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From Groups to Leibniz Algebras: Common Approaches, Parallel Results

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Abstract

In this article, we study (locally) nilpotent and hyper-central Leibniz algebras. We obtained results similar to those in group theory. For instance, we proved a result analogous to the Hirsch-Plotkin Theorem for locally nilpotent groups.

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1 Introduction

The concept of nilpotency arises in many algebraic disciplines and plays a key role there. One of the sources of its origin were triangular matrices. The ring theoretical concept of a commutator of two triangular matrices led to the zero-triangular matrices, the nilpotency in associative rings, the lower central series, and the concept of nilpotency in Lie algebras. The concept of a group-theoretical commutator of two nonsingular triangular matrices led to unitriangular matrices, and to the concept of the lower central series in the group of matrices. At the first stage, this commonality of origin brought some parallelism in approaches, however then the specificity of each theory introduces its own modifications. Nevertheless, it turned out that in many cases, the same approaches led to comparable results in groups and Lie algebras. This parallelism runs through the book [1], it was noted in many articles devoted to Lie algebras, in particular, in the paper [17]. One of the interesting generalizations of Lie algebras is Leibniz algebras. Therefore, the following question naturally arises: Which of the group-theoretical concepts and results have analogs in Leibniz algebras? An algebra L over a field F is said to be a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if it satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] \qquad \text{for all } a,b,c \in L. \quad \text{(LI)}$$

Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a,a] = 0$ for every element $a \in L$. By this reason, we may consider Leibniz algebras as "non-anticommutative" analogs of Lie algebras. Leibniz algebras appeared first in the papers of A.M. Bloh [4],[5],[6],... in which he called them the D-algebras. However, at that time these researches were not in demand, and they have not been properly developed. Real interest in Leibniz algebras arose only after two decades. This happened thanks to J.L. Loday [12], who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Leibniz who discovered and proved the "Leibniz rule" for differentiation of functions.

The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. The theory of Leibniz algebras develops quite intensively now, however, it should be noted that most of the obtained results refer to finite-dimensional Leibniz algebras, and in the greater part of the latter, algebras over fields of characteristic zero are only considered. This also applies to nilpotent Leibniz algebras. The concept of nilpotency for the Leibniz algebras is introduced as follows. Let L be a Leibniz algebra over a field F. If A, B are subspaces of L, then $[A,B]$ will denote a subspace, generated by all elements $[a,b]$ where $a \in A, b \in B$. We note that if A is an ideal of L, then $[A,A]$ is also an ideal of L.

If M is non-empty subset of L, then $\langle M \rangle$ denotes the subalgebra of L generated by M.

Let L be a Leibniz algebra. We define the lower central series of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L, \gamma_2(L) = [L, L]$, and recursively,

$$\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$$

for all ordinals α, while

$$\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$$

for limit ordinals λ. It is possible to shows that every term of this series is an ideal of L. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \dots, L] \dots]$.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* c if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$. We denote by $ncl(L)$ the nilpotency class of L.

In some algebraic structures, another definition of nilpotency based on the concept of the (upper) central series is used. In fact, suppose that L is a nilpotent Leibniz algebra and $\gamma_{k+1}(L) = \langle 0 \rangle$. For each factor $\gamma_j(L)/\gamma_{j+1}(L)$ we have

$$[L, \gamma_j(L)] = \gamma_{j+1}(L) \quad \text{and} \quad [\gamma_j(L), L] \leqslant \gamma_{j+1}(L),$$

and this leads us to the following concepts. Let A, B be the ideal of L such that $A \leqslant B$. The factor B/A is called *central* (in L) if

$$[L, B], [B, L] \leqslant A.$$

The center $\zeta(L)$ of a Leibniz algebra L is defined in the following way:

$$\zeta(L) = \{x \in L \,|\, [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

Clearly, $\zeta(L)$ is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$. Starting from the center we can define the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \dots \leqslant \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \dots \zeta_\gamma(L) = \zeta_\infty(L)$$

of Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively

$$\zeta_{\alpha+1}(L)/\zeta_{\alpha}(L) = \zeta(L/\zeta_{\alpha}(L))$$

for all ordinals α, while

$$\zeta_{\lambda}(L) = \bigcup_{\mu<\lambda} \zeta_{\mu}(L)$$

for limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_{\infty}(L)$ of this series is called the *upper hypercenter* of L. A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter. Denote by $zl(L)$ the length of upper central series of L. In the paper [11], the connection between the lower and upper central series in nilpotent Leibniz algebras has been considered. It was proved that in this case, the lengths of the lower and upper central series coincide. Moreover, they are the least among the lengths of all other central series.

The concepts of upper and lower central series introduced here immediately lead to the following classes of Leibniz algebras.

A Leibniz algebra L is said to be *hypercentral* if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be *hypocentral* if it coincides with the lower hypercenter.

In the case of finite dimensional algebras, these two concepts coincide, but in general, these two classes are very different. Thus, for finitely generated hypercentral Leibniz algebras we have the following theorem.

Theorem A. *Let* L *be a finitely generated Leibniz algebra over a field* F*. If* L *is hypercentral, then* L *is nilpotent. Moreover,* L *has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group theoretical result proved by A.I. Mal'cev (see [13]).

At the same time, a finitely generated hypocentral Leibniz algebra can have infinite dimension. Thus, a cyclic Leibniz algebra $\langle a \rangle$ where an element a has infinite depth is hypocentral and has infinite dimension (see [8]).

A Leibniz algebra L is said to be *locally nilpotent* if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras. We obtained the following characterization of hypercentral Leibniz algebras.

Theorem B. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if for each element $a \in L$ and every countable subset $\{x_n | n \in \mathbb{N}\}$ of elements of L there exists a positive integer k such that all commutators $[x_1, \dots, x_j, a, x_{j+1}, \dots, x_k]$ are zeros for all $j, 0 \leqslant j \leqslant k$.*

Corollary. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if every subalgebra of L having finite or countable dimension is hypercentral.*

These results are analogues to the results proved for groups by S.N. Chernikov (see [7]).

Let L be a Leibniz algebra. If A, B are nilpotent ideals of L, then their sum $A + B$ is a nilpotent ideal of L (see [3], Lemma 1.5). In this connection, the following question arises: is an analogous assertion valid for locally nilpotent ideals? As it was shown by B. Hartley (see [9]), for Lie algebras this assertion takes place. Our next result gives a positive answer to this question.

Theorem C. *Let L be a Leibniz algebra over a field F, A, B be locally nilpotent ideals of L. Then $A + B$ is locally nilpotent.*

Corollary C1. *Let L be a Leibniz algebra over a field F and S be a family of locally nilpotent ideals of L. Then a subalgebra generated by S is locally nilpotent.*

Corollary C2. *Let L be a Leibniz algebra over a field F. Then L has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over field F. The greatest locally nilpotent ideal of L is called the *locally nilpotent radical* of L and will be denoted by $\mathrm{Ln}(L)$.

These results are the analogues to the results for groups proved by K.A. Hirsch (see [10]) and B.I. Plotkin (see [15]); see also the survey [16].

The subalgebra $\mathrm{Nil}(L)$ generated by all nilpotent ideals of L is called the *nil-radical* of L. Clearly $\mathrm{Nil}(L)$ is an ideal of L. If $L = \mathrm{Nil}(L)$, then L is called a Leibniz *nil-algebra*. Every nilpotent Leibniz algebra is a nil-algebra, but the converse is not true even for a Lie algebra. Every Leibniz nil-algebra is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil-algebra, which is not hypercentral (see, for example, [1], Chapter 6).

