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Products of Irreducible Characters Having Complex-Valued Constituents

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Abstract

First, we prove that when a finite solvable group G has a faithful irreducible character χ such that $\chi\overline{\chi}$ has two irreducible constituents, both must be real-valued. Then, we study the situation where $\chi\overline{\chi}$ has exactly three distinct nonprincipal irreducible constituents, two of which are complex conjugates. In this case, we prove that G has derived length bounded above by 6.

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1 Introduction

Throughout, we let G be a finite solvable group and denote the irreducible characters of G by $\mathrm{Irr}(G)$. Let $\chi \in \mathrm{Irr}(G)$ be faithful and, as usual, denote its complex conjugate by $\overline{\chi}$, which is also an irreducible character of G. We are interested in studying how the number of irreducible constituents of $\chi\overline{\chi}$ affects the structure of G. We will write 1_G for the principal character of G. Then we know that the principal character has multiplicity 1 in $\chi\overline{\chi}$, since $1 = [\chi, \chi] = [\chi\overline{\chi}, 1_G]$, where the second equality follows from comments on page 48 of [7].

In [2], Adan-Bante was able to completely classify those solvable groups which had some faithful character $\chi \in \mathrm{Irr}(G)$ such that

$$\chi\overline{\chi} = 1_G + m\alpha.$$

In the same paper, she also proved some facts about the solvable groups with $\chi \in \mathrm{Irr}(G)$ faithful and such that

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2,$$

including that the derived length $\mathrm{dl}(G) \leqslant 18$. We proved in [6] that in this case, $\mathrm{dl}(G) \leqslant 8$, and that this is the best possible bound. In this paper, we will go further and show that both α_1 and α_2 must be real-valued characters, i.e., they cannot be complex conjugates of each other. It should be noted that this work has appeared in the first author's dissertation [5].

Theorem 1.1 *Let* G *be a finite solvable group with a faithful character* $\chi \in \mathrm{Irr}(G)$ *such that*

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2,$$

where $\alpha_1, \alpha_2 \in \mathrm{Irr}(G)$ *are distinct nonprincipal characters and* m_1 *and* m_2 *are strictly positive integers. Then both* α_1 *and* α_2 *are real-valued characters.*

When this is complete, we will use Theorem 1.1 to study the situation that G is a solvable group, $\chi \in \mathrm{Irr}(G)$ is faithful, and

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2}.$$

We are most interested in the possible derived lengths of such groups, bounding that derived length, and finding examples of groups with such a character χ.

At this point, the only known group satisfying the conditions of Theorem 1.2 is A_4, the alternating group on 4 letters. Notice that $\mathrm{dl}(A_4) = 2$, which means that there is a gap between our known example and what we prove here. This suggests that either our bound can be improved or that there are more examples with higher derived lengths that have yet to be discovered. Regardless, the existence of even one example shows that it is impossible to extend Theorem 1.1 to the case when $\chi\overline{\chi}$ has three nonprincipal irreducible constituents. It should also be noted that Adan-Bante proved a result in [1] which applies in this situation. However, because the number of nonprincipal irreducible constituents is so small in this case, that bound is quite weak.

Theorem 1.2 *Let* G *be a finite solvable group and let* $\chi \in \mathrm{Irr}(G)$ *be a faithful character. Assume that*

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2}$$

where $\alpha_1, \alpha_2 \in \mathrm{Irr}(G)$ *are nonprincipal characters and* m_1 *and* m_2 *are strictly positive integers. Then* $\mathrm{dl}(G) \leqslant 6$ *and* $\ker(\alpha_i)$ *is an abelian group for* $i = 1, 2$*, with* $\ker(\alpha_i) = Z(G)$ *for exactly one* i*.*

In the next section, we will prove Theorem 1.1. Sections 3 and 4 will be devoted to the proof of Theorem 1.2. This proof will require the use of several new lemmas and propositions.

