so there exists $z \in B_G \cap Z(G)$ such that $o(z) = p$. Now

$$A\langle z\rangle/\langle z\rangle \cap B/\langle z\rangle = (A \cap B)\langle z\rangle/\langle z\rangle = 1_{G/\langle z\rangle}$$

so, by induction, we have

$$A\langle z\rangle/\langle z\rangle\Omega_1(B/\langle z\rangle) \leqslant G/\langle z\rangle.$$

We let $\widetilde{B}/\langle z\rangle = \Omega_1(B/\langle z\rangle)$. Then $\Omega_1(B) \leqslant \widetilde{B} \leqslant B$. In particular, we have $\Omega_1(B) = \Omega_1(\widetilde{B})$. Now

$$A\langle z\rangle/\langle z\rangle\Omega_1(B/\langle z\rangle) = A\langle z\rangle/\langle z\rangle(\widetilde{B}/\langle z\rangle),$$

so

$$A\langle z\rangle\widetilde{B} = A\widetilde{B} \leqslant G.$$

Hence, if $\widetilde{B}$ is a proper subgroup of B, then

$$|A\widetilde{B}| < |AB| = |G|$$

so, by induction, we have

$$A\Omega_1(B) = A\Omega_1(\widetilde{B}) \leqslant A\widetilde{B} \leqslant G,$$

and are done. We thus assume that $\widetilde{B} = B$, so $\Omega_1(B/\langle z\rangle) = B/\langle z\rangle$. Since

$$c(B/\langle z\rangle) \leqslant c(B) < \frac{p}{2}$$

and $B \neq \langle z\rangle$ (as otherwise the result is trivial), we see by Theorem 2.1 that $\exp(B/\langle z\rangle) = p$. By Lemma 2.2, we then have

$$\Omega_1(A\langle z\rangle/\langle z\rangle)(B/\langle z\rangle) \trianglelefteq G/\langle z\rangle.$$

Since $A \cap B = 1$, we further have

$$\Omega_1(A\langle z\rangle/\langle z\rangle) = \Omega_1(A)\langle z\rangle/\langle z\rangle,$$

so

$$\Omega_1(A\langle z\rangle/\langle z\rangle)(B/\langle z\rangle) = \Omega_1(A)\langle z\rangle/\langle z\rangle(B/\langle z\rangle).$$

It follows that $\Omega_1(A)B \trianglelefteq G$. If $B \trianglelefteq G$, then $\Omega_1(B) \trianglelefteq G$ and so $A\Omega_1(B) \leqslant G$. If $B \ntrianglelefteq G$, then we let $g \in G \setminus N_G(B)$ and see, by comparison of orders, that $\Omega_1(A)B = BB^g$. But

$$|\Omega_1(A)B : B| = |\Omega_1(A)B : B^g| = p,$$

so B and B^g are both normal in $\Omega_1(A)B$. In addition, we have

$$c(B^g) = c(B) < \frac{p}{2},$$

so

$$c(\Omega_1(A)B) < \frac{p}{2} + \frac{p}{2} = p.$$

Again by Theorem 2.1, we see that

$$\Omega_1(\Omega_1(A)B) = \{x \in \Omega_1(A)B \mid x^p = 1\}.$$

Now if $\exp(B) = p$, then $B = \Omega_1(B)$ and we are done. We thus assume that $\exp(B) \neq p$. Since $\exp(B/\langle z\rangle) = p$, we then have $\exp(B) = p^2$. In particular, we have $\Omega_1(\Omega_1(A)B) \neq \Omega_1(A)B$, so

$$|\Omega_1(A)B : \Omega_1(\Omega_1(A)B)| \geqslant p.$$

On the other hand, since $c(B) < p/2$, we can apply Lemma 2.3 to see that $|B : \Omega_1(B)| = p$. In addition,

$$\Omega_1(A) \leqslant \Omega_1(\Omega_1(A)B) \quad \text{and} \quad \Omega_1(A) \cap \Omega_1(B) \leqslant A \cap B = 1.$$

Since $\Omega_1(A)$ normalises B, and hence normalises $\Omega_1(B)$, we have

$$\Omega_1(A)\Omega_1(B) \leqslant G \quad \text{and} \quad |\Omega_1(A)B : \Omega_1(A)\Omega_1(B)| = p.$$

But

$$\Omega_1(A)\Omega_1(B) \leqslant \Omega_1(\Omega_1(A)B),$$

so we conclude, by comparison of orders, that

$$\Omega_1(A)\Omega_1(B) = \Omega_1(\Omega_1(A)B) \trianglelefteq G.$$

It then follows that $A\Omega_1(B) = A\Omega_1(A)\Omega_1(B) \leqslant G$. □

Corollary 2.5 *Let p be an odd prime and let $G = AB$ be a finite p-group for non-trivial subgroups A and B such that A is cyclic and $c(B) < \frac{p}{2}$. If $A \cap B = 1$, then*

(i) $\Omega_1(B)^G \leqslant \Omega_1(A)\Omega_1(B) \leqslant G$;

(ii) $\exp(\Omega_1(B)^G) = p$.

Proof — By Lemma 2.4, we have $A\Omega_1(B) \leqslant G$. Noting that $c(B) < p/2$, we see, by Theorem 2.1, that $\exp(\Omega_1(B)) = p$. Hence, by Lemma 2.2, we have

$$\Omega_1(A)\Omega_1(B) \trianglelefteq A\Omega_1(B).$$

In particular, $\Omega_1(A)\Omega_1(B)$ is a subgroup of G. But $\Omega_1(B) \trianglelefteq B$, so

$$\Omega_1(B)^G = \Omega_1(B)^{BA\Omega_1(B)} = \Omega_1(B)^{A\Omega_1(B)} \leqslant \Omega_1(A)\Omega_1(B).$$

If $\Omega_1(B) \trianglelefteq G$, then $\exp(\Omega_1(B)^G) = \exp(\Omega_1(B)) = p$. If $\Omega_1(B) \ntrianglelefteq G$, then, by comparison of orders, we have $\Omega_1(B)^G = \Omega_1(A)\Omega_1(B)$. Hence $|\Omega_1(B)^G : \Omega_1(B)| = |\Omega_1(A)| = p$, so $\Omega_1(B) \trianglelefteq \Omega_1(B)^G$. In addition, letting $g \in G \setminus N_G(\Omega_1(B))$, we see, by comparison of orders, that

$$\Omega_1(B)^G = \Omega_1(B)\Omega_1(B)^g.$$

Thus $\Omega_1(B)^G$ is the product of two normal subgroups both of class less than $\frac{p}{2}$. It follows that $c(\Omega_1(B)^G) < p$. Since $\Omega_1(B)^G$ is generated by elements of order p, we again apply Theorem 2.1 to see that $\exp(\Omega_1(B)^G) = p$. □

We use Lemma 2.4 and Corollary 2.5 to prove the following more general result.

Theorem 2.6 *Let p an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic, $c(B) < \frac{p}{2}$ and $\exp(B) = p^k$ (where $k \geqslant 1$). If $A \cap B = 1$ then, for $1 \leqslant t \leqslant k$, we have:*

(i) $A\Omega_t(B) \leqslant G$;

(ii) $\Omega_t(B)^G \leqslant \Omega_t(A)\Omega_t(B) \leqslant G$;

(iii) $\exp(\Omega_t(B^G)) = p^t$.

Proof — The result holds for $t = 1$ by Lemma 2.4 and Corollary 2.5. Suppose that we have already shown that the result holds for some t with $1 \leqslant t < k$. Since $\Omega_t(B)^G \leqslant \Omega_t(A)\Omega_t(B) \leqslant G$, we have

$$\Omega_t(B)^G = \Omega_s(A)\Omega_t(B),$$

for some $s \leqslant t$. We let $W = \Omega_t(B)^G$ and note that $\exp(W) = p^t$. Hence if $g \in B$ is such that $g^p \in W$, then $g^{p^{t+1}} = 1$, so $g \in \Omega_{t+1}(B)$. Thus $\Omega_1(BW/W) = \Omega_{t+1}(B)W/W$.