Note the following important properties of locally nilpotent Leibniz algebras.

Theorem D. *Let* L *be a locally nilpotent Leibniz algebra over a field* F.

(i) *If* A, B *are ideals of* L *such that* $A \leqslant B$ *and the factor* B/A *is* L*-chief, then* B/A *is central in* L *(that is* $B/A \leqslant \zeta(L/A)$*). In particular, we have that* $\dim_F(B/A) = 1$.

(ii) *If* A *is a maximal subalgebra of* L, *then* A *is an ideal of* L.

Let L be a Leibniz algebra over a field F and H a subalgebra of L. The *idealizer* of H is defined by the following rule:

$$\mathbb{I}_L(H) = \{x \in L \,|\, [h, x], [x, h] \in H \text{ for all } h \in H\}.$$

It is possible to prove that the idealizer of H is a subalgebra of L. If L is a hypercentral (in particular, nilpotent) Leibniz algebra, then $H \neq \mathbb{I}_L(H)$ (see Proposition 1.10 below). This leads us to the following class of Leibniz algebras.

Let L be a Leibniz algebra over field F. We say that L *satisfies the idealizer condition* if $\mathbb{I}_L(A) \neq A$ for every proper subalgebra A of L.

A subalgebra A is called *ascendant* in L, if there is an ascending chain of subalgebras

$$A = A_0 \leqslant A_1 \leqslant \dots A_\alpha \leqslant A_{\alpha+1} \leqslant \dots A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant. The last our result is the following

Theorem E. *Let* L *be a Leibniz algebra over a field* F. *If* L *satisfies the idealizer condition then* L *is locally nilpotent.*

This result is an analogue to the result proved for groups in [14] by B.I. Plotkin.

Again, it should be noted that Leibniz algebras with the idealizer condition will form a subclass of the class of locally nilpotent Leibniz algebras, since this is already the case for Lie algebras (see, for example, [1], Chapter 6).

2 On hypercentral Leibniz algebras

Proposition 2.1 *Let* L *be a finitely generated Leibniz algebra over a field* F*. Let* H *be an ideal of* L *having finite codimension. Then* H *is finitely generated as an ideal.*

Proof — Let

$$M = \{a_1, \dots, a_n\}$$

be a finite subset generated L, and let B be a subspace of L such that $L = B \oplus H$. Let $\mathrm{codim}_F(H) = d$. Then $\dim_F(B) = d$. Choose in B some basis $\{b_1, \dots, b_d\}$. Denote by pr_B (respectively pr_H) the canonical projection of L on B (respectively H). Let E be the ideal, generated by the elements

$$\{pr_H(a_j), pr_H([a_j, b_m]), pr_H([b_m, a_j]) | 1 \leqslant j \leqslant n, 1 \leqslant m \leqslant d\}.$$

By such choice H includes E, and E is a finitely generated as an ideal of L. If x is an arbitrary element of $E + B$, then $x = u + b$ where $u \in E$ and $b \in B$. Furthermore

$$b = \alpha_1 b_1 + \ldots + \alpha_d b_d$$

for suitable elements $\alpha_1, \dots, \alpha_d \in F$. We have

$$[b, a_j] = [\alpha_1 b_1 + \ldots + \alpha_d b_d, a_j] = \alpha_1 [b_1, a_j] + \ldots + \alpha_d [b_d, a_j] =$$

$$\alpha_1(pr_H([b_1, a_j]) + pr_B([b_1, a_j]) + \ldots + \alpha_d(pr_H([b_d, a_j]) + pr_B([b_d, a_j]) =$$

$$\alpha_1 pr_H([b_1, a_j]) + \ldots + \alpha_d pr_H([b_d, a_j]) + \alpha_1 pr_B([b_1, a_j]) + \ldots + \alpha_d pr_B([b_d, a_j]);$$

$$[a_j, b] = [a_j, \alpha_1 b_1 + \ldots + \alpha_d b_d] = \alpha_1 [a_j, b_1] + \ldots + \alpha_d [a_j, b_d] =$$

$$\alpha_1 (pr_H([a_j, b_1]) + pr_B([a_j, b_1]) + \ldots + \alpha_d (pr_H([a_j, b_d]) + pr_B([a_j, b_d]) =$$

$$\alpha_1 pr_H([a_j, b_1]) + \ldots + \alpha_d pr_H([a_j, b_d]) + \alpha_1 pr_B([a_j, b_1]) + \ldots + \alpha_d pr_B([a_j, b_d]).$$

The elements

$$\Sigma 1 \leqslant m \leqslant d(\alpha_m pr_H([b_m, a_j]) + \alpha_m pr_B([b_m, a_j])),$$

and

$$\Sigma 1 \leqslant m \leqslant d(\alpha_m pr_H([a_j, b_m]) + \alpha_m pr_B([a_j, b_m]))$$

clearly belong to $E + B$. It follows that $E + B$ is an ideal of A. Since

$$a_j = pr_H(a_j) + pr_B(a_j) \in E + B, 1 \leqslant j \leqslant n,$$

then

$$E + B = A = H + B.$$

The inclusion $E \leqslant H$ and the equation $H \cap B = \langle 0 \rangle$ imply that $H = E$. In particular, H is a finitely generated as an ideal. □

Corollary 2.2 *Let* L *be a finitely generated Leibniz algebra over a field* F. *If* L *is nilpotent, then* L *has finite dimension.*

Proof — Let

$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_n = L$$

be the upper central series of L. Proposition 2.1 shows that Z_{n-1} is finitely generated as an ideal, since L/Z_{n-1} is abelian and the dimension $\dim_F(L/\zeta_{n-1}(L))$ is finite. The inclusion

$$Z_{n-1}/Z_{n-2} \leqslant \zeta(L/Z_{n-2})$$

implies that Z_{n-1}/Z_{n-2} is finitely generated as a subalgebra. In turn out, it follows that $\dim_F(Z_{n-1}/Z_{n-2})$ is finite. Then $\dim_F(L/Z_{n-2})$ is finite. Using the similar arguments and ordinary induction we prove that $\dim_F(L)$ is finite. □

Proof of Theorem A — Let

$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots \leqslant Z_\alpha \leqslant Z_{\alpha+1} \leqslant \ldots Z_\gamma = \zeta_\infty(L) = L$$

be the upper central series of L. Since L is finitely generated, γ is not a limit ordinal. Suppose that γ is infinite, then $\gamma = \kappa + n$ for some limit ordinal $\kappa \geqslant \omega$. Then L/Z_κ is a nilpotent finitely generated Leibniz algebra, and Corollary 2.2 shows that L/Z has finite dimension. Then Proposition 2.1 implies that Z_κ is finitely generated as an ideal. Let

$$W = \{w_1, \ldots, w_m\}$$

be a finite subset such that Z_κ is generated by W as ideal. From the equation

$$Z_\kappa = \bigcup_{\beta<\kappa} Z_\beta$$

we obtain that $w_j \in Z_{\beta(j)}$ for some $\beta(j) < \tau, 1 \leqslant j \leqslant m$. Let σ be the greatest ordinal from the set

$$\{\beta(1), \dots, \beta(m)\}.$$

Then $w_j \in Z_\sigma$ for all $j, 1 \leqslant j \leqslant m$. Since Z_σ is an ideal, it follows that Z_σ includes the ideal of L, generated by elements $w_1, \dots, w_m$. But the last coincides with Z_κ, so that $Z_\kappa \leqslant Z_\sigma$, and we obtain a contradiction. This contradiction shows that γ must be finite. In other words, L is nilpotent and we can apply Corollary 2.2. □

Corollary 2.3 *Let L be a hypercentral Leibniz algebra over a field F. Then L is locally nilpotent.*

We will obtain some properties of hypercentral Leibniz algebras.