2 Proof of Theorem 1.1

To prove Theorem 1.1, we will need two lemmas. The first includes more information than is needed at this point, since we will be using it again for Lemma 3.5.

Lemma 2.1 *Let* G *be a solvable group. Let* V *be a vector space of orders* q^n *for* $q^n = p^2$*, where* p *is a odd prime and* $p \leqslant 71$*, or* $q^n \in \{3^4, 5^4, 7^4, 3^6, 3^{10}\}$*. Assume that* V *is a faithful, irreducible, and primitive* G*-module. Finally, assume that* $G \nleqslant \Gamma(V)$*. Then* G *contains the central involution of* $\mathrm{GL}(n, q)$*.*

Proof — Since V is a primitive faithful G module, it restricts homogeneously to every abelian normal subgroup of G. Thus, every abelian normal subgroup of G has a faithful, irreducible module. This implies that every normal abelian subgroup of G is cyclic, and we may apply Corollary 1.10 of [10] to G. This result will be used extensively in what follows.

Set $F = F(G)$. First, assume that $\dim(V) = 2$. Then by Theorem 2.11 of [10], $F = QT$ where $Q \simeq Q_8$, the quaternion group of order 8, and $T \leqslant Z(\mathrm{GL}(2, p))$. Also, $Q \cap T = Z(Q) \simeq \mathbb{Z}_2$. Thus, if $\dim(V) = 2$, the central involution is contained in G.

Next, assume that $\dim(V) = 4$. Since V is a faithful F-module, we know that q does not divide $|F|$, where $q \in \{3, 5, 7\}$. Since G is not a subgroup of $\Gamma(V)$, we may assume that F is nonabelian by Corollary 2.3 of [10]. Let Z be the socle of $Z(F)$. Then by the first conclusion of Corollary 1.10 in [10], there exists normal subgroups Q and T of G

such that $F = QT$ and $Q \cap T = Z$. Also, the Sylow r-subgroups of Q are extra-special or cyclic of prime order by the second conclusion of Corollary 1.10. Hence, by Corollary 2.6 of [10], we have that there exists an integer e such that

$$e^2 = |F : T| = |Q : Z|.$$

Notice that e divides $\dim_{\mathbb{Z}_q}(W)$, where $V = f \cdot W$ for some integer f. Since q cannot divide e, we have that $e \in \{1, 2, 4\}$. Furthermore, $e \neq 1$, since otherwise Corollary 2.3 (b) of [10] implies that G is a subgroup of $\Gamma(V)$, which we have removed by assumption. So, $|F|$ is even and $|Z(F)|$ is also even. Since $Z(F)$ is cyclic, this implies it has a unique subgroup L of order 2. Now, since V is irreducible, the involution $x \in L$ can either fix everything in V or nothing. If x fixes everything, then x is in the kernel of the action of G on V. However, this is a contradiction since we are assuming that V is a faithful module. Therefore, x fixes nothing and sends every element $v \in V$ to its inverse $-v$. In particular, x is the central involution of $GL(V)$ and $x \in G$.

Next, assume that $\dim(V) = 6$. Then, by a similar argument, we see that $e = 2$, and $|F|$ is again even. Then we may apply the previous argument to F to get that G must contain the central involution of $GL(6, 3)$.

Lastly, assume that $\dim(V) = 10$. The previous argument works in the case that $e = 2$ or 10 but fails when $e = 5$. However, by Theorem 1.2 of [4] and comments made on page 1 of that paper, we get that G has a minimal normal nonabelian subgroup N with even order. From this, we can use a similar argument for all possible e, which completes the proof. □

We are now ready to prove the following lemma, which will make Theorem 1.1 possible.

