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Some Theorems of Fitting Type *

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Abstract

Let $\mathfrak{W}$ be the class consisting of all groups G such that, if N is a normal subgroup of the term $G^{(\alpha)}$ of the derived series of G, for some ordinal α, then N is normal in G. Clearly, $\mathfrak{W}$ contains the class T of groups in which normality is a transitive relation studied by many authors. In this article we will prove, among other results, that a group which is the product of finitely many normal (generalized) soluble $\mathfrak{W}$-subgroups is locally supersoluble.

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1 Introduction

A fundamental theorem proved by H. Fitting in the first part of his Habilitationsschrift [4] states that if H and K are normal nilpotent subgroups of a group G, then their product HK is a normal nilpotent subgroup of G. On the other hand, it is well known that a similar result is not true for supersoluble groups. The first example of non-supersoluble group which is the product of two normal supersoluble subgroups was constructed by B. Huppert [6] (see also Example 1, p.186 of [1]). Recall that a group G is said to be *supersoluble* if

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it has a finite normal series containing the subgroups $\{1\}$ and G all of whose factors are cyclic. It can be proved that the commutator subgroup of a supersoluble group is nilpotent (Wendt's Theorem) and that all finitely generated nilpotent groups are supersoluble.

Let $\mathfrak{W}$ be the class consisting of all groups G such that, if N is a normal subgroup of the term $G^{(\alpha)}$ of the derived series of G, for some ordinal α, then N is normal in G. Examples of $\mathfrak{W}$-groups are all simple groups and all symmetric groups S_n for $n \geqslant 5$. Finite soluble groups with this property were introduced and investigated by I. Weidig [11], while the infinite case was considered by the author [10]. Clearly, a soluble $\mathfrak{W}$-group is *hypercyclic* (i.e., it has an ascending normal series with cyclic factors), and hence locally supersoluble. Moreover, since every group of power automorphisms is abelian, it follows that soluble $\mathfrak{W}$-groups have nilpotent commutator subgroup (recall that an automorphism of a group G is called a *power automorphism* if it maps every subgroup of G onto itself).

Note also that $\mathfrak{W}$ contains the class T of groups in which the normality is a transitive relation (i.e., the groups in which every subnormal subgroup is normal). The structure of finite soluble T-groups has been described by W. Gaschütz [5], while D.J.S. Robinson [7] investigated infinite soluble groups with the property T. In particular, it turns out that soluble T-groups are metabelian and that any finitely generated soluble T-group either is finite or abelian (see Theorem 2.3.1 and Theorem 3.3.1 of [7], respectively).

The main result of this article states that the product of a normal finitely generated (generalized) soluble $\mathfrak{W}$-subgroup and a subnormal supersoluble subgroup of a group is supersoluble. As a consequence, it will be proved that a group which is the product of finitely many normal (generalized) soluble $\mathfrak{W}$-subgroups is locally supersoluble. This result generalizes that one obtained by J. Cossey [3] for finite groups which are the product of normal soluble T-subgroups.

Most of our notation is standard and can be found in [9].

2 Statements and proofs

The class $\mathfrak{W}$ is obviously closed with respect to homomorphic images; on the other hand, $\mathfrak{W}$ is not subgroup-closed, since each finite group of order n can be embedded in the symmetric group S_n.

Our first result shows that if G is a $\mathfrak{W}$-group, then $G^{(3)} = G^{(4)}$. In particular, soluble $\mathfrak{W}$-groups have derived length at most 3. Recall that a group G is said to be *hypoabelian* if it has a descending series with abelian factors, or equivalently if G does not contain perfect non-trivial subgroups.