Lemma 2.4 *Let L be a hypercentral Leibniz algebra over a field F and A be a non-zero ideal of L. Then the intersection $A \cap \zeta(L)$ is non - zero.*

Proof — Let

$$\langle 0\rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \zeta_2(L) \leqslant \dots \leqslant \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \dots \zeta_\gamma(L) = L$$

be the upper central series of L. Choose the least ordinal β such that $A \cap \zeta_\beta(L) \neq \langle 0\rangle$. Clearly β is not a limit ordinal. So $\beta - 1$ exists. Let

$$0 \neq a \in A \cap \zeta_\beta(L)$$

and let x be an arbitrary element of L. Since A is an ideal of L, it follows that $[x, a], [a, x] \in A$. On the other hand, since $a \in \zeta_\beta(L)$, we obtain that $[x, a], [a, x] \in \zeta_{\beta-1}(L)$, so that

$$[x, a], [a, x] \in A \cap \zeta_{\beta-1}(L) = \langle 0\rangle.$$

It follows that $a \in \zeta(L)$. □

Now we obtain the following characterization of hypercentral Leibniz algebras.

Proof of Theorem B — Let

$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \dots Z_\alpha \leqslant Z_{\alpha+1} \dots Z_\gamma = L$$

be the upper central series of L. Then there exists the least ordinal κ such that $a \in Z_\kappa$. By such a choice $a \notin Z_\alpha$ for all $\alpha < \kappa$. For the proof we will use an induction by κ. If $\kappa = 1$, then

$$[x_1, a] = [a, x_1] = 0.$$

Assume now that $\kappa > 1$, and we have already proved our assertion for all elements of Z_α where $\alpha < \kappa$. If κ is a limit ordinal, then $a \in Z_\beta$ for some $\beta < \kappa$, and we come to contradiction. Thus κ is not limit ordinal, so that $\kappa - 1$ exists. Since

$$Z_\kappa / Z_{\kappa-1} \leqslant \zeta(L/Z_{\kappa-1}),$$

then

$$[x_1, a], [a, x_1] \in Z_{\kappa-1}.$$

Hence for elements $[x_1, a], [a, x_1]$ we can use the induction hypothesis. An application of induction hypothesis shows that there are the positive integers m, s such that

$$[x_1, \dots, x_j, a, x_{j+1}, \dots, x_m] = 0, \quad 1 \leqslant j \leqslant m,$$

$$[x_1, \dots, x_m, a] = 0,$$

$$[x_1, \dots, x_j, a, x_{j+1}, \dots, x_s] = 0, \quad 1 \leqslant j \leqslant s - 1,$$

$$[a, x_1, \dots, x_s] = 0$$

at any arrangement of parentheses.

Put $k = \max\{m, s\}$, then

$$[x_1, \dots, x_j, a, x_{j+1}, \dots, x_k] = 0$$

for all $j, 0 \leqslant j \leqslant k$.

To prove the sufficiency of the condition, we first note that this condition is inherited by each factor-algebra of L. Hence it suffices to prove that $\zeta(L) \neq \langle 0 \rangle$. Assume that this is false. Let a be an arbitrary non-zero element of L. Since $\zeta(L) = \langle 0 \rangle$, $a \notin \zeta(L)$. It follows that there exists an element x_1 such that $[x_1, a] \neq 0$ or $[a, x_1] \neq 0$. For definiteness, we assume that $[x_1, a] \neq 0$. Since $[x_1, a] \notin \zeta(L)$, there exists an element x_2 such that $[x_2, [x_1, a]] \neq 0$ or $[[a, x_1], x_2] \neq 0$. Using the same arguments we find for every positive integer k the elements $x_1, x_2, \dots, x_k$ such that there exists a non-zero commutator $[x_1, \dots, x_j, a, x_{j+1}, \dots, x_k]$, a contradiction. Therefore $\zeta(L) \neq \langle 0 \rangle$.□

Proof of Corollary to Theorem B — If L is hypercentral, then each of its subalgebra is hypercentral and hence every countable dimensional subalgebra is certainly hypercentral.

Conversely, suppose that every countable dimensional subalgebra of L is hypercentral, but L is not hypercentral. Factoring by the hypercentre we may suppose that $\zeta(L) = \langle 0 \rangle$. Using the arguments from the proof of Theorem B we find an element a and a countable subset $\{x_n | n \in \mathbb{N}\}$ such that for every positive integer n there exists a non-zero commutator

$$[x_1, \dots, x_j, a, x_{j+1}, \dots, x_n].$$

The subalgebra $\langle a, x_n | n \in \mathbb{N} \rangle$ has finite or countable dimension and hence is hypercentral. Theorem B shows that there exists a positive integer k such that all commutators

$$[x_1, \dots, x_j, a, x_{j+1}, \dots, x_k]$$

are zeros, and we obtain a contradiction. □

Let H be a subalgebra of L. The *left idealizer* or the *left normalizer* of H in L is defined as the following:

$$\mathbb{I}_L^{\text{left}}(H) = \{x \in L | [x, h] \in H \text{ for all } h \in H\}.$$

Clearly, the term *left normalizer* arise from group theory. Similarly, the *right idealizer* is defined as the following:

$$\mathbb{I}_L^{\text{right}}(H) = \{x \in L | [h, x] \in H \text{ for all } h \in H\}.$$

Then $\mathbb{I}_L(H) = \mathbb{I}_L^{\text{left}}(H) \cap \mathbb{I}_L^{\text{right}}(H)$.

The left idealizer of H is a subalgebra of L. Indeed, let $x, y \in \mathbb{I}_L^{\text{left}}(H)$ and $h \in H$, $\alpha \in F$; then

$$[x - y, h] = [x, h] - [y, h] \in H, [\alpha x, h] = \alpha[x, h] \in H,$$

and

$$[[x, y], h] = [x, [y, h]] - [y, [x, h]] \in H.$$

The idealizer of H is also a subalgebra of L. Indeed, let $x, y \in \mathbb{I}_L(H)$, $h \in H$, $\alpha \in F$. As above, we can show that $x - y, x, [x, y] \in \mathbb{I}_L(H)$. Further,

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] \in H,$$

$\alpha \in F$, then

$$[x - y, h] = [x, h] - [y, h] \in H, [\alpha x, h] = \alpha[x, h] \in \mathbb{I}_L(H).$$

However, the right idealizer need not be a subalgebra. The corresponding example has been constructed in [2].

Proposition 2.5 *Let L be a hypercentral Leibniz algebra over a field F. Then $\mathbb{I}_L(H) \neq H$ for every proper subalgebra H of L.*

Proof — Let

$$\langle 0 \rangle = Z_0 \leqslant Z_1 \leqslant \ldots Z_\alpha \leqslant Z_{\alpha+1} \ldots Z_\gamma = L$$

be the upper central series of L. There exists an ordinal α such that

$$Z_\alpha < H$$

but H does not include $Z_{\alpha+1}$. Choose an element $x \in Z_{\alpha+1} \backslash H$. For every element $h \in H$ we have $[x, h], [h, x] \in Z_\alpha$. The inclusion $Z_\alpha \leqslant H$ implies that $[x, h], [h, x] \in H$. This shows that $\mathbb{I}_L(H) \neq H$, moreover

$$\mathbb{I}_L^{\mathrm{right}}(H) \neq H \neq \mathbb{I}_L^{\mathrm{left}}(H).$$

The proof is complete. □

Corollary 2.6 *Let L be a nilpotent Leibniz algebra over a field F. Then*

$$\mathbb{I}_L(H) \neq H$$

for every proper subalgebra H of L.