Lemma 2.2 *Let G be a finite solvable group. Let V be a symplectic vector space of dimension $2n$ over $GF(q)$, where q is a prime power. Assume that V is a faithful irreducible G-module and the action of G on V preserves the symplectic form. Also assume that G acts with two orbits on $V^\# = V - \{0\}$. Call them O_1 and O_2 and set $e^2 = |V| = q^{2n}$. Then if $v \in O_i$, so is $-v$.*

Proof — Notice that our hypotheses imply the hypotheses of Theorem 4 of [6]. Therefore, we will study the cases given there. We know that v and $-v$ are in orbits of equal size. So, if O_1 and O_2 have different sizes, we obtain the conclusion. Hence, we may assume that the orbits have equal size, which implies that $|G|$ is divisible by

$$\frac{1}{2}(e^2-1)=\frac{1}{2}(q^{2n}-1).$$

Assume Theorem 4(i) of [6]. Then G is isomorphic to a subgroup of $\Gamma(V)$, $N = G \cap \Gamma_0(V)$ is a normal cyclic subgroup of G, and G/N acts faithfully on N. Also, since N is cyclic, we know that

$$|\mathrm{Aut}(N)| = \varphi(|N|),$$

where φ is the Euler φ-function, and since G/N acts faithfully on N, we have that $|G| \leqslant |N|\varphi(|N|)$.

Suppose that $|N|$ is not a prime number and is not equal to 4. Then

$$\varphi(|N|) \leqslant |N| - 3.$$

By Lemma 2.2.3 of [2], $|N| \leqslant e+1$. So $\varphi(|N|) \leqslant e-2$, which implies that $|G| < e^2-1$. Therefore, $|G| = \frac{1}{2}(e^2-1)$, G acts Frobeniusly on $V^{\#}$, and $G = N$. This implies that elements are in the same orbits as their inverses.

Now suppose that $|N| = p$, a prime. Then $p \leqslant e+1$. If $p < e+1$, then $p \leqslant e$, and

$$|G| \leqslant |N|\varphi(|N|) = p(p-1) \leqslant e(e-1) < e^2-1,$$

which yields the same results as the last paragraph. So assume that $p = e+1$. Notice that $p(p-2) = e^2-1$ and $\frac{1}{2}p(p-2)$ in an integer dividing $|G|$. However, $e \geqslant 3$ implies that $p > 4$ is an odd prime. Hence, $p(p-2)$ is an odd integer, and $\frac{1}{2}p(p-2)$ is not an integer. This is a contradiction.

So assume that $|N| = 4$. Then $|G|$ divides 8, and thus $\frac{1}{2}(e^2-1)$ divides 8. Hence, $\frac{1}{2}(e^2-1)$ is 1, 2, 4, or 8. Since V is a symplectic vector space, $|V| = e^2 = r^2$ for some prime power r. Thus, $|V^{\#}| = 8$ and $G \leqslant SL(2,3)$. Therefore, $G \simeq Q_8$, the quaternion group of order 8. However, since Q_8 acts on a vector space of order 9 transitively, this case is impossible.

Next, assume Theorem 4 (ii) of [6]. Then the vector space V has two spaces of imprimitivity V_1 and V_2 with $V = V_1 \oplus V_2$. If $H = N_G(V_1)$, we know that $H/C_H(V_1)$ acts transitively on $V_1^{\#}$ and $v \in V_1$ is in the same orbit as its inverse.

Finally, assume Theorem 4 (iii) of [6]. Then by Lemma 2.1, we know that G contains the central involution of $GL(2n, q)$ and therefore ν and $-\nu$ are in the same orbit, as desired. □

Using this lemma, it is now possible to prove Theorem 1.1.

PROOF OF THEOREM 1.1 — First, notice that $\chi\overline{\chi}$ is a real-valued character. So, if $\ker(\alpha_1) \neq \ker(\alpha_2)$ and α_i is a constituent of $\chi\overline{\chi}$, then $\overline{\alpha_i}$ is also a constituent of $\chi\overline{\chi}$ for $i = 1, 2$ with $\ker(\overline{\alpha_i}) = \ker(\alpha_i)$. In our case, if $\ker(\alpha_1) \neq \ker(\alpha_2)$, then $\alpha_i = \overline{\alpha_i}$, and α_i is a real-valued character. So, let $Z = Z(G)$ and assume that $\ker(\alpha_1) = \ker(\alpha_2)$. Then by Lemma 4.2.4 of [2], we know that

$$Z = \ker(\chi\overline{\chi}) = \ker(\alpha_1) \cap \ker(\alpha_2) = \ker(\alpha_1) = \ker(\alpha_2).$$