Now

$$\begin{aligned} AW/W \cap BW/W &= (A \cap BW)W/W = (A \cap \Omega_s(A)\Omega_t(B)B)W/W \\ &= \Omega_s(A)(A \cap B)W/W = 1_{G/W}. \end{aligned}$$

We can then apply Lemma 2.4 to see that

$$(AW/W)\Omega_1(BW/W) \leqslant G/W.$$

Hence

$$(AW/W)\Omega_{t+1}(B)W/W \leqslant G/W,$$

so

$$A\Omega_{t+1}(B)W \leqslant G.$$

But

$$W = \Omega_s(A)\Omega_t(B) \subseteq A\Omega_{t+1}(B),$$

so

$$A\Omega_{t+1}(B) \leqslant G.$$

Since A is cyclic, we then have $\Omega_r(A)\Omega_{t+1}(B) \leqslant G$ for all r. By Corollary 2.5, we further have

$$\Omega_1(BW/W)^{G/W} \leqslant \Omega_1(AW/W)\Omega_1(BW/W).$$

Now

$$\begin{aligned} A \cap W = A \cap \Omega_s(A)\Omega_t(B) &= \Omega_s(A)(A \cap \Omega_t(B)) \\ &= \Omega_s(A) \leqslant \Omega_t(A). \end{aligned}$$

Hence

$$\Omega_1(AW/W) = \Omega_{s+1}(A)W/W \leqslant \Omega_{t+1}(A)W/W.$$

We then have

$$\begin{aligned} \Omega_{t+1}(B)^G W/W &= \Omega_1(BW/W)^{G/W} \\ &\leqslant (\Omega_{t+1}(A)W/W)(\Omega_{t+1}(B)W/W). \end{aligned}$$

It follows that

$$\Omega_{t+1}(B)^G \leqslant \Omega_{t+1}(A)\Omega_{t+1}(B)W = \Omega_{t+1}(A)\Omega_{t+1}(B) \leqslant G.$$

We finally note that $\exp(W) = p^t$ by assumption. Moreover, by Corollary 2.5, we see that

$$\exp(\Omega_{t+1}(B)^G W/W) = \exp(\Omega_1(BW/W)^{G/W}) = p.$$

But $t + 1 \leqslant k$, so $\exp(\Omega_{t+1}(B)) = p^{t+1}$. We thus conclude that $\exp(\Omega_{t+1}(B)^G) = p^{t+1}$. □

Remark 2.7 We note that a result of Huppert (see [2], Satz 3, or [1], Corollary 3.1.9) shows that if the p-group $G = AB$ is the product of the cyclic subgroups A and B, then G is the *totally permutable* product of A and B, that is $A_1B_1 \leqslant G$ for each $A_1 \leqslant A$ and $B_1 \leqslant B$. Since A and B are cyclic p-groups, this can be restated as $\Omega_s(A)\Omega_t(B) \leqslant G$ for all values of s and t. In general, we cannot expect that G will be a totally permutable product if A and B are non-cyclic subgroups. However, if p is odd, then in the case where A is cyclic, $c(B) < \frac{p}{2}$ and $A \cap B = 1$, it is a straightforward consequence of Theorem 2.6 (i) that $\Omega_s(A)\Omega_t(B) \leqslant G$, for all values of s and t. This can be viewed as a partial analogue to Huppert's result for products of cyclic subgroups.

The question now arises as to whether the results of Theorem 2.6 and Remark 2.7 also hold when $A \cap B \neq 1$. The following example shows that this is not always the case.

Example 2.8 We let p be a prime and let $A = \langle x \rangle \simeq C_{p^n}$, where $n \geqslant 3$. We further let $\langle y_1, \dots, y_p \rangle$ be an elementary abelian p-group of rank p. Now let $\langle x \rangle$ act on $\langle y_1, \dots, y_p \rangle$ as follows: $y_i^x = y_{i+1}$, $i = 1, \dots, p-1$ and $y_p^x = y_1$. We see that this action defines an automorphism of order p on $\langle y_1, \dots, y_p \rangle$. We let G be the semi-direct product of $\langle y_1, \dots, y_p \rangle$ by $\langle x \rangle$. Thus G can be expressed as follows:

$$G = \left\langle \begin{array}{c} y_1, \dots, y_p \\ x \end{array} \;\middle|\; \begin{array}{l} y_1^p = \dots = y_p^p = 1 = x^{p^n}; \\ [y_i, y_j] = 1,\ 1 \leqslant i < j \leqslant p \\ y_i^x = y_{i+1},\ i = 1, \dots, p-1;\ y_p^x = y_1 \end{array} \right\rangle.$$

We note that x^p centralises $\langle y_1, \dots, y_p \rangle$ and that the group $G/\langle x^p \rangle$ is isomorphic to the wreath product $C_p \operatorname{wr} C_p$. We let $A = \langle x \rangle$ and let $B = \langle y_2, \dots, y_p, x^p y_1 \rangle$. In particular

$$B = \langle y_2, \dots, y_p \rangle \times \langle x^p y_1 \rangle,$$

where $\langle y_2, \dots, y_p \rangle$ is elementary abelian of rank $p-1$ and $\langle x^p y_1 \rangle \simeq C_{p^{n-1}}$. Now

$$A \cap B = \langle x^{p^2} \rangle \simeq C_{p^{n-2}},$$

so

$$|AB| = \frac{|A||B|}{|A \cap B|} = p^{n+p} = |G|.$$

Hence $G = AB$. But, for $1 \leqslant t \leqslant n-2$,

$$\Omega_t(B) = \langle y_2, \dots, y_p, x^{p^{n-t}} \rangle = \Omega_t(A)\Omega_t(B),$$

whereas

$$\Omega_t(B)^G = \langle y_1, y_2, \dots, y_p, x^{p^{n-t}} \rangle.$$

Hence $\Omega_t(A)\Omega_t(B)$ is a proper subgroup of $\Omega_t(B)^G$. In particular,

$$\Omega_t(B)^G \nleqslant \Omega_t(A)\Omega_t(B).$$

We further note that, for $1 \leqslant t \leqslant n-2$, $\langle A, \Omega_t(B) \rangle = G$, but that

$$|A\Omega_t(B)| = p^{n+p-1}.$$

It follows that $A\Omega_t(B)$ is *not* a subgroup of G. Thus G also provides an example where the results of Lemma 2.4 and Theorem 2.6 (i) and (ii) fail when $A \cap B \neq 1$.

Having explored the limitations of Theorem 2.6, we note that if we relax the assumption that $A \cap B = 1$ in the statement of that theorem, then we have the following more general result which, in particular, generalises [6] Theorem 6.

Theorem 2.9 *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic, $c(B) < \frac{p}{2}$ and $\exp(B) = p^k$ (where $k \geqslant 1$). Then, for $1 \leqslant t \leqslant k$, we have*

(i) $|\Omega_t(B)^G : \Omega_t(B)| \leqslant p^t$;

(ii) $\exp(\Omega_t(B)^G) = p^t$.

In particular, we have $B^G \leqslant \Omega_k(A)B \trianglelefteq G$.