For finite dimensional Leibniz algebras this result was proved in the paper [2] (see Lemma 2.2).

3 On some properties of locally nilpotent ideals

Let $x_1, \dots, x_n$ be elements of a Leibniz algebra L. If we write the complex commutator as $[x_1, \dots, x_n]$, then it means that the arrangement of the square brackets here is arbitrary.

Lemma 3.1 *Let L be a Leibniz algebra over a field F, a be an element of L and Y be a finite subset of L. Then a subalgebra A, generated by the subset*

$$\{x, [a, y], [z, a] \mid x, y, z \in Y\}$$

contains all elements

$$[a, y_1, \dots, y_k], [x_1, \dots, x_n, a], [x_1, \dots, x_n, a, y_1, \dots, y_k],$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in Y$.

Proof — For the elements $[a, y_1, \dots, y_k]$ we will use induction by k. If $k = 2$, then we have only the following elements:

$$[[a, y_1], y_2] \quad \text{and} \quad [a, [y_1, y_2]].$$

Here the first element belongs to A. For the second element we have $[a, [y_1, y_2]] = [[a, y_1], y_2] + [y_1, [a, y_2]] \in A$.

Let $k > 2$ and suppose that we already proved our statement for complex commutators of weight k. For the element $[a, y_1, \dots, y_k]$ we have the following alternatives: $[a, [y_1, \dots, y_k]]$, $[[a, y_1, \dots, y_{k-1}], y_k]$ or $[[a, y_1, \dots, y_j], [y_{j+1}, \dots, y_k]]$ (here we have written only external direct brackets). By induction hypothesis, the elements

$$[a, y_1, \dots, y_{k-1}] \quad \text{and} \quad [a, y_1, \dots, y_j]$$

belong to A, which implies that the elements

$$[[a, y_1, \dots, y_{k-1}], y_k] \quad \text{and} \quad [[a, y_1, \dots, y_j], [y_{j+1}, \dots, y_k]]$$

belong to A. For the element $[a, [y_1, \dots, y_k]]$ we have the following variants of the parenthesis arrangement:

$$[a, [y_1, [y_2, \dots, y_k]]], [a, [[y_1, \dots, y_{k-1}], y_k]]$$

and

$$[a, [[y_1, \dots, y_j], [y_{j+1}, \dots, y_k]].$$

For each of these elements we obtain successively

$$[a,[y_1,[y_2,\dots,y_k]]] = [[a,y_1],[y_2,\dots,y_k]] + [y_1,[a,[y_2,\dots,y_k]]],$$
$$[a,[[y_1,\dots,y_{k-1}],y_k]] = [[a,[y_1,\dots,y_{k-1}]],y_k] + [[y_1,\dots,y_{k-1}],[a,y_k]],$$
$$[a,[[y_1,\dots,y_j],[y_{j+1},\dots,y_k]]$$
$$= [a,[y_1,\dots,y_j]],[y_{j+1},\dots,y_k]] + [[y_1,\dots,y_j],[a,[y_{j+1},\dots,y_k]]].$$

By induction hypothesis, the elements

$$[a,[y_2,\dots,y_k]],[a,[y_1,\dots,y_{k-1}]],[a,[y_1,\dots,y_j]],[a,[y_{j+1},\dots,y_k]]$$

belong to A, and so it follows that A contains the elements

$$[a,[y_1,[y_2,\dots,y_k]]],[a,[[y_1,\dots,y_{k-1}],y_k]],$$

$$[a,[[y_1,\dots,y_j],[y_{j+1},\dots,y_k]].$$

Consider now the elements $[x_1,\dots,x_n,a]$. For these elements we also will use induction by n. If $n=2$, then we have only the following elements:

$$[[x_1,x_2],a] \quad \text{and} \quad [x_1,[x_2,a]].$$

The second element belongs to A. For the first element we obtain

$$[[x_1,x_2],a] = [[x_1,[x_2,a]] - [x_2,[x_1,a]] \in A.$$

Let $n>2$ and suppose that we already proved our statement for complex commutators of weight n. For the element $[x_1,\dots,x_n,a]$ we have the following variants of the parenthesis arrangement: $[[x_1,\dots,x_n],a]$, $[x_1,[x_2,\dots,x_n,a]]$ and $[[x_1,\dots,x_j],[x_{j+1},\dots,x_n,a]]$ (here we have written only external direct brackets). By induction hypothesis, the elements $[x_2,\dots,x_n,a]$ and $[x_{j+1},\dots,x_n,a]$ belong to A, which implies that the elements $[x_1,[x_2,\dots,x_n,a]]$ and $[[x_1,\dots,x_j],[x_{j+1},\dots,x_n,a]]$ belong to A. For the element $[[x_1,\dots,x_n],a]$ we have the following variants of parenthesis arrangement:

$$[[[x_1,\dots,x_{n-1}],x_n],a],[[x_1,[x_2,\dots,x_n]],a],$$

and

$$[[[x_1,\dots,x_j],[x_{j+1},\dots,x_n]],a].$$

For each of these elements we obtain successively

$$[[[x_1,\dots,x_{n-1}],x_n],a] = [[x_1,\dots,x_{n-1}],[x_n,a]] - [x_n,[[x_1,\dots,x_{n-1}],a]].$$

$$[[x_1,[x_2,\dots,x_n]],a] = [x_1,[[x_2,\dots,x_n],a]] - [[x_2,\dots,x_n],[x_1,a]].$$

$$[[[x_1,\dots,x_j],[x_{j+1},\dots,x_n]],a]$$

$$= [[x_1,\dots,x_j],[[x_{j+1},\dots,x_n],a]] - [[x_{j+1},\dots,x_n],[[x_1,\dots,x_j],a]].$$

By induction hypothesis, the elements

$$[[x_1,\dots,x_{n-1}],a],[[x_2,\dots,x_n],a],[[x_{j+1},\dots,x_n],a],[[x_1,\dots,x_j],a]$$

belong to A, which implies that A contains the elements

$$[[[x_1,\dots,x_{n-1}],x_n],a],[[x_1,[x_2,\dots,x_n]],a]$$

and

$$[[[x_1,\dots,x_j],[x_{j+1},\dots,x_n]],a].$$

Finally, consider the elements

$$[x_1,\dots,x_n,a,y_1,\dots,y_k].$$

In this case, we will use induction by $n+k$. Let $n+k=2$, i.e. $n=k=1$. Here we have only two commutators $[x_1,[a,y_1]]$ and $[[x_1,a],y_1]$, which clearly belong to A. Suppose that we already proved our statement for the case when $n+k=d>2$. Let now

$$n+k=d+1.$$

As for the element $[x_1,\dots,x_n,a,y_1,\dots,y_k]$ we have the following variants of the parenthesis arrangement

$$[x_1,[x_2,\dots,x_n,a,y_1,\dots,y_k]],[[x_1,\dots,x_n,a,y_1,\dots,y_{k-1}],y_k],$$

$$[[x_1,\dots,x_j],[x_{j+1},\dots,x_n,a,y_1,\dots,y_k]],[[x_1,\dots,x_n],[a,y_1,\dots,y_k]],$$

$$[[x_1,\dots,x_n,a],[y_1,\dots,y_k]],[[x_1,\dots,x_n,a,y_1,\dots,y_m],[y_{m+1},\dots,y_k]]$$