Let E/Z be a chief factor of G. Then by Proposition 4.2.5 of [2], E/Z is a fully ramified section of G with respect to χ_E and $\lambda \in \mathrm{Irr}(Z)$ such that $[\chi_Z, \lambda] \neq 0$. Also, G/E acts symplectically and faithfully on E/Z with two orbits on $(E/Z)^{\#}$. Suppose the α_i are complex-valued characters that are not real-valued. Then since $\chi\overline{\chi}$ is real-valued, $\alpha_2 = \overline{\alpha_1}$. Hence the orbits of G/E on $\mathrm{Irr}(E/Z)^{\#}$ are inverses of each other, i.e., if the nontrivial orbits are $\mathcal{O}_1$ and $\mathcal{O}_2$ and $\theta \in \mathcal{O}_1$, then $\overline{\theta} \in \mathcal{O}_2$. However, by Lemma 2.2, we know that θ and $\overline{\theta}$ must belong to the same orbit. Thus, we have a contradiction of the assumption that α_i are complex, which completes the proof. □

3 The abelian case

For this section, we will assume that

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2}.$$

For simplicity, we introduce the following hypothesis.

Hypothesis 3.1 *Let G be a finite solvable group with $\chi \in \mathrm{Irr}(G)$ a faithful character. Assume*

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2},$$

where the $\alpha_i \in \mathrm{Irr}(G)^{\#}$ are distinct and the m_i are strictly positive integers for $i = 1, 2$. Set $Z = Z(G)$.

Our first lemma will be useful in determining when restrictions of χ are irreducible.

Lemma 3.2 *Let* G *be a finite solvable group. Let* $\chi \in \mathrm{Irr}(G)$ *be a faithful character. Assume*

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} m_i\alpha_i,$$

where the $\alpha_i \in \mathrm{Irr}(G)^{\#}$ *are distinct and the* $m_i \in \mathbb{N}$. *Let* N *be a normal subgroup of* G. *Then* $\chi_N \in \mathrm{Irr}(N)$ *if and only if* $N \nleqslant \ker(\alpha_i)$ *for all* $i = 1, 2, \ldots, n$.

Proof — Notice that

$$[\chi_N, \chi_N] = [(\chi\overline{\chi})_N, 1_N]$$

$$= [1_N + \sum_{i=1}^{n} m_i(\alpha_i)_N, 1_N] = 1 + \sum_{i=1}^{n} m_i[(\alpha_i)_N, 1_N].$$

Thus, $[\chi_N, \chi_N] = 1$ if and only if $[(\alpha_i)_N, 1_N] = 0$ for all $i = 1, 2, \ldots, n$. Therefore, $[\chi_N, \chi_N] = 1$ if and only if $N \nleqslant \ker(\alpha_i)$ for all i. Since $[\chi_N, \chi_N] = 1$ if and only if $\chi_N \in \mathrm{Irr}(N)$, the result follows. □

In [2], Adan-Bante was able to show that the intersection of the kernels of the irreducible constituents is equal to the center of the solvable group G. In particular, when there are only two nonprincipal irreducible constituents of the product $\chi\overline{\chi}$, then the center of the group must be equal to one of their kernels. The same is true in this situation, and we will establish this fact in the following lemma.

Lemma 3.3 *Let* G *be a finite solvable group with* $\chi \in \mathrm{Irr}(G)$ *faithful such that*

$$\chi\overline{\chi} = 1_G + m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3,$$

with $\ker(\alpha_2) = \ker(\alpha_3)$. *Let* $Z = Z(G)$. *Then either* $\ker(\alpha_1) = Z$ *or* $\ker(\alpha_2) = Z$. *In particular, if we have Hypothesis 3.1, then* $\ker(\alpha_1) = Z$ *or* $\ker(\alpha_2) = Z$.