Proof — We first show that the result holds for $t = 1$. We let s be such that

$$A \cap B = \Omega_s(A).$$

We can assume that

$$\Omega_{s+1}(A) \nleqslant B,$$

as otherwise $A = \Omega_s(A)$ and the result is trivial. In particular, we have

$$\Omega_s(A) \simeq C_{p^s} \quad \text{and} \quad \Omega_{s+1}(A) \simeq C_{p^{s+1}}.$$

We let $W = \Omega_s(A)^G$. Then

$$W = \Omega_s(A)^{AB} = \Omega_s(A)^B \leqslant B.$$

Since $c(B) < p/2$, we see, by Theorem 2.1, that $\exp(\Omega_s(B)) = p^s$. In particular, we have $\exp(W) = p^s$. Now,

$$\begin{aligned} AW/W \cap B/W &= (AW \cap B)/W \\ &= (A \cap B)W/W = \Omega_s(A)W/W = 1_{G/W}. \end{aligned}$$

Hence, by Corollary 2.5, we have

$$\Omega_1(B/W)^{G/W} \leqslant \Omega_1(AW/W)\Omega_1(B/W) \leqslant G/W.$$

Now $A \cap W = A \cap B = \Omega_s(A)$, so

$$\Omega_1(AW/W) = \Omega_{s+1}(A)W/W.$$

Since $\Omega_1(B/W) \leqslant B/W$, we then have

$$\Omega_1(B)^G W/W \leqslant \Omega_1(B/W)^{G/W} \leqslant (\Omega_{s+1}(A)W/W)(B/W).$$

It follows that

$$\Omega_1(B)^G \leqslant \Omega_{s+1}(A)WB = \Omega_{s+1}(A)B.$$

Now $|\Omega_{s+1}(A)B : B| = p$, so $B \trianglelefteq \Omega_{s+1}(A)B$. But $\Omega_1(B)$ is characteristic in B, so $\Omega_1(B) \trianglelefteq \Omega_{s+1}(A)B$.

Since $\Omega_1(B)^G \leqslant \Omega_{s+1}(A)B$, we have $\Omega_1(B)^G \leqslant (\Omega_{s+1}(A)B)_G$. In addition,

$$(\Omega_{s+1}(A)B)_G = \bigcap_{a\in A, b\in B} (\Omega_{s+1}(A)B)^{ba}$$

$$= \bigcap_{a\in A} (\Omega_{s+1}(A)B)^a = \bigcap_{a\in A} \Omega_{s+1}(A)B^a.$$

Hence

$$\Omega_{s+1}(A) \leqslant (\Omega_{s+1}(A)B)_G.$$

Letting

$$B_1 = (\Omega_{s+1}(A)B)_G \cap B,$$

we then have

$$(\Omega_{s+1}(A)B)_G = \Omega_{s+1}(A)((\Omega_{s+1}(A)B)_G \cap B) = \Omega_{s+1}(A)B_1 \trianglelefteq G.$$

If $\Omega_1(B)^G \leqslant B_1$, then

$$\Omega_1(B)^G \leqslant \Omega_1(B_1) \leqslant \Omega_1(B),$$

so $\Omega_1(B) \trianglelefteq G$. In this case our result holds trivially, so we assume that $\Omega_1(B)^G \nleqslant B_1$. We have $\Omega_s(A)^G \leqslant B$, so $\Omega_s(A) \leqslant B_G \leqslant B_1$. Thus $|\Omega_{s+1}(A)B_1 : B_1| = p$, so $B_1 \trianglelefteq \Omega_{s+1}(A)B_1$. Hence, letting $g \in G$ be such that $\Omega_1(B)^g \nleqslant B_1$, we see, by comparison of orders, that

$$\Omega_{s+1}(A)B_1 = \Omega_1(B)^g B_1.$$

Moreover,

$$\Omega_1(B) \trianglelefteq \Omega_{s+1}(A)B,$$

so $\Omega_{s+1}(A)B_1$ is the product of the normal subgroups $\Omega_1(B)^g$ and B_1. Hence

$$c(\Omega_{s+1}(A)B_1) \leqslant c(\Omega_1(B)^g) + c(B_1) < \frac{p}{2} + \frac{p}{2} = p.$$

By Theorem 2.1, we then have $\exp(\Omega_1(\Omega_{s+1}(A)B_1)) = p$. It follows that

$$\Omega_1(\Omega_{s+1}(A)B_1) \cap B_1 = \Omega_1(B_1) = \Omega_1(B).$$

But $|\Omega_{s+1}(A)B_1 : B_1| = p$, so

$$\begin{aligned} &|\Omega_1(\Omega_{s+1}(A)B_1) : \Omega_1(B)| \\ &= |\Omega_1(\Omega_{s+1}(A)B_1) : \Omega_1(\Omega_{s+1}(A)B_1) \cap B_1| \leqslant p. \end{aligned}$$

But $\Omega_1(B)^g \nleqslant B_1$ so, by comparison of orders, we have

$$\Omega_1(B)^g\Omega_1(B) = \Omega_1(\Omega_{s+1}(A)B_1) \trianglelefteq G.$$

Hence

$$\Omega_1(B)^G = \Omega_1(\Omega_{s+1}(A)B_1).$$

In addition, we see that $|\Omega_1(\Omega_{s+1}(A)B_1)| = p|\Omega_1(B)|$, so

$$|\Omega_1(B)^G : \Omega_1(B)| = p.$$

Since

$$\exp(\Omega_1(B)^G) = \exp(\Omega_1(\Omega_{s+1}(A)B_1)) = p,$$

our result is thus established for $t = 1$. Now suppose that $k > 1$ and that we have shown that the result holds for some t with $1 \leqslant t < k$. We let $H = \Omega_t(B)^G$, and have $\exp(H) = p^t$. Thus $B \cap H = \Omega_t(B)$. Hence $\Omega_1(BH/H) = \Omega_{t+1}(B)H/H$ and we apply the result for $t = 1$ to see that

$$|(\Omega_{t+1}(B)H/H)^{G/H} : \Omega_{t+1}(B)H/H| \leqslant p$$

and that $\exp((\Omega_{t+1}(B)H/H)^{G/H}) = p$. Now

$$(\Omega_{t+1}(B)H/H)^{G/H} = \Omega_{t+1}(B)^G H/H$$

and $H = \Omega_t(B)^G \leqslant \Omega_{t+1}(B)^G$, so $\exp(\Omega_{t+1}(B)^G/\Omega_t(B)^G) = p$. But $\exp(\Omega_t(B))^G = p^t$, so $\exp(\Omega_{t+1}(B)^G) = p^{t+1}$. We further have

$$|\Omega_{t+1}(B)^G/H : \Omega_{t+1}(B)H/H| \leqslant p,$$

so

$$|\Omega_{t+1}(B)^G : \Omega_{t+1}(B)H| \leqslant p.$$

But $\Omega_t(B) \leqslant \Omega_{t+1}(B)$ and $|H : \Omega_t(B)| \leqslant p^t$. In addition we obtain $\Omega_t(B) \leqslant \Omega_{t+1}(B) \cap H$, so

$$\begin{aligned}|\Omega_{t+1}(B)H| = \frac{|\Omega_{t+1}(B)||H|}{|\Omega_{t+1}(B) \cap H|} &\leqslant \frac{|\Omega_{t+1}(B)||H|}{|\Omega_t(B)|} \\ = \frac{|H|}{|\Omega_t(B)|}|\Omega_{t+1}(B)| &\leqslant p^t|\Omega_{t+1}(B)|.\end{aligned}$$

Thus

$$\begin{aligned}&|\Omega_{t+1}(B)^G : \Omega_{t+1}(B)| \\ &= |\Omega_{t+1}(B)^G : \Omega_{t+1}(B)H||\Omega_{t+1}(B)H : \Omega_{t+1}(B)| \leqslant p \cdot p^t = p^{t+1}.\end{aligned}$$

We thus see that if the result holds for $1 \leqslant t < k$, then it also holds for $t + 1$. Hence our result is established for all values of t such that $1 \leqslant t \leqslant k$.

We finally note that $\Omega_k(B) = B$ so that, in particular, $\exp(B^G) = p^k$. But $B^G = (A \cap B^G)B$, so $\exp(A \cap B^G) \leqslant p^k$. Hence $A \cap B^G \leqslant \Omega_k(A)$, so $B^G \leqslant \Omega_k(A)B$. Since G/B^G is cyclic, then $B^G \leqslant \Omega_k(A)B \trianglelefteq G$. □

We note that, for $p = 3$, the restriction $c(B) < \frac{p}{2}$ in the statement of Theorem 2.9 requires the second "factor" B to be abelian. Theorem 2.11, the final result of this section, addresses the special case where $p = 3$, $c(B) = 2$ and $\exp(B) = 3$. We present the result in a more general form, as the proof may be of independent interest. We first derive a generalisation of [4] Lemma 1.