(here we have written only external direct brackets). By induction hypothesis, the elements

$$[x_2, \dots, x_n, a, y_1, \dots, y_k], [x_1, \dots, x_n, a, y_1, \dots, y_{k-1}],$$
$$[x_{j+1}, \dots, x_n, a, y_1, \dots, y_k], [a, y_1, \dots, y_k]],$$
$$[x_1, \dots, x_n, a], [x_1, \dots, x_n, a, y_1, \dots, y_m]$$

belong to A, which implies that all of the above elements belong to A. □

Lemma 3.2 *Let L be a Leibniz algebra over a field F, a be an element of L, and Y be a finite subset of L such that $[a, a] \in Y$. Let X be the set of elements*

$$[a, y_1, \dots, y_k], [x_1, \dots, x_n, a], [x_1, \dots, x_n, a, y_1, \dots, y_k]$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in Y$. Let B be the subalgebra generated by a and Y, and put C to be the subalgebra generated by a and X. Then C is an ideal of B.

Proof — Clearly, if $z \in Y$, then the elements

$$[z, a, y_1, \dots, y_k], [a, y_1, \dots, y_k, z], [z, x_1, \dots, x_n, a],$$
$$[x_1, \dots, x_n, a, z], [z, x_1, \dots, x_n, a, y_1, \dots, y_k],$$
$$[x_1, \dots, x_n, a, y_1, \dots, y_k, z]$$

belong to X. Next, we must consider the following elements:

$$[a, a, y_1, \dots, y_k], [a, y_1, \dots, y_k, a], [x_1, \dots, x_n, a, a],$$
$$[a, x_1, \dots, x_n, a, y_1, \dots, y_k], [x_1, \dots, x_n, a, y_1, \dots, y_k, a].$$

We consider each of these elements and again use induction on the number of elements in these complex commutators. For the element $[a, a, y_1]$ we have only two possibilities: $[[a, a], y_1] = 0$ and $[a, [a, y_1]] \in C$. If $k > 1$, then for element $[a, a, y_1, \dots, y_k]$ we have the following possibilities:

$$[a, [a, y_1, \dots, y_k]], [[a, a], [y_1, \dots, y_k]] = 0,$$
$$[[a, a, y_1, \dots, y_j], [y_{j+1}, \dots, y_k]].$$

By induction hypothesis all these elements belong to C.

For element $[a, y_1, a]$ we have only two possibilities: $[a, [y_1, a]]$ and $[[a, y_1], a]$. Both these elements belong to C. If $k > 1$, then we come to the following elements:

$$[a, [y_1, \dots, y_k, a]], [[a, y_1, \dots, y_j], [y_{j+1}, \dots, y_k, a]], [[a, y_1, \dots, y_k], a].$$

By induction hypothesis, all of these elements belong to C.

For element $[x_1, a, a]$ we have only two possibilities: $[[x_1, a], a] \in C$ and $[x_1, [a, a]]$. For the last element we obtain

$$[x_1, [a, a]] = [[x_1, a], a] + [a, [x_1, a]],$$

so again $[x_1, [a, a]] \in C$. If $k > 1$, then we come to the following elements:

$$[[x_1, \dots, x_n, a], a], [[x_1, \dots, x_n], [a, a]], [[x_1, \dots, x_j], [x_{j+1}, \dots, x_n, a, a]].$$

By induction hypothesis, the first and third elements belong to C. For the second element we obtain

$$[[x_1, \dots, x_n], [a, a]] = [[[x_1, \dots, x_n], a], a] + [a, [[x_1, \dots, x_n], a]] \in C.$$

For elements $[a, x_1, a, y_1]$ we have the following variants of the parenthesis arrangement:

$$[a, [x_1, a, y_1], [[a, x_1], [a, y_1]] \in C, [[a, x_1, a], y_1].$$

By what we have proved above $[a, x_1, a] \in C$, so that $[[a, x_1, a], y_1] \in C$. If $k > 1$, then we come to the following elements:

$$[a, [x_1, \dots, x_n, a, y_1, \dots, y_k]], [[a, x_1, \dots, x_j], [x_j, \dots, x_n, a, y_1, \dots, y_k]],$$
$$[[a, x_1, \dots, x_n], [a, y_1, \dots, y_k]], [[a, x_1, \dots, x_n, a], [y_1, \dots, y_k]],$$
$$[[a, x_1, \dots, x_n, a, y_1, \dots, y_m], [y_{m+1}, \dots, y_k]]$$

By what we have proved above and induction hypothesis all of these elements belong to C.

For elements $[x_1, a, y_1, a]$ we have the following variants of the parenthesis arrangement:

$$[[x_1, a, y_1], a], [[x_1, a], [y_1, a]] \in C, [x_1, [a, y_1, a]].$$

By what we have proved above $[a, y_1, a] \in C$, so that $[x_1, [a, y_1, a]] \in C$. If $k > 1$, then we come to the following elements:

$$\begin{gathered}
[[x_1, \dots, x_n, a, y_1, \dots, y_k], a], \\
[x_1, \dots, x_n, a, y_1, \dots, y_j], [y_{j+1}, \dots, y_k, a]], \\
[[x_1, \dots, x_n, a], [y_1, \dots, y_k, a]], [[x_1, \dots, x_n], [a, y_1, \dots, y_k, a]], \\
[[x_1, \dots, x_j], [x_{j+1}, \dots, x_n, a, y_1, \dots, y_k, a]].
\end{gathered}$$

By what we have proved above and induction hypothesis all of these elements belong to C. □

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H^{\mathrm{left}}(M) = \{a \in H \mid [a, M] = 0\}, \mathrm{Ann}_H^{\mathrm{right}}(M) = \{a \in H \mid [M, a] = 0\}.$$

The subset $\mathrm{Ann}_H^{\mathrm{left}}(M)$ is called the *left annihilator* or *left centralizer* of M in subalgebra H; the subset $\mathrm{Ann}_H^{\mathrm{right}}(M)$ is called the *right annihilator* or *right centralizer* of M in subalgebra H. The intersection

$$\begin{aligned}
\mathrm{Ann}_H(M) &= \mathrm{Ann}_H^{\mathrm{left}}(M) \cap \mathrm{Ann}_H^{\mathrm{right}}(M) \\
&= \{a \in H \mid [a, M] = \langle 0 \rangle = [M, a]\}
\end{aligned}$$

is called the *annihilator* or *centralizer* of M in subalgebra H.

It is not hard to see that all of these subsets are the subalgebras of L. Moreover, if M is a left ideal of L, then $\mathrm{Ann}_L^{\mathrm{left}}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathrm{Ann}_H^{\mathrm{left}}(M), b \in M$. Then

$$\begin{gathered}
[[a, x], b] = [a, [x, b]] - [x, [a, b]] = 0 - [x, 0] = 0, \text{ and} \\
[[x, a], b] = [x, [a, b]] - [a, [x, b]] = [x, 0] - 0 = 0.
\end{gathered}$$

If M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, a $\mathrm{Ann}_L(M), b \in M$. Using the above arguments, we obtain that $[[a, x], b] = [[x, a], b] = 0$. Further,

$$[b, [a, x]] = [[b, a], x]] + [a, [b, x]]] = [0, x] + 0 = 0$$

and

$$[b, [x, a]] = [[b, x], a] + [x, [b, a]]] = 0 + [x, 0] = 0.$$

Lemma 3.3 *Let* L *be a Leibniz algebra over a field* F *and* A *be an ideal of* L. *If* $\dim_F(A) = n$ *is finite, then* $\dim_F(A/\mathrm{Ann}_L(A))$ *is also finite.*

Proof — By Proposition 3.2 of the paper [11], $\mathrm{Ann}_L^{\mathrm{left}}(A)$ has finite codimension.