Proposition 2.1 *Let G be an hypoabelian $\mathfrak{W}$-group. Then the commutator subgroup G' of G is nilpotent with class at most 2. In particular, if G is a T-group, then it is metabelian.*

Proof — Let α be any ordinal number. Then G acts as a group of power automorphisms on $G^{(\alpha)}/G^{(\alpha+1)}$, and so G' is contained in the centralizer $C_G(G^{(\alpha)}/G^{(\alpha+1)})$. Since the group G is hypoabelian, it follows that G' is hypocentral and $\gamma_i(G') = G^{(i)}$ for each non-negative integer i. In particular, we have

$$\gamma_3(G') = G^{(3)} = (\gamma_2(G'))' \leqslant \gamma_4(G'),$$

so that $\gamma_3(G') = \{1\}$ and G' is nilpotent with class at most 2.

Finally, suppose that G is a T-group. As G' is nilpotent, then G acts on it as a group of power automorphisms. Therefore $G/C_G(G')$ is abelian and hence G is metabelian. □

Let p be an odd prime, and let

$$P = \langle a, b, c \mid a^p = b^p = c^p = [a, c] = 1,\ [b, c] = a \rangle$$

be an extraspecial group of order p^3 and exponent p. If x is the automorphism of P defined by the positions

$$a^x = a,\ b^x = b^{-1},\ c^x = c^{-1},$$

the semidirect product $G = \langle x \rangle \ltimes P$ is a soluble $\mathfrak{W}$-group with derived length 3. Clearly the direct product of G and an infinite cyclic group is a finitely generated $\mathfrak{W}$-group which is neither abelian nor finite.

Moreover, it is worth noting that a finitely generated $\mathfrak{W}$-group with non-periodic commutator subgroup is metabelian (see [10, Theorem 7]).

In the following we will use a result of Beidleman and Smith (see Proposition 7 of [2]) stating the supersolublity of a group which is the product of a normal finitely generated nilpotent subgroup and subnormal supersoluble subgroup.

Theorem 2.2 *Let* G *be any group. If* H *is a normal finitely generated hypoabelian* $\mathfrak{W}$*-subgroup of* G *and* K *is a subnormal supersoluble subgroup of* G*, then the product* HK *is supersoluble.*

Proof — Clearly, we may suppose that $G = HK$. By Proposition 2.1 the commutator subgroup H' of H is nilpotent with class at most 2, so that it enough to prove the statement for the factor group $G/H^{(2)}$ (see [8]). Therefore we may suppose that H is metabelian.

We argue by induction on defect d of K in G. First let K be normal in G. Since K is supersoluble, it contains a finite normal series

$$\{1\} = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_n = K$$

all of whose factors are cyclic. From this series we may construct the following series of $H' \cap K$:

$$\Sigma_1 : \{1\} = H' \cap K_0 \leqslant H' \cap K_1 \leqslant \ldots \leqslant H' \cap K_n = H' \cap K.$$

Clearly, every term of Σ_1 is normal in K and all its factors are cyclic. Moreover, as H is a $\mathfrak{W}$-group, we have that $H' \cap K_i$ is H-invariant (and hence G-invariant) for all $i = 0, 1, \ldots, n$. Since $H'/H' \cap K$ is a finitely generated abelian group and H is a $\mathfrak{W}$-group, then there is an H-invariant series

$$\Sigma_2 : H' \cap K = X_0 \leqslant X_1 \leqslant \ldots \leqslant X_m = H'$$

whose factors are cyclic. But $[X_i, K] \leqslant [H', K] \leqslant H' \cap K \leqslant X_i$, and hence we have also that every term of Σ_2 is G-invariant. The above considerations show that $\Sigma_1 \cup \Sigma_2$ is a G-invariant series of H' whose factor are cyclic. On the other hand, G/H' is supersoluble by the quoted result of Beidleman and Smith and hence so does G.