Lemma 2.10 *Let p be a prime and let $G = N_1N_2$ be a finite p-group for subgroups N_1 and N_2 such that $|G : N_1| = |G : N_2| = p$. Let $c = \max\{c(N_1), c(N_2)\}$. Then:*

(i) $|G'| \leqslant p|N_1'N_2'|$;

(ii) *if* $c \geqslant 2$, *then* $d(G) \leqslant c$.

Proof — We let $H = N_1'N_2'$ and let $W = N_1 \cap N_2$. Since

$$|G : N_1| = |G : N_2| = p,$$

we have

$$N_i \trianglelefteq G \quad \text{and} \quad G/N_i \simeq C_p \quad (i = 1, 2).$$

Hence $G/W \simeq C_p \times C_p$, so $H \leqslant G' \leqslant W$. Now N_1/H and N_2/H are abelian, so $W/H \leqslant Z(G/H)$. We let $x_i \in N_i \setminus W \quad (i = 1, 2)$. Then $G = \langle x_1, x_2, W\rangle$. Since

$$W/H \leqslant Z(G/H) \quad \text{and} \quad G/W \simeq C_p \times C_p,$$

we see that

$$\langle [x_1, x_2]\rangle H/H \leqslant Z(G/H)$$

and that $[x_1, x_2]^p \in H$. It follows that $G' = \langle [x_1, x_2]\rangle H$ and that

$$|G'| \leqslant p|H| = p|N_1'N_2'|.$$

Thus (i) is established.

For (ii), we let $Z = Z_{c-2}(W)$. We have

$$N_i' \leqslant Z_{c-1}(N_i) \cap W \leqslant Z_{c-1}(W) \quad (i = 1, 2),$$

so $H \leqslant Z_{c-1}(W)$. In particular, we have

$$HZ/Z \leqslant Z(W/Z).$$

But

$$G'Z/HZ = \langle [x_1, x_2]\rangle HZ/HZ,$$

so $G'Z/HZ$ is cyclic. Hence $G'Z/Z$ is abelian, so $G^{(2)} \leqslant Z$. Since $c(Z) \leqslant c - 2$, we then have

$$G^{(c)} \leqslant Z^{(c-2)} = 1,$$

as desired. □

Theorem 2.11 *Let p be an odd prime and let $G = AB$ be a finite p-group for subgroups A and B such that A is cyclic, $c(B) = 2$ and $\exp(B) = p$. Then $\Omega_2(A)B \trianglelefteq G$.*

Proof — We use induction on $|G|$. We can assume that

$$\Omega_2(A) \neq \Omega_3(A),$$

as otherwise $\Omega_2(A) = A$ and the result is trivial. In particular, we have $\Omega_3(A) \simeq C_{p^3}$. By [7] Lemma 1, either $A_G \neq 1$ or $B_G \neq 1$. Hence either $1 \neq Z(G) \cap A$ or $1 \neq Z(G) \cap B$. If $1 \neq Z(G) \cap B$, then we let

$$1 \neq b \in Z(G) \cap B.$$

Since $\exp(B) = p$, we have $\langle b \rangle \simeq C_p$. If $\langle b \rangle \nleqslant A$ then, by induction, we have

$$\Omega_2(A\langle b\rangle/\langle b\rangle)B/\langle b\rangle = (\Omega_2(A)\langle b\rangle/\langle b\rangle)B/\langle b\rangle \trianglelefteq G/\langle b\rangle.$$

It follows that $\Omega_2(A)B \trianglelefteq G$, as desired. If $\langle b \rangle \leqslant A$, then $\langle b \rangle = \Omega_1(A)$. By induction, we have

$$\Omega_2(A\langle b\rangle/\langle b\rangle)B/\langle b\rangle = (\Omega_3(A)/\langle b\rangle)B/\langle b\rangle \trianglelefteq G/\langle b\rangle.$$

Hence $\Omega_3(A)B \trianglelefteq G$.

If B^G is a proper subgroup of $\Omega_3(A)B$, then $B^G \leqslant \Omega_2(A)B$. Since G/B^G is cyclic, we then have $\Omega_2(A)B \trianglelefteq G$ and are done. We thus may assume that $B^G = \Omega_3(A)B$. If $B \trianglelefteq B^G$, then

$$B^G/B = \Omega_3(A)B/B \simeq \Omega_3(A)/(\Omega_3(A) \cap B) = \Omega_3(A)/\Omega_1(A) \simeq C_{p^2}.$$

We let $\{B^{g_1}, \dots, B^{g_n}\}$ be the set of conjugates of B in G. Since each conjugate of B is normal in B^G and

$$B^G/B_G = B^G/\bigcap_{i=1}^{n} B^{g_i},$$

we see that B^G/B_G is isomorphic to a subgroup of

$$B^G/B^{g_1} \times \dots \times B^G/B^{g_n}$$

which, in turn, is isomorphic to $C_{p^2} \times \dots \times C_{p^2}$. Hence B^G/B_G is abelian. Moreover, since $B^G/B \simeq C_{p^2}$, we see that

$$\exp(B^G/B_G) = p^2.$$

On the other hand, B^G/B_G is abelian and is generated by conjugates of B/B_G. Since $\exp(B) = p$, it follows that $\exp(B^G/B_G) = p$, so a contradiction arises. We may thus assume that $B \ntrianglelefteq B^G$.

Now
$$|B^G : \Omega_2(A)B| = |\Omega_3(A) : \Omega_2(A)| = p,$$
so $\Omega_2(A)B \trianglelefteq B^G$. Let $\Omega_3(A) = \langle x_1 \rangle$. Bearing in mind that $\Omega_1(A) \leqslant B$, we see, by comparison of orders, that $\Omega_2(A)B = BB^{x_1}$, where

$$|\Omega_2(A)B : B| = |\Omega_2(A)B : B^{x_1}| = p.$$

In addition, we have

$$c(B) = c(B^{x_1}) = 2 \quad \text{and} \quad \exp(B) = \exp(B^{x_1}) = p.$$

We let $W = \Omega_2(A)B$. By Lemma 2.10, we see that $d(W) = 2$, so W' is abelian. But $W/B \simeq C_p$, so $W' \leqslant B$. Hence $\exp(W') = p$, so W' is elementary abelian. In addition, W/W' is the product of the elementary abelian subgroups BW'/W' and $B^{x_1}W'/W'$, so W/W' is also elementary abelian. We note further that if A normalises W, then

$$B^G = \Omega_3(A)B \leqslant W = \Omega_2(A)B,$$

which is ruled out. Letting $A = \langle x \rangle$, we can thus assume, by comparison of orders, that $B^G = \Omega_3(A)B = WW^x$. Now

$$|B^G : W| = |B^G : W^x| = p,$$

so both W and W^x are normal in B^G. Hence both W' and $(W^x)'$ are normal elementary abelian subgroups of B^G. Thus $c(W'(W')^g) \leqslant 2$ and, by Lemma 2.1, $\exp(W'(W')^x) = p$. In addition, $B^G/W'(W')^x$ is the product of the normal elementary abelian subgroups $W/W'(W')^x$ and $W^x/W'(W')^x$, so we similarly see that

$$\exp(B^G/W'(W')^x) = p.$$

But then

$$p^3 = \exp(B^G) \leqslant \exp(B^G/W'(W')^x) \times \exp(W'(W')^x) = p^2,$$

so a contradiction arises. We thus conclude that if $1 \neq Z(G) \cap B$, then $\Omega_2(A)B \trianglelefteq G$.