For an arbitrary element $v \in L$ we consider the mapping

$$r_v : A \longrightarrow A,$$

defined by the rule $r_v(x) = [x, v], x \in A$. For every $x, y \in A$ and $\lambda \in F$ we have

$$r_v(x + y) = r_v(x) + r_v(y), \quad r_v(\alpha x) = \alpha r_v(x),$$

which implies that r_v is a linear mapping. We also note that

$$\beta r_v = r_{\beta v} \quad \text{ and } r_v + r_w = r_v + w$$

for all $v, w \in L$, and $\beta \in F$.

Consider now the mapping

$$f : \; L \longrightarrow \mathrm{End}_F(A)$$

defined by the rule $f(v) = r_v$ for each element $v \in L$. We have

$$f(v + w) = r_{v+w} = r_v + r_w = f(v) + f(w),$$

and

$$f(\beta v) = r_{\beta v} = \beta r v = \beta f(v)$$

for all $v, w \in L$ and $\beta \in F$. It shows that mapping f is linear. The fact that $\dim_F(A)$ is finite implies that $\dim_F(\mathrm{End}_F(A))$ is finite, so that $\mathrm{Im}(f)$ is finite dimensional. We have

$$\mathrm{Ker}(f) = \{v \,|\, v \in L \text{ and } r_v \text{ is a zero mapping}\}.$$

But $0 = r_v(x)$ for each $x \in A$ means that $[x, v] = 0$ for each $x \in A$, which implies that $v \in Ann_L^{right}(A)$. Thus

$$\mathrm{Ker}(f) = Ann_L^{right}(A),$$

so we obtain that

$$A/Ann_L^{right}(A)$$

has finite dimension. In turn it implies that

$$Ann_L(A) = Ann_L^{left}(A) \bigcap Ann_L^{right}(A)$$

has finite codimension. □

Lemma 3.4 *Let* L *be a locally nilpotent Leibniz algebra over a field* F *and* A *be an ideal of* L*. If* $\dim_F(A) = n$ *is finite, then a hypercenter with a finite number* k *includes* A*. Moreover,* $k \geqslant n$*.*

Proof — Since $\dim_F(A)$ is finite, A has a finite L-composition series

$$\langle 0 \rangle = A_0 \leqslant A_1 \leqslant \ldots \leqslant A_t = A.$$

Suppose that the center of L does not include A_1. Then

$$C = Ann_L(A) \neq L.$$

Let B be a complement to C in A, i.e.

$$L = C \oplus B.$$

By Lemma 3.3 $\dim_F(B)$ is finite. Let D be a subalgebra, generated by A_1 and B. Since $\dim_F(A)$ and $\dim_F(B)$ are finite, D is finitely generated as a subalgebra. Then D is nilpotent. The choice of D yields that A_1 is a D-chief factor of D. Lemma 2.4 proves that

$$A_1 \cap \zeta(D) \neq \langle 0 \rangle.$$

Since $A_1 \cap \zeta(D)$ is D-invariant,

$$A_1 \cap \zeta(D) = A_1,$$

that is $A_1 = \zeta(D)$, and we obtain a contradiction. This contradiction shows that $A_1 \leqslant \zeta(D)$.

Using the similar arguments and ordinary induction, we prove that $A \leqslant \zeta_k(L)$ for some positive integer k. $\square$

Lemma 3.5 *Let L be a Leibniz algebra over a field F, A and B be the ideals of L. Suppose that B is locally nilpotent, $\mathrm{dim}_F(A)$ is finite and A is nilpotent. Then $A + B$ is locally nilpotent.*

Proof — Suppose first that $A \cap B = \langle 0 \rangle$. It follows that $[A, B] = \langle 0 \rangle$. Let C be an arbitrary finitely generated subalgebra of B. Then we have

$$\zeta_k(A \oplus C) = \zeta_k(A) \oplus \zeta_k(C)$$

for all positive integer k. If $ncl(A) = n$, then

$$\zeta_n(A \oplus C) = A \oplus \zeta_n(C).$$

It follows that

$$(A \oplus C)/\zeta_n(A \oplus C) = (A \oplus C)/(A \oplus \zeta_n(C)) \cong C/\zeta_n(C),$$

in particular, $(A \oplus C)/\zeta_n(A \oplus C)$ is nilpotent. It follows that $A \oplus C$ is also nilpotent, which implies that $A \oplus B$ is locally nilpotent.

Suppose now that $A \cap B = C \neq \langle 0 \rangle$. Since A is nilpotent, there is a number k such that $C \leqslant \zeta_k(A)$. The fact that A is an ideal in L implies that C is an ideal in B. Using Lemma 3.4 we obtain that there is a number t such that $C \leqslant \zeta_t(B)$. Put $m = \max\{k, t\}$. Then $C \leqslant \zeta_m(A + B)$. Since

$$A/C + B/C = A/C \oplus B/C,$$

by what we have proved above we obtain that

$$(A + B)/\zeta_m(A + B)$$

is locally nilpotent. It follows that $A + B$ is locally nilpotent. $\square$

Proposition 3.6 *Let L be a Leibniz algebra over a field F, A, B be locally nilpotent ideals of L. If U is a finitely generated subalgebra of A such that $(U + B)/B$ is abelian, then $U + B$ is locally nilpotent.*

Proof — Let

$$a \in U \backslash B.$$

The fact that $(U + B)/B$ is abelian implies that $[a, a] \in B$. Choose arbitrary finite subsets $Y \subseteq B$ such that $[a, a] \in B$. Let X be the set of elements

$$[a, y_1, \dots, y_k], [x_1, \dots, x_n, a], [x_1, \dots, x_n, a, y_1, \dots, y_k]$$

for all $x_1, \dots, x_n, y_1, \dots, y_k \in Y$, and denote by C the subalgebra generated by a and X. Let D be the subalgebra generated by $\{a\} \cup Y$. Then Lemma 3.1 shows that a subalgebra of B, generated by the subset

$$S = \{x, [a, y], [z, a] \mid x, y, z \in Y\}$$

includes a subset X. Since Y is finite, S is also finite. The fact that B is locally nilpotent implies that a subalgebra $\langle S \rangle$ is nilpotent. Then there exists a positive integer m such that all commutators

$$[a, y_1, \dots, y_k], [x_1, \dots, x_n, a], [x_1, \dots, x_n, a, y_1, \dots, y_k]$$

are zeros whenever $k, n \geqslant m$. In other words, a subset X is finite and hence subalgebra C is finitely generated. Since $a \in A$ and A is an ideal, $X \subseteq A$. Then the fact that A is locally nilpotent implies that C is nilpotent. By Lemma 3.2, C is an ideal of D. The inclusions $\langle a \rangle \leqslant D$ and $D \leqslant \langle a \rangle + B$ imply that

$$D = \langle a \rangle + V$$

where $V = D \cap B$. It follows that $D = C + V$. Here C is a nilpotent ideal of D and V is a locally nilpotent ideal of D and Lemma 3.5 implies that D is locally nilpotent. Since D is a finitely generated subalgebra, D is nilpotent. In turn out it follows that a subalgebra $\langle a \rangle + B$ is locally nilpotent.