Finally, let $d > 1$. As $K^G = (H \cap K^G)K$ is the product of a normal finitely generated $\mathfrak{W}$-subgroup and a subnormal supersoluble subgroup of defect $d - 1$, the inductive hypothesis yields that K^G is supersoluble. Therefore $G = HK^G$ is supersoluble by the first part of the proof. □

Note that the latter result generally is not true if H is a subnormal subgroup with defect 2. To prove this we consider the group quoted in our introduction and constructed by R. Baer in [1, Example 1, p.186]). Let $A = \langle a \rangle \times \langle b \rangle$ be a non-cyclic group of order 9. Denote by x and y the automorphisms of A defined by the positions

$$a^x = b^{-1},\ b^x = a; \quad a^y = b,\ b^y = a.$$

Clearly, $|x| = 4$, $|y| = 2$ and $y^{-1}xy = x^{-1}$, so that $X = \langle x, y \rangle$ is isomorphic to the dihedral group D_8 of order 8. Put

$$G = X \ltimes A.$$

Since X acts irreducibly over A, it follows that G is not supersoluble. Let $L = \{1, x^2, y, x^2y\}$ and $M = \{1, x^2, yx, yx^3\}$ be the non-cyclic subgroups of order 4 of X. Clearly,

$$|G : AL| = 2 = |G : AM| \quad \text{and} \quad G = (AL)(AM).$$

Moreover, it is easy to show that AL and AM are supersoluble.

Put $H = \langle y \rangle \ltimes \langle a^{-1}b \rangle$ and $K = AM$. Since

$$(a^{-1}b)^y = (a^{-1}b)^{-1} = (a^{-1}b)^{x^2},$$

then $H \triangleleft AL$ and hence it is subnormal in G of defect 2. On the other hand, it is clear that $G = HK$.

Corollary 2.3 *Let G be a group which is the product of finitely many normal finitely generated hypoabelian $\mathfrak{W}$-subgroups. Then G is supersoluble.*

Proof — The statement follows immediately from Theorem 2.2. □

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of groups such that $\mathfrak{X} \leqslant \mathfrak{Y}$. We say that $\mathfrak{Y}$ is N_0-*closed with respect to* $\mathfrak{X}$ if for every normal subgroups H and K of a group G such that H is an $\mathfrak{X}$-group and K is an $\mathfrak{Y}$-group, the product HK is an $\mathfrak{Y}$-group. In particular, if $\mathfrak{X}=\mathfrak{Y}$, we have that the class $\mathfrak{X}$ is N_0-*closed* (see [9, Chapter 1]). Let $\mathfrak{X}$ be a subgroup-closed class of groups. Recall that the class $L\mathfrak{X}$ of *locally $\mathfrak{X}$-groups* consists of all groups whose finitely generated subgroups are $\mathfrak{X}$-groups.

Lemma 2.4 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be subgroup-closed classes of groups such that $\mathfrak{X} \leqslant \mathfrak{Y}$ and every $\mathfrak{Y}$-group is finitely generated. If $\mathfrak{Y}$ is N_0-closed with respect to $\mathfrak{X}$, then the class $L\mathfrak{Y}$ is N_0-closed with respect to $L\mathfrak{X}$*

Proof — The statement follows arguing as in the proof of Hirsch-Plotkin-Baer theorem (see [9, Theorem 2.31]). □

Theorem 2.5 *Let* G *be any group. If* H *is a normal hypoabelian* $\mathfrak{W}$*-subgroup of* G *and* K *is a subnormal locally supersoluble subgroup of* G*, then the product* HK *is locally supersoluble.*

Proof — Clearly, we may suppose that $G = HK$. By Proposition 2.1 the commutator subgroup H' of H is nilpotent with class at most 2. Since the factor group $G/H^{(2)}$ satisfies the same hypotheses of G, we may suppose that H is metabelian (see [8]). Denote by $\mathfrak{W}_1$ the class of finitely generated metabelian $\mathfrak{W}$-groups. Clearly, $\mathfrak{W}_1$ is subgroup-closed and by Theorem 2.2 the class of supersoluble groups is N_0-closed with respect to $\mathfrak{W}_1$. Now the statement follows from Lemma 2.4 arguing by induction on the defect of K in G. □

Corollary 2.6 *Let* G *be a group which is the product of finitely many normal hypoabelian* $\mathfrak{W}$*-subgroups. Then* G *is locally supersoluble.*

Proof — The statement follows immediately from Theorem 2.5. □

The Corollary 2.6 shows in particular that a group is locally supersoluble if it is the product of finitely many normal soluble T-subgroups. This result was proved by Cossey [3] for finite groups.