Finally, if $Z(G) \cap B = 1$, then $1 \neq Z(G) \cap A$, so $\Omega_1(A) \leqslant Z(G)$. But then $\Omega_1(A)B = \Omega_1(A) \times B$ has class 2 and exponent p. We let

$$\widehat{B} = \Omega_1(A)B.$$

Then $G = A\widehat{B}$, where $1 \neq Z(G) \cap \widehat{B}$. Arguing as above, we can again show by induction that $\Omega_2(A)\widehat{B} = \Omega_2(A)B \trianglelefteq G$. □

3 An example

We present an example to show that there exist factorised 3-groups $G = AB$ where A is a cyclic subgroup and B is a subgroup of exponent 3 and class 2, but for which $\Omega_1(A)B \ntrianglelefteq G$ and for which $\exp(B^G) \neq 3$. Our example shows, in particular, that the requirement that $c(B) < \frac{p}{2}$ in the statement of Theorem 2.9 is not always redundant.

Example 3.1 We let U be the direct product of a non-abelian group of order 27 and exponent 3 with a cyclic group of order 3, presented as follows:

$$U = \left\langle \begin{array}{c} b_1,\ b_2\ ,b_3 \\ z \end{array} \;\middle|\; \begin{array}{l} b_1^3 = b_2^3 = b_3^3 = z^3 = 1;\ [b_1, z] = [b_2, z] = 1 \\ b_1^{b_2} = b_1 z;\ [b_1, b_3] = [b_2, b_3] = 1 \end{array} \right\rangle.$$

Thus $\langle b_1, b_2, z\rangle = \langle b_1, b_2\rangle$ is non-abelian of order 27 and exponent 3, while

$$\langle b_3\rangle \simeq C_3 \quad \text{and} \quad \langle b_3\rangle \leqslant Z(U).$$

We further have $U = \langle b_1, b_2\rangle \times \langle b_3\rangle$. We let $\langle a\rangle \simeq C_{27}$ and define an action of $\langle a\rangle$ on U by

$$b_1^a = b_1 b_2,\ b_2^a = b_2 b_3,\ b_3^a = b_3,\ z^a = z.$$

We have

$$(b_1^a)^{b_2^a} = (b_1 b_2)^{b_2 b_3} = b_1^{b_2} b_2 = b_1 z b_2 = b_1 b_2 z = b_1^a z^a.$$

Since the remaining relations are easily seen to be satisfied, we see that this action of a defines an automorphism of U.

We further note that

$$b_1^{a^2} = (b_1 b_2)^a = b_1 b_2 b_2 b_3 = b_1 b_2^2 b_3.$$

Thus

$$b_1^{a^3} = (b_1 b_2^2 b_3)^a = b_1 b_2 b_2^2 b_3^2 b_3 = b_1.$$

Since we also have $b_2^{a^3} = b_2$ and $b_3^{a^3} = b_3$, we see that U is centralised by a^3. We form the semi-direct product of U by $\langle a \rangle$ and denote this semi-direct product by U_1. We then identify a^9 with z to form the group $W = U_1/\langle z a^{-9} \rangle$. Thus W can be expressed as

$$\left\langle \begin{array}{c|l} b_1,\ b_2, & b_1^3 = b_2^3 = b_3^3 = z^3 = a^{27} = 1;\ [b_1, z] = [b_2, z] = 1; \\ b_3, & b_1^{b_2} = b_1 z;\ [b_1, b_3] = [b_2, b_3] = 1; \\ a,\ z & b_1^a = b_1 b_2;\ b_2^a = b_2 b_3;\ b_3^a = b_3;\ z^a = z;\ a^9 = z \end{array} \right\rangle.$$

We now let $\langle b_4 \rangle \simeq C_3$ and let b_4 act on W as follows:

$$b_1^{b_4} = b_1,\ b_2^{b_4} = b_2 z,\ b_3^{b_4} = b_3 z,\ z^{b_4} = z,\ a^{b_4} = a b_1^{-1} a^{-3}.$$

We show that the action of b_4 defines an automorphism, of order 3, of W. We note that

$$(b_1^{b_4})^{a^{b_4}} = b_1^{a b_1^{-1} a^{-3}} = (b_1 b_2)^{b_1^{-1} a^{-3}} = b_1 b_2^{b_1^{-1}}.$$

Now $b_1^{b_2} = b_1 z$, so $[b_1, b_2] = z$ and $[b_2, b_1] = z^{-1}$. Hence $b_2^{b_1} = b_2 z^{-1}$, so

$$b_2^{b_1^{-1}} = b_2^{b_1^2} = b_2 z^{-2} = b_2 z.$$

It follows that

$$(b_1^{b_4})^{a^{b_4}} = b_1 b_2 z = b_1^{b_4} b_2^{b_4}.$$

We further have

$$(b_2^{b_4})^{a^{b_4}} = (b_2 z)^{a b_1^{-1} a^{-3}}$$

$$= (b_2^a z)^{b_1^{-1} a^{-3}} = (b_2 b_3 z)^{b_1^{-1} a^{-3}} = b_2^{b_1^{-1}} b_3 z = b_2 z b_3 z.$$

Hence

$$(b_2^{b_4})^{a^{b_4}} = b_2 z b_3 z = b_2^{b_4} b_3^{b_4}.$$

Next, we check that $(a^{b_4})^{27} = 1$ and that $(a^{b_4})^9 = z^{b_4} = z$. We note first that

$$\begin{aligned}(a^{b_4})^3 &= (ab_1^{-1}a^{-3})^3 = (ab_1^{-1})^3a^{-9} \\ &= a^3(b_1^{-1})^{a^2}(b_1^{-1})^a b_1^{-1}a^{-9} \\ &= a^3(b_2^{-1}b_1^{-1})^a b_2^{-1}b_1^{-1}b_1^{-1}a^{-9} \\ &= a^3b_3^{-1}b_2^{-1}b_2^{-1}b_1^{-1}b_2^{-1}b_1^{-1}b_1^{-1}a^{-9} \\ &= a^3b_3^{-1}b_2^{-3}(b_1^{-1})^{b_2^{-1}}b_1^{-2}a^{-9} \\ &= a^3b_3^{-1}b_1^{-1}zb_1^{-2}a^{-9} = a^3b_3^{-1}zb_1^{-3}a^{-9}.\end{aligned}$$

Thus

$$(a^{b_4})^3 = a^3b_3^{-1}za^{-9}.$$

But $a^9 = z$, so

$$(a^{b_4})^3 = a^3b_3^{-1}.$$

It follows that

$$(a^{b_4})^9 = a^9b_3^{-3} = a^9 = z = z^{b_4}.$$

In addition, we have

$$(a^{b_4})^{27} = ((a^{b_4})^9)^3 = (a^9)^3 = a^{27} = 1.$$

Since the remaining relations are straightforward to verify, this confirms that b_4 defines an automorphism of W.

We finally check that $o(b_4) = 3$ in $\mathrm{Aut}(W)$. It is evident that

$$b_1^{b_4^3} = b_1, \quad b_2^{b_4^3} = b_2, \quad b_3^{b_4^3} = b_3 \quad \text{and} \quad z^{b_4^3} = z.$$

Thus we need only confirm that $a^{b_4^3} = a$. Now $a^3 \in Z(W)$ and, from above, $(a^3)^{b_4} = (a^{b_4})^3 = a^3b_3^{-1}$, so $(a^{-3})^{b_4} = a^{-3}b_3$. Hence

$$a^{b_4^2} = (ab_1^{-1}a^{-3})^{b_4} = ab_1^{-1}a^{-3}b_1^{-1}a^{-3}b_3 = ab_1^{-2}(a^{-3})^2b_3.$$

It follows that

$$\begin{aligned}a^{b_4^3} &= a^{b_4}(b_1^{-2})^{b_4}((a^{-3})^{b_4})^2b_3^{b_4} = ab_1^{-1}a^{-3}b_1^{-2}(a^{-3}b_3)^2b_3z \\ &= ab_1^{-3}a^{-3}a^{-6}b_3^2b_3z = aa^{-9}b_3^3z = aa^{-9}z.\end{aligned}$$

But $z = a^9$, so $a^{b_4^3} = a$. We conclude that $o(b_4) = 3$ in $\mathrm{Aut}(W)$.