The fact that $(U + B)/B$ is abelian and finitely generated implies that $\dim_F(U + B)/B$ is finite, so that

$$(U + B)/B = (a_1 F + \dots + a_n F + B)/B$$

for some elements $a_1, \dots, a_n \in U$. Then

$$B_1 = a_1 F + B$$

is an ideal of $U + B$. By what we have proved above B_1 is locally nilpotent. We have now that $(U + B_1)/B_1$ is abelian, and

$$\dim_F(U + B1)/B_1 \leqslant \dim_F(U + B)/B.$$

Using ordinary induction we obtain that $U+B$ is locally nilpotent. □

Proof of Theorem C — Let M be an arbitrary finite subset of A and D be a subalgebra of A, generated by M. Then

$$(D+B)/B = (\langle M\rangle + B)/B,$$

which shows that $(D+B)/B$ is nilpotent and finitely generated. In particular, Corollary 2.2 yields that $\mathrm{dim}_F((D+B)/B)$ is finite. Let

$$\langle 0\rangle = Z_0/B \leqslant Z_1/B \leqslant \ldots \leqslant Z_n/B = (D+B)/B$$

be the upper central series of $(D+B)/B$. For the proof we will apply induction by n. If $n=1$, then $(D+B)/B$ is abelian. Proposition 3.6 implies that D is locally nilpotent. Suppose now that $n>1$. Ideal Z_1/B of $(D+B)/B$ is abelian and finite dimensional. Using Proposition 3.6 we obtain that Z_1 is locally nilpotent. Since

$$\mathrm{ncl}((D+B)/Z_1) < \mathrm{ncl}((D+B)/B)$$

we can apply induction hypothesis and obtain that $D+B$ is locally nilpotent.

Let now S be an arbitrary finite subset of $A+B, S=\{y_1,\ldots,y_k\}$. Then $y_j = a_j + b_j$ for some elements $a_j \in A, b_j \in B, 1 \leqslant j \leqslant k$. We have now

$$\langle S\rangle \leqslant \langle a_j, b_j | 1 \leqslant j \leqslant k\rangle.$$

It turn out

$$\langle a_j, b_j | 1 \leqslant j \leqslant k\rangle \leqslant \langle a_1,\ldots,a_n\rangle + B.$$

By what we have proved above, the last subalgebra is locally nilpotent. It follows that $\langle S\rangle$ is nilpotent, so that $A+B$ is locally nilpotent. □

The *left* (respectively *right*) *center* $\zeta^{\mathrm{left}}(L)$ (respectively $\zeta^{\mathrm{right}}(L)$) of L is defined by the rule

$$\zeta^{\mathrm{left}}(L) = \{x \in L | [x,y] = 0 \text{ for each element } y \in L\}.$$

(respectively,

$$\zeta^{\mathrm{right}}(L) = \{x \in L [y,x] = 0 \text{ for each element } y \in L\}).$$

The left center of L is an ideal of L, moreover $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$, so that $L/\zeta^{\mathrm{left}}(L)$ is a Lie algebra. The right center is an subalgebra of L, and, in general, the left and right centers are different, moreover, they even may have different dimensions. Example 2.1 of paper [11] shows it.

Lemma 3.7 *Let* L *be a locally nilpotent Leibniz algebra over a field* F*. If* M *is a minimal ideal of* L*, then* $M \leqslant \zeta(L)$*. In particular,* $\dim_F(M) = 1$.

Proof — Let $K = \mathrm{Leib}(L)$, then either $M \leqslant K$ or $M \cap K = \langle 0 \rangle$. Consider first the last case. Factor-algebra L/K is a Lie algebra. Since

$$(M + K)/K$$

is a minimal ideal of L/K, Lemma 10 of paper [11] shows that

$$(M + K)/K \leqslant \zeta(L/K).$$

It follows that $[M, L], [L, M] \leqslant K$. On the other hand, the fact that M is an ideal of L implies that

$$[M, L], [L, M] \leqslant M,$$

so that

$$[M, L], [L, M] \leqslant M \cap K = \langle 0 \rangle.$$

This means that $M \leqslant \zeta(L)$.

Consider now the case when $M \leqslant K$. Suppose the contrary, let $\zeta(L)$ does not include M. Since $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L), [M, L] = \langle 0 \rangle$. Being non-central, M contains an element b, and L has an element x such that

$$[x, b] = c \neq 0.$$

Since M is an ideal, $c \in M$. Then the fact that M is a minimal ideal of L implies that $M = \langle c \rangle^L$. Hence there exists a finite subset S of L such that $b \in \langle c \rangle^S$. Let $H = \langle b, x, S \rangle$ and $D = \langle b \rangle^H \leqslant H$. Then

$$c = [x, b] \in [H, D]$$

Let u, v be arbitrary elements of H, $d \in D$. The inclusion $D \leqslant \zeta^{\mathrm{left}}(L)$ implies that $[[u, d], v] = 0$. We have

$$[v, [u, d]] = [[v, u], d] + [u, [v, d]].$$

Since D is an ideal of H, $[v, d] \in D$, so that $[u, [v, d]] \in [H, D]$. It follows that $[v, [u, d]] \in [H, D]$, and therefore, $[H, D]$ is an ideal of H. Then from $c \in [H, D]$ we obtain that

$$\langle c \rangle^H \leqslant [H, D].$$

Since $b \in \langle c \rangle^S \leqslant \langle c \rangle^H$, $b \in [H, D]$ and it implies that $\langle b \rangle^H = D = [H, D]$. Being finitely generated, subalgebra H is nilpotent. Then there is a positive integer t such that

$$\gamma_t(H) = [H, [H, [\ldots [H, H] \ldots] = \langle 0 \rangle.$$

Then $[H, [H, [\ldots [H, D] \ldots] = \langle 0 \rangle$. On the other hand, $D = [H, D]$ implies that $[H, [H, D]] = [H, D] = D$. Using an ordinary induction we obtain that $[H, [H, [H, \ldots [H, D] \ldots] = D$, and we obtain a contradiction. This contradiction proves the inclusion $M \leqslant \zeta(L)$. □

Proof of Theorem D — **(i)** In fact, B/A is a minimal ideal of factor-algebra L/A. Then Lemma 3.7 implies that $B/A \leqslant \zeta(L/A))$.

(ii) Since A is a maximal subalgebra of L, then $L = \langle A, x \rangle$ for each element $x \notin L$. Then the fact that A is an ideal of L implies that a factor-algebra L/A is cyclic. If we suppose that L/A is not a Lie algebra, then $\mathrm{Leib}(L/A)$ is a non-zero proper ideal of L/A, and we obtain a contradiction. This contradiction shows that L/A is a Lie algebra. Being cyclic L/A is abelian, which implies that $[L, L] \leqslant A$.