Proof — Here, we follow the style of Lemma 4.2.4 in [2]. Since α_2 and α_3 have the same kernel, we need only consider $\ker(\alpha_1)$ and $\ker(\alpha_2)$. Assume that Z is proper in both kernels. Then let R/Z be a chief factor of G with $R \leqslant \ker(\alpha_1)$. Note that $R \nleqslant \ker(\alpha_2)$. Also, let S/Z be a chief factor of G with $S \leqslant \ker(\alpha_2)$. Set $T = RS$.

Then $R \cap S = Z$ since $\ker(\alpha_1) \cap \ker(\alpha_2) = Z$. This implies that $T \nleqslant \ker(\alpha_i)$ for $i = 1, 2$. By Lemma 3.2, $\chi_T \in \mathrm{Irr}(T)$.

Notice that T/R is also a chief factor of G. Let $\psi \in \mathrm{Irr}(R)$ such that $[\chi_R, \psi] \neq 0$. Then by Theorem 6.18 of [7], either $\chi_R = e\psi$ for some $\psi \in \mathrm{Irr}(R)$ and $e^2 = |T : R|$ or $\psi^T = \chi_T$. Both of these cases imply that $\chi(g) = 0$ if $g \in T - R$. Similarly, $\chi(g) = 0$ if $g \in T - S$. Also, since R/Z and S/Z are chief factors of G, they are elementary abelian. Since R and S are normal subgroups of G with intersection Z and product T, we have that $T/Z = R/Z \times S/Z$. Thus, Lemma 4.2.2 of [2] gives us

$$\chi\overline{\chi}(g) = \begin{cases} 0 & \text{if } g \in T - R, \\ 0 & \text{if } g \in T - S, \\ \chi(1)^2 & \text{if } g \in R \cap S = Z. \end{cases}$$

Lemma 4.2.2 of [2] also tells us that $\chi\overline{\chi}$ is a multiple of the regular character of RS/Z. Thus, $\chi(1)^2 \geqslant |T : Z|$. However, since $\chi_T \in \mathrm{Irr}(T)$ it is also true that $\chi(1)^2 \leqslant |T : Z|$. Hence, equality holds and $(\chi\overline{\chi})_T = 1_Z^T$ is the regular character of T/Z. Since all characters of $\mathrm{Irr}(T/Z)$ are linear, they appear with multiplicity 1 in $1_Z^T = (\chi\overline{\chi})_T$ and $m_i = 1$ for $i = 1, 2$.

Notice that

$$[(\chi\overline{\chi})_R, 1_R] = [((\chi\overline{\chi})_T)_R, 1_R] = [(1_Z^T)_R, 1_R] = |T : R|.$$

Since $R \leqslant \ker(\alpha_1)$ and $R \neq \ker(\alpha_2)$, we have

$$1 + \alpha_1(1) = |T : R|.$$

In a similar fashion, $1 + \alpha_2(1) + \alpha_3(1) = |T : S|$. This yields the equation

$$|T : Z| = \chi(1)^2 = 1 + \alpha_1(1) + \alpha_2(1) + \alpha_3(1) = |T : R| + |T : S| - 1.$$

Also since $T = RS$ and $R \cap S = Z$, we can rewrite this as

$$|T : R||R : Z| - |T : R| - |R : Z| + 1 = 0.$$

Thus,

$$(|T : R| - 1)(|R : Z| - 1) = 0.$$

But since T/R and R/Z are chief factors of G, neither of them can have size 1. Therefore,

$$Z = \ker(\alpha_1) \text{ or } Z = \ker(\alpha_2) = \ker(\alpha_3),$$

as desired. In particular, if we consider the situation outlined in Hypothesis 3.1, we see that since α_2 and $\overline{\alpha_2}$ are complex conjugates, they have the same kernel. Thus, if we assume Hypothesis 3.1, we get the conclusion. □