We let G be the semi-direct product of W by $\langle b_3 \rangle$. Then G can be expressed as

$$\left\langle \begin{array}{c|l} b_1, b_2, & a^{27} = z^3 = b_i^3 = 1,\ i = 1, \ldots, 4;\ [b_i, z] = 1,\ i = 1, \ldots, 4 \\ b_3, b_4, & [b_1, b_2] = [b_2, b_4] = [b_3, b_4] = z;\ [b_1, b_3] = [b_2, b_3] = [b_1, b_4] = 1 \\ a, z & b_1^a = b_1 b_2;\ b_2^a = b_2 b_3;\ b_3^a = b_3;\ z^a = z;\ a^9 = z; a^{b_4} = a b_1^{-1} a^{-3} \end{array} \right\rangle.$$

We have $G = AB$, where $A = \langle a \rangle \simeq C_{27}$ and $B = \langle b_1, b_2, b_3, b_4, z \rangle$. We note that $\Omega_1(A) = \langle a^9 \rangle = \langle z \rangle \leqslant B$. We further see that $B' = \langle z \rangle$ and that B has class 2 and exponent 3. We note that $[a, b_4] = b_1^{-1} a^{-3}$. Hence $a^{-3} \in B^G \setminus B$. Thus

$$\Omega_1(A)B = B \ntrianglelefteq G.$$

In fact we have $B^G = \Omega_2(A)B$, in accordance with Theorem 2.11. In particular, we see that $\exp(B^G) = 9$.

We note that $b_2^{b_1 b_4} = (b_2 z^{-1})^{b_4} = b_2 z z^{-1} = b_2$. Thus

$$[\langle b_1, b_2 \rangle, \langle b_3, b_1 b_4 \rangle] = 1.$$

In addition, we have $b_3^{b_1 b_4} = b_3^{b_4} = b_3 z$. Hence $B = \langle b_1, b_2 \rangle \langle b_3, b_1 b_4 \rangle$ is the central product of $\langle b_1, b_2 \rangle$ and $\langle b_3, b_1 b_4 \rangle$, both of which are non-abelian subgroups of order 27 and exponent 3. In particular, we see that B is an extraspecial 3-group of order 3^5 and exponent 3.

4 Bounds for derived length

We first establish a bound for the derived length of p-groups of the type treated in Theorem 2.9.

Theorem 4.1 *Let* p *be an odd prime and let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is cyclic,* $c(B) < p/2$ *and* $\exp(B) = p^k$ $(k \geqslant 1)$. *Then* $d(G) \leqslant 1 + k + d(B)$.

Proof — By Theorem 2.9, we have $\Omega_k(A)B \trianglelefteq G$. In addition, the group $G/\Omega_k(A)B$ is isomorphic to a factor group of A, so that $G' \leqslant \Omega_k(A)B$. Now, for $i = 1, \ldots, k$, we have

$$|\Omega_i(A)B : \Omega_{i-1}(A)B| \leqslant |\Omega_i(A) : \Omega_{i-1}(A)| \leqslant p,$$

so

$$\Omega_{i-1}(A)B \trianglelefteq \Omega_i(A)B$$

and $\Omega_i(A)B/\Omega_{i-1}(A)B$ is isomorphic to a factor group of the cyclic group $\Omega_i(A)/\Omega_{i-1}(A)$. Hence

$$(\Omega_i(A)B)' \leqslant \Omega_{i-1}(A)B$$

for $i = 1, \ldots, k$, so $G^{(1+k)} \leqslant B$. It follows that $G^{(1+k+d(B))} = 1$. □

An alternative bound for derived length can be established in the case where B has exponent p.

Theorem 4.2 *Let* p *be a prime such that* $p \geqslant 5$ *and let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is cyclic,* $2 \leqslant c(B) < \frac{p}{2}$ *and* $\exp(B) = p$. *Then* $d(G) \leqslant 1 + c(B)$.

Proof — We let $c = c(B)$. By Theorem 2.9, we have

$$B^G \leqslant \Omega_1(A)B \trianglelefteq G.$$

Now $|\Omega_1(A)B : B| \leqslant p$, so either $B \trianglelefteq G$ or $B \ntrianglelefteq G$ and $B^G = \Omega_1(A)B$. In the first case G/B is abelian, so $G^{(1+c)} = 1$. In the second case we let $g \in G \setminus N_G(B)$. By comparison of orders we have

$$B^G = \Omega_1(A)B = BB^g.$$

We further see that

$$|\Omega_1(A)B : B| = |\Omega_1(A)B : B^g| = p,$$

so both B and B^g are normal subgroups of index p in $\Omega_1(A)B$. We can then apply Lemma 2.10 to see that $(\Omega_1(A)B)^{(c)} = 1$. Since $G/\Omega_1(A)B$ is abelian, it follows that $G^{(1+c)} = 1$. □

The particular case of Theorem 4.2 in which B has class 2 and exponent p yields the following corollary.

Corollary 4.3 *Let* p *be a prime such that* $p \geqslant 5$ *and let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is cyclic,* $c(B) = 2$ *and* $\exp(B) = p$. *Then* $G^{(3)} = 1$.

The following generalisation of [4] Theorem 5 will enable us to extend Corollary 4.3 to the case where $p = 3$. For $p \geqslant 5$, it will further allow us to extend Corollary 4.3 to the case where $c(B) = 2$ and $\exp(B) = p^2$. The proof is based on that of [4] Theorem 5. However, it is somewhat simpler than the original and differs significantly in detail.

Theorem 4.4 *Let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is abelian and* $B' \leqslant Z(B)$*. If* $|B^G : B| = p^2$ *then either* $(B')^G$ *is abelian; or there exists a subgroup* $B_1 \leqslant G$ *such that* $B_1' \leqslant Z(B_1)$*,* $|B_1^G : B_1| = p$ *and* G/B_1^G *is abelian.*

Proof — We assume that $(B')^G$ is non-abelian. Thus B is non-abelian and $c(B) = 2$. By say [4] Lemma 4, B has defect at least three in G. But G is a finite p-group and $|B^G : B| = p^2$, so the defect of B is exactly three. Letting $W = B^G$, we have $W = (A \cap W)B$ and see that there exist $w, x \in A \cap W$ such that

$$N_W(B) = \langle w \rangle B \quad \text{and} \quad W = \langle w, x \rangle B.$$

In addition, we can assume that

$$|W : \langle w \rangle B| = |\langle w \rangle B : B| = p,$$

so

$$B \trianglelefteq \langle w \rangle B \trianglelefteq \langle w, x \rangle B = W = B^G \trianglelefteq G.$$

Now

$$B' \leqslant W' \trianglelefteq G \quad \text{and} \quad W/\langle w \rangle B \simeq C_p,$$

so $W' \leqslant \langle w \rangle B$. Hence $(B')^G \leqslant \langle w \rangle B$. Now $c(B) = 2$, so $B \leqslant C_W(B')$. If $B \neq C_W(B')$, then B is a proper subgroup of

$$N_{C_W(B')}(B) = C_W(B') \cap N_W(B).$$

By comparison of orders, it follows that

$$\langle w \rangle B \leqslant N_{C_W(B')}(B) \leqslant C_W(B'),$$

so B' is centralised by $\langle w \rangle B$. But $(B')^G \leqslant \langle w \rangle B$, so

$$B' \leqslant (B')^G \cap Z(\langle w \rangle B) \leqslant Z((B')^G) \trianglelefteq G.$$