Indeed the groups which are product of finitely many normal soluble T-subgroups satisfy other interesting properties too. Firstly we need a preliminary lemma. Recall that a group G is said to be a *Baer group* if all its cyclic subgroups are subnormal. It is well known that a Baer group is locally nilpotent.

Lemma 2.7 *Let* G *be a Baer group. If* $G = H_1 \dots H_n$ *is the product of finitely many normal* T*-subgroups, then* G *is nilpotent with class at most* $n + 1$.

Proof — Clearly H_i is a Dedekind group for all $i = 1, \dots, n$. It follows that G is nilpotent by Fitting's theorem. Moreover, as H_i' is a normal subgroup of G of order at most 2, it is central and hence

$$H_1' \dots H_n' \leqslant Z(G).$$

On the other hand, it is clear that the factor group $G/H_1' \dots H_n'$ is nilpotent with class at most n. Therefore the nilpotent class of G is at most $n + 1$. □

Note that the latter result is not true for locally nilpotent groups as the consideration of locally dihedral 2-group D_{2^∞} shows.

Theorem 2.8 *Let*

$$G = H_1 \dots H_n$$

be a group which is the product of finitely many normal hypoabelian T*-subgroups. Then* G *is hypercyclic and* G' *is contained in*

$$\langle F_i \cap F_j \mid i, j = 1, \dots, n\rangle,$$

where each F_i *is the Fitting subgroup of* H_i*. In particular,* G' *is nilpotent with class at most* n*.*

Proof — By Proposition 2.1 we have that H_i is metabelian for all $i = 1, \dots, n$. It follows that

$$H_1' \cap \dots \cap H_n'$$

has a G-invariant ascending series all of whose factors are cyclic. Moreover, as the factor group

$$H_1 \cap \dots \cap H_n / H_1' \cap \dots \cap H_n'$$

is abelian, there is an ascending G-invariant series with cyclic factors containing the subgroups $H_1' \cap \dots \cap H_n'$ and $H_1 \cap \dots \cap H_n$. It follows that $H_1 \cap \dots \cap H_n$ is hypercyclically embedded in G. Finally, recalling that a soluble T-group is hypercyclic, we have that the direct product

$$G/H_1 \times \dots \times G/H_n$$

is hypercyclic. It follows that $G/H_1 \cap \dots \cap H_n$ is hypercyclic, and so does G by the first part of the proof.

Now let x and y be elements in H_i and H_j, respectively. We will prove that $[x, y] \in F_i \cap F_j$. Clearly, $[x, y] = x^{-1}x^y = (y^{-1})^x y \in H_i \cap H_j$ and the elements y and y^x act as power automorphisms over the abelian group H_j'. Let $z \in H_j'$. Then $z^y = z^n$, for some integer n. On the other hand,

$$(y^x)^{-1} z y^x = x^{-1}y^{-1}(xzx^{-1})yx.$$

Since a power automorphism of an abelian group is *homogeneous* (see Lemma 4.1.2 of [7]) and $\bar{z} = xzx^{-1} \in H_j'$, we have that $\bar{z}^y = \bar{z}^n$. Therefore

$$x^{-1}\bar{z}^n x = x^{-1}(xzx^{-1})^n x = z^n.$$

Thus

$$z^y = z^n = z^{y^x}$$

and so $z^{(y^{-1})^x y} = z$. It follows that $[x,y] \in C_{H_j}(H_j') = F_j$ (see Lemma 2.2.2 of [7]). Arguing in the same way, it can be proved that

$$[x,y] \in C_{H_i}(H_i') = F_i.$$

Thus $[x,y] \in F_i \cap F_j$ and hence $G' \leqslant \langle F_i \cap F_j \mid i,j = 1,\dots,n\rangle$ as required.

Finally, G' is nilpotent by Fitting's theorem and Lemma 2.7 yields that its nilpotent class is at most n. □

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