Assume now that A is not an ideal of L. Then by above remarked A does not include $[L, L]$. It follows that there exists an element x in $[L, L]$ such that $x \in A$. The fact that A is maximal subalgebra implies that

$$L = \langle A, x \rangle.$$

Since $x \in [L, L]$, there exists a finite subset M of L such that

$$x \in [\langle M \rangle, \langle M \rangle].$$

If $y \in M$, then the equality $L = \langle A, x \rangle$ shows that A includes a finitely generated subalgebra H_y such that

$$y \in \langle H_y, x \rangle.$$

Put

$$H = \langle H_y \mid y \in M \rangle.$$

Then H is finitely generated and $M \subseteq \langle H, x\rangle$. Put

$$B = \langle H, x\rangle.$$

Since B is finitely generated, it is nilpotent. The inclusion $H \leqslant A$ implies that $x \notin H$. Among all of the subalgebras B including H and not containing the element x, we choose a maximal subalgebra D. Taking into account the equality $B = \langle H, x\rangle$, we obtain that D is a maximal subalgebra of B. Since B is nilpotent, it follows that D is an ideal of B. By above remarked $[B, B] \leqslant D$. On the other hand, the inclusion $M \subseteq B$ implies

$$[\langle M\rangle, \langle M\rangle] \leqslant [B, B].$$

Then from $x \in [\langle M\rangle, \langle M\rangle]$ we obtain that

$$x \in [B, B] \leqslant D,$$

which contradicts the choice of D. This contradiction proves the result. □

Corollary 3.8 *Let* L *be a hypercentral Leibniz algebra over a field* F*. Then every maximal subalgebra of* L *is an ideal of* L*.*

4 The idealizer condition for Leibniz algebras

We note the following trivial characterization of algebras satisfying idealizer condition.

Proposition 4.1 *Let* L *be a Leibniz algebra over a field* F*. Then* L *satisfies the idealizer condition if and only if every subalgebra of* L *is ascendant in* L*.*

Corollary 4.2 *Let* L *be a Leibniz algebra over a field* F*. If* L *satisfies the idealizer condition then every maximal subalgebra of* L *is an ideal of* L*.*

Lemma 4.3 *Let* L *be a finite dimensional Leibniz algebra over a field* F*. If* L *satisfies the idealizer condition then* L *is nilpotent.*

Proof — In fact, let M be a maximal subalgebra of L. Since

$$\mathbb{I}_L(M) \neq M, \mathbb{I}_L(M) = L.$$

In other words, M is an ideal of L. Thus every maximal subalgebra of L is an ideal. By Theorem 5.3 of paper [2] L is nilpotent. □

Corollary 4.4 *Let* $L = \langle a \rangle$ *be a cyclic Leibniz algebra over a field* F*. If* L *satisfies the idealizer condition then*

$$L = Fa \oplus Fa_1 \oplus \ldots \oplus Fa_n$$

where

$$[a_j, a] = [a_j, a_m] = 0$$

for all j, m *with*

$$1 \leqslant j, m \leqslant n, [a, a] = a_1, [a, a_1] = a_2, \ldots, [a, a_{n-1}] = a_n,$$

$[a, a_n] = 0$*. In particular,* L *is nilpotent.*

Proof — Put

$$[a, a] = a_1, [a, a_1] = a_2, \ldots, [a, a_n] = a_{n+1}, n \in \mathbb{N}.$$

Suppose first that element a has infinite depth, that all elements

$$\{a, a_n \mid n \in \mathbb{N}\}$$

are linearly independent. We note that

$$\mathrm{Leib}(L) = \bigoplus_{j \in \mathbb{N}} Fa_j.$$

Consider a subalgebra $\langle a - a_1 \rangle$. We have

$$[a - a_1, a - a_1] = [a, a] - [a, a_1] - [a_1, a] + [a_1, a_1] = a_1 - a_2,$$

$$[a - a_1, a_1 - a_2] = [a, a_1] - [a, a_2] - [a_1, a_1] + [a_1, a_2] = a_2 - a_3,$$

$$[a - a_1, a_j - a_{j+1}] = [a, a_1] - [a, a_2] - [a_1, a_1] + [a_1, a_2] =$$

$$a_{j+1} - a_{j+2}, j \in \mathbb{N}.$$

These equalities shows that

$$\langle a - a_1 \rangle = F(\langle a - a_1 \rangle) \oplus \left(\bigoplus_{j \in \mathbb{N}} F(a_j - a_{j+1}) \right).$$

It implies that subalgebra $\langle a - a_1 \rangle$ has a codimension 1. In particular, this subalgebra is maximal in L. If we suppose that

$$\mathbb{I}_L(\langle a - a_1 \rangle)\langle a - a_1 \rangle,$$

then $\mathbb{I}_L(\langle a - a_1 \rangle) = L$, which means that $\langle a - a_1 \rangle$ is an ideal of L. It follows that

$$[a - a_1, a] \neq \langle a - a_1 \rangle.$$

However $[a - a_1, a] = [a, a] = a_1$. In this case,

$$(a - a_1) + a_1 = a \in \langle a - a_1 \rangle,$$

and we obtain a contradiction. This contradiction shows that element a has finite depth. In this case, L has finite dimension (see Theorem 1.1 of the paper [8]), and we can apply Lemma 4.3. □

Corollary 4.5 *Let* L *be a Leibniz algebra over a field* F*. If* L *satisfies the idealizer condition then* L *has an ascending series*

$$\langle 0 \rangle = A_0 \leqslant A_1 \leqslant \ldots A_\alpha \leqslant A_{\alpha+1} \leqslant \ldots A_\gamma = L$$

of subalgebras, whose factors are cyclic and finite dimensional.

Lemma 4.6 *Let* L *be a Leibniz algebra over a field* F, K *be an ideal of* L*, and* A *an ascendant subalgebra of* L*. If* A, K *are locally nilpotent, then* $A + K$ *is also locally nilpotent.*

Proof — Let

$$A = A_0 \leqslant A_1 \leqslant \ldots A_\alpha \leqslant A_{\alpha+1} \leqslant \ldots A_\gamma = L$$

be the ascendant series of subalgebras, connecting A and L. To avoid introducing new notation, we can put

$$L = A + K.$$

We have

$$A_1 = A_0 + (K \cap A_1).$$

Since K is ideal of $L, K \cap A_1$ is an ideal of A_1. Then the fact that A_0 is an ideal of A_1 together with Theorem C prove that A_1 is locally nilpotent. Suppose that we have already proved that the subalgebras $A_\beta + K$ are locally nilpotent for all $\beta < \alpha$. If α is a limit ordinal, then

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta$$

and

$$K + A_\alpha = \bigcup_{\beta<\alpha} (K + A_\beta).$$

This equality shows that $K + A_\alpha$ is locally nilpotent. Suppose now that α is not a limit ordinal. Then $\alpha - 1$ exists. We have

$$A_\alpha = A + (K \cap A_\alpha) = A_{\alpha-1} + (K \cap A_\alpha).$$

Again, $A_{\alpha-1}$ and $K \cap A_\alpha$ are the ideals of A_α, so using again Theorem C we obtain that A_α is locally nilpotent. For $\alpha = \gamma$ we obtain the result. $\square$

Proof of Theorem E — By Corollary 4.5, L has an ascending series

$$A = A_0 \leqslant \ A_1 \leqslant \ldots A_\alpha \leqslant A_{\alpha+1} \ldots A_\gamma = L$$

of subalgebras, whose factors are cyclic and finite dimensional. It follows that A_n has finite dimension and therefore it is nilpotent (Lemma 4.3). Suppose that we have already proved that the subalgebras A_β are locally nilpotent for all $\beta < \alpha$. If α is a limit ordinal, then

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta,$$

so that A_α obviously is locally nilpotent. Suppose now that α is not a limit ordinal. Then $\alpha - 1$ exists. Let a be an element of L such that

$$A_\alpha = \langle a \rangle + A_{\alpha-1}.$$

Since $A_{\alpha-1}$ is an ideal of A_α and a subalgebra $\langle a \rangle$ is ascendant by Proposition 4.1, Lemma 4.6 implies that A_α is locally nilpotent.

For $\alpha = \gamma$ we obtain the result. □

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