Hence $(B')^G$ is abelian, in contradiction to our assumption. We conclude that $B = C_W(B')$. In particular, for $g \in G$ we then have

$$B^g = C_{W^g}((B')^g) = C_W((B')^g).$$

If $(B')^G \leqslant B$, then

$$B' \leqslant Z(B) \cap (B')^G \leqslant Z((B')^G),$$

so again $(B')^G$ is abelian, which is excluded. Hence $(B')^G \nleqslant B$. Now the conjugates of B' are

$$\{(B')^{ba} \mid a \in A, b \in B\} = \{(B')^a \mid a \in A\}.$$

In addition, for $a \in N_A(\langle w \rangle B)$, we have

$$(B^{'})^a \leqslant ((\langle w \rangle B)')^a = (\langle w \rangle B)' \leqslant B.$$

Hence, if $(B')^{\widetilde{a}} \leqslant B$ for all $\widetilde{a} \in A \setminus N_A(\langle w \rangle B)$, then

$$(B')^G = \langle (B')^a \mid a \in A \rangle \leqslant B.$$

But this is ruled out, so there exists $y \in A \setminus N_A(\langle w \rangle B)$ such that

$$(B')^y \nleqslant B.$$

Thus

$$(\langle w \rangle B)^y = \langle w \rangle B^y \neq \langle w \rangle B.$$

In particular, $B^y \nleqslant \langle w \rangle B$, as otherwise $\langle w \rangle B^y = \langle w \rangle B$. Since

$$|\langle w \rangle B : B| = |\langle w, x \rangle B : \langle w \rangle B| = p$$

and $x \notin N_W(B)$, we see, by comparison of orders, that

$$\langle w \rangle B = BB^x = B(B')^y.$$

In addition, we have

$$W = \langle w, x \rangle B = \langle w \rangle BB^y = B(B')^y B^y = BB^y.$$

Hence

$$p^2|B| = |W| = \frac{|B||B^y|}{|B \cap B^y|} = \frac{|B|}{|B \cap B^y|}|B^y|,$$

so
$$|B : B \cap B^y| = p^2.$$
Now
$$|(B')^y : (B')^y \cap B| = |B(B')^y : B| = |\langle w \rangle B : B| = p.$$
Therefore, since $B = C_W(B')$, we have
$$\begin{aligned}|(B')^y : C_{(B')^y}(B')| &= |(B')^y : (B')^y \cap C_W(B')| \\ &= |(B')^y : (B')^y \cap B| = p.\end{aligned}$$
In particular, we see that B' and $(B')^y$ do not centralise each other. Thus, since $C_W((B')^y) = B^y$, we have $B' \nleqslant B^y$. Hence, by comparison of orders, we have $\langle w \rangle B^y = B^y B'$. We can then argue as above to see that
$$|B' : B' \cap B^y| = p.$$
Moreover, since $(B')^G \leqslant \langle w \rangle B = N_W(B)$, we have $B' \trianglelefteq (B')^G$. Thus we also have $(B')^y \trianglelefteq (B')^G$, so B' and $(B')^y$ normalise each other. We note that
$$(B')^y \cap B = (B')^y \cap B \cap B^y \leqslant (B')^y \cap B'(B \cap B^y) \leqslant (B')^y \cap B,$$
and so
$$(B')^y \cap B'(B \cap B^y) = (B')^y \cap B.$$

We define the subgroup $H \leqslant W$ by
$$H = \langle B', (B')^y, B \cap B^y \rangle.$$
Since B' and $(B')^y$ normalise each other and since both subgroups are centralised by $B \cap B^y$, we see that
$$H = (B')^y B'(B \cap B^y).$$
Thus
$$\begin{aligned}|H| = \frac{|(B')^y||B'(B \cap B^y)|}{|(B')^y \cap B'(B \cap B^y)|} &= \frac{|(B')^y||B'(B \cap B^y)|}{|(B')^y \cap B|} \\ &= p|B'(B \cap B^y)| = p\frac{|B'||B \cap B^y|}{|B' \cap B \cap B^y|} \\ &= p\frac{|B'|}{|B' \cap B^y|}|B \cap B^y| = p^2|B \cap B^y|.\end{aligned}$$

But $|B : B \cap B^y| = p^2$, so it follows that $|H| = |B|$.

Since B' and $(B')^y$ normalise each other, we have

$$[B', (B')^y] \leqslant B' \cap (B')^y \leqslant Z(B) \cap Z(B^y) \leqslant Z(BB^y) = Z(W).$$

In addition,

$$(B \cap B^y)' \leqslant B' \cap (B')^y \leqslant Z(W).$$

Hence, bearing in mind that $B \cap B^y$ centralises both B' and $(B')^y$, we have

$$H' = [B', (B')^y](B \cap B^y)' \leqslant Z(W) \cap B' \cap (B')^y \leqslant Z(H).$$

It follows that $c(H) \leqslant 2$. Moreover $H' \leqslant Z(W) \cap B'$, so $H' \leqslant B'$.

Now

$$H = B'(B \cap B^y)(B')^y \leqslant B(B')^y = \langle w \rangle B.$$

But $(B')^y \nleqslant B$ so, by comparison of orders, we have $\langle w \rangle B = BH$. In addition,

$$|\langle w \rangle B : H| = |\langle w \rangle B : B| = p,$$

so $H \trianglelefteq \langle w \rangle B$. By Lemma 2.10, we then see that $|(\langle w \rangle B)'| \leqslant p|H'B'|$. But $H' \leqslant B'$, so $|(\langle w \rangle B)'| \leqslant p|B'|$.

For x as above with

$$W = \langle w, x \rangle B,$$

we see that, if $(B')^x = B'$, then $x \in N_W(B')$. But then x normalises $C_W(B') = B$, which is ruled out. Hence B' is a proper subgroup of $B'(B')^x$. We have

$$B'(B')^x \leqslant (\langle w \rangle B)'$$

and, from above, $|(\langle w \rangle B)'| \leqslant p|B'|$. Thus, by comparison of orders, we have

$$(\langle w \rangle B)' = B'(B')^x.$$

It follows that

$$\begin{aligned} C_W((\langle w \rangle B)') &= C_W(B'(B')^x) \\ &= C_W(B') \cap C_W((B')^x) = B \cap B^x. \end{aligned}$$

Since $\langle w \rangle B \trianglelefteq W$, we have

$$C_W((\langle w \rangle B)') \trianglelefteq W,$$

so $B \cap B^x \trianglelefteq W$. Now

$$|\langle w \rangle B : B| = |\langle w \rangle B : B^x| = p,$$

so $B' \leqslant B \cap B^x$. Thus, if y normalises $B \cap B^x$, we have

$$(B')^y \leqslant B \cap B^x \leqslant B,$$

which is ruled out. Hence $B \cap B^x \ntrianglelefteq W\langle y \rangle$.

Now $B^y \leqslant W$ and $B \cap B^x \trianglelefteq W$. In addition, $(B')^y \nleqslant B$, so that $(B')^y \nleqslant B \cap B^x$. Thus

$$B^y(B \cap B^x)/(B \cap B^x)$$

is a non-abelian group. Since

$$|B : B \cap B^x| = |BB^x : B| = |\langle w \rangle B : B| = p,$$

we have

$$|W : B \cap B^x| = |W : B||B : B \cap B^x| = p^2 p = p^3.$$

But

$$B^y(B \cap B^x)/(B \cap B^x)$$

is non-abelian so, by comparison of orders, we have

$$B^y(B \cap B^x)/(B \cap B^x) = W/(B \cap B^x).$$

In particular, $W = B^y(B \cap B^x)$.

We see from above that $(\langle w \rangle B)' = B'[\langle w \rangle, B] = B'(B')^x$. Therefore, if $B \leqslant C_G([\langle w \rangle, B])$, then B centralises $(\langle w \rangle B)'$. In particular B centralises $(B')^x$, so $B \leqslant C_W((B')^x) = B^x$. But this is ruled out, so $B \nleqslant C_G([\langle w \rangle, B])$. Now,

$$[\langle w \rangle, B] = [\langle w \rangle, AB] = [\langle w \rangle, G] \trianglelefteq G.$$

In addition, $B \cap B^x = C_W((\langle w \rangle B)') \leqslant C_G([\langle w \rangle, B])$ so, by normality,

$$(B \cap B^x)^G \leqslant C_G([\langle w \rangle, B]).$$

Thus, in particular, we have $B \nleqslant (B \cap B^x)^G$.

We now let $T = (B \cap B^x)^G$. Then $T \leqslant W$ but, since $B \nleqslant T$, we have $T \neq W$. From above,

$$W/(B \cap B^x) = B^y(B \cap B^x)/(B \cap B^x)$$

is a non-abelian group of order p^3, so $Z(W/(B \cap B^x)) \simeq C_p$. We have

$$|W| = p^2|B| = |(A \cap W)B| = \frac{|A \cap W||B|}{|A \cap W \cap B|} = \frac{|A \cap W||B|}{|A \cap B|}.$$

Thus $|A \cap W : A \cap B| = p^2$. In addition, we have

$$A \cap B = (A \cap B)^x = A \cap B^x,$$

so

$$A \cap B = A \cap B \cap B^x = A \cap W \cap B \cap B^x.$$

Hence

$$|(A \cap W)(B \cap B^x)| = \frac{|A \cap W||B \cap B^x|}{|A \cap W \cap B \cap B^x|} = \frac{|A \cap W||B \cap B^x|}{|A \cap B|} = p^2|B \cap B^x|.$$

In addition, we have

$$|\langle w \rangle B : B \cap B^x| = |\langle w \rangle B : B||B : B \cap B^x| = p^2.$$

Hence

$$|\langle w \rangle B/(B \cap B^x)| = |(A \cap W)(B \cap B^x)/(B \cap B^x)| = p^2.$$

Now $A \cap W \nleqslant \langle w \rangle B$, since otherwise $W = (A \cap W)B \leqslant \langle w \rangle B$. Thus, by comparison of orders, we have

$$\begin{aligned} Z(W/(B \cap B^x)) &= \langle w \rangle B/(B \cap B^x) \cap (A \cap W)(B \cap B^x)/(B \cap B^x) \\ &= (\langle w \rangle B \cap (A \cap W))(B \cap B^x)/(B \cap B^x) \\ &= (\langle w \rangle (A \cap W \cap B))(B \cap B^x)/(B \cap B^x) \\ &= \langle w \rangle (A \cap B)(B \cap B^x)/(B \cap B^x). \end{aligned}$$

But

$$A \cap B = A \cap B^x \leqslant B \cap B^x,$$

so
$$Z(W/(B\cap B^x)) = \langle w\rangle(B\cap B^x)/(B\cap B^x) \quad (\simeq C_p).$$
Now $B\cap B^x \ntrianglelefteq W\langle y\rangle$, so $B\cap B^x$ is a proper subgroup of T. By normality, we then have
$$1 \neq T/(B\cap B^x)\cap Z(W/(B\cap B^x)),$$
so
$$\langle w\rangle(B\cap B^x)/(B\cap B^x) \leqslant T/(B\cap B^x).$$
Since $B \nleqslant T$ and $|B : B\cap B^x| = p$, we have $B\cap T = B\cap B^x$. Hence
$$BT/T \simeq B/(B\cap T) = B/(B\cap B^x) \simeq C_p.$$
Thus, by conjugation,
$$(BT/T)^{yT} = B^yT/T \simeq C_p.$$
Therefore, if
$$T/(B\cap B^x) = \langle w\rangle(B\cap B^x)/(B\cap B^x),$$
then
$$|B^yT : B\cap B^x| = |B^yT : T||T : B\cap B^x| = p^2,$$
so B^yT is a proper subgroup of W. But we have already shown that $W = B^y(B\cap B^x)$, so $W = B^yT$ and a contradiction arises. Hence $\langle w\rangle(B\cap B^x)/(B\cap B^x)$ is a proper subgroup of $T/(B\cap B^x)$. Since $T \neq W$, we then see, by comparison of orders, that $|W : T| = p$. Now $G = AW$, so
$$G/T = (AT/T)(W/T).$$
We have $W/T \simeq C_p$ and $W/T \trianglelefteq G/T$, so $W/T \leqslant Z(G/T)$. But AT/T is abelian, so G/T is the product of an abelian subgroup and a central subgroup. Hence G/T is abelian.

Now
$$\langle w\rangle(B\cap B^x) \leqslant T = (B\cap B^x)^G \leqslant C_G([\langle w\rangle, B]) \leqslant C_G([\langle w\rangle, B\cap B^x]).$$
In addition,
$$(B\cap B^x)' \leqslant B'\cap(B')^x,$$
which is centralised by $BB^x = \langle w\rangle B$. Thus, in particular, $\langle w\rangle(B\cap B^x)$

centralises $(B \cap B^x)'$. Now,

$$(\langle w \rangle (B \cap B^x))' = (B \cap B^x)'[\langle w \rangle, B \cap B^x],$$

and both $(B \cap B^x)'$ and $[\langle w \rangle, B \cap B^x]$ are centralised by $\langle w \rangle (B \cap B^x)$, so

$$(\langle w \rangle (B \cap B^x))' \leqslant Z(\langle w \rangle (B \cap B^x)).$$

Since

$$\langle w \rangle (B \cap B^x)/(B \cap B^x) \simeq C_p \quad \text{and} \quad |T/(B \cap B^x)| = p^2,$$

we have $|T : \langle w \rangle (B \cap B^x)| = p$. We finally let $B_1 = \langle w \rangle (B \cap B^x)$. Then

$$B_1' \leqslant Z(B_1) \quad \text{and} \quad B_1 \leqslant T = (B \cap B^x)^G,$$

so that $B_1^G = T$. Hence $|B_1^G : B_1| = p$ and we see from above that $G/B_1^G = G/T$ is abelian, as desired. □

Corollary 4.5 *Let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is abelian and* $B' \leqslant Z(B)$*. If* $|B^G : B| \leqslant p^2$*, then* $G^{(3)} = 1$.

Proof — If $|B^G : B| \leqslant p$, then B has subnormal defect at most two, and the result follows from [4] Lemma 4. If $|B^G : B| = p^2$ then, by Theorem 4.4, either $(B')^G$ is abelian and the result follows from [4] Lemma 3; or G has a subgroup B_1, of class at most two, such that $|B_1^G : B_1| = p$ and such that G/B_1^G is abelian. In the latter case $B_1 \trianglelefteq B_1^G$ and, letting $g \in G \setminus N_G(B_1)$, we see that $B_1^G = B_1 B_1^g$ is the normal product of two subgroups of class at most two and index p. If B_1 is abelian, then it is clear that $(B_1^G)^{(2)} = 1$. If $c(B_1) = 2$, then we can apply Lemma 2.10 to see that $(B_1^G)^{(2)} = 1$. Since G/B_1^G is abelian, we then conclude that $G^{(3)} = 1$. □

We use Corollary 4.5 to extend the result of Corollary 4.3.

Theorem 4.6 *Let* p *be an odd prime and let* $G = AB$ *be a finite* p*-group for subgroups* A *and* B *such that* A *is cyclic and* $c(B) = 2$*. If* $\exp(B) = p$ *or if* $p \geqslant 5$ *and* $\exp(B) = p^2$*, then* $G^{(3)} = 1$.

Proof — We apply Theorems 2.9 and 2.11 to see that in each case

$$B \leqslant \Omega_2(A)B \trianglelefteq G.$$

Hence $|B^G : B| \leqslant |\Omega_2(A)B : B| \leqslant p^2$. The result then follows from Corollary 4.5 □

5 Conclusion

Taken together with [5] and [6], this paper provides some initial steps in the direction of a theory of the structure of factorised finite p-groups $G = AB$, where A is a cyclic subgroup and B is a non-cyclic subgroup. A key feature of such groups is that each subgroup of A is permutable with B, that is, if $A_1 \leqslant A$ then $A_1B \leqslant G$. However, this need not be the case if A is non-cyclic. It remains an open question as to the extent to which the above results can be generalised to factorised groups where neither "factor" is cyclic.

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