It remains to go through all the possible equalities between $\ker(\alpha_1)$, $\ker(\alpha_2)$, and $Z = Z(G)$. We will begin with the case that all three subgroups are equal. In fact, this is impossible. Then we will consider the case when one of the kernels in an abelian group properly containing $Z(G)$. Lastly, we will assume that one of the kernels is a nonabelian group, properly containing $Z(G)$, which is equal to the other kernel. For the case when $Z(G) = \ker(\alpha_1) = \ker(\alpha_2)$, we will need the following theorem, proved by Foulser in [4] and simplified by Dornhoff in [3].

For this theorem, we must consider a primitive permutation group $\overline{G}$ acting on a finite set Ω. A standard reference for this situation is page 39 of [10]. First, let $\overline{G}$ be a solvable primitive permutation group on Ω with $\alpha \in \Omega$ having as its point stabilizer $G = \overline{G}_\alpha$. Then $\overline{G}$ has a unique minimal normal subgroup V such that

$$\overline{G} = GV,\ G \cap V = 1,\ C_{\overline{G}}(V) = V,$$

and V acts regularly on Ω. This means that

$$|\Omega| = |V| = q^n$$

is a prime power. Also, the mapping that takes $v \in V$ to $v\alpha$ is a G permutation isomorphism between V and Ω. In this case, G acts on V by conjugation. Since V is the unique minimal normal subgroup of $\overline{G}$, it follows that V is irreducible and faithful as a module of G. Because of the permutation isomorphism, the rank of $\overline{G}$ on Ω is the number of orbits, including the trivial orbit, of G on V.

Theorem 3.4 *Let $\overline{G}$ be a primitive solvable permutation group of rank 4; write $\overline{G} = GV$ where V is a minimal normal subgroup of $\overline{G}$ and G is the stabilizer of a point in Ω, the set upon which $\overline{G}$ acts. Then one of the following holds:*

(i) *V has order $|V| = q^n$ for a prime q and $\overline{G}$ is permutation-isomorphic to a subgroup $A\Gamma(q^n)$, the affine semi-linear group. In particular, $\mathrm{dl}(G) \leqslant 2$.*

(ii) G *is an imprimitive linear group with* $V = \oplus_{i=1}^{r} V_i$ *where the* V_i *are imprimitivity spaces and* $r = 2$ *or* 3. *Here* $H = N_G(V_1)$ *and* $H/C_H(V_1)$ *is a linear group that acts transitively on* $V_1^{\#} = V_1 - \{0\}$. *Thus,* $\mathrm{dl}(G) \leqslant 5$.

(iii) G *acts as a primitive linear group on* V *and* $\overline{G}$ *has one of the degrees* q^n *for* $q^n = p^2$, *where* p *is a prime and* $p \leqslant 71$, *or* $q^n \in \{2^4, 3^4, 5^4, 7^4, 2^6, 3^6, 2^8, 2^{10}, 3^{10}, 2^{12}\}$.

Proof — See Theorem 1.2 of [4]. To see the second conclusion of (ii), we know that $\mathrm{dl}(G) \leqslant 5$ since

$$G \leqslant (H/C_H(V_1)) \wr C_r,$$

by the argument of Lemma 1.4 from [9]. However, this wreath product has derived length bounded above by 5, since from Theorem 6.8 of [10], $\mathrm{dl}(H/C_H(V_1)) \leqslant 4$. □

Using Theorem 3.4, we are able to prove the following lemma, which will make the situation where $\ker(\alpha_1) = \ker(\alpha_2) = Z(G)$ impossible.

Lemma 3.5 *Let* G *be a finite solvable group. Let* V *be a symplectic vector space of dimension* $2n$ *over* $GF(q)$ *for some prime power* q. *Assume that* V *is a faithful irreducible* G*-module. Suppose the action of* G *on* V *preserves the symplectic form, and that* G *acts on* $V^{\#}$ *with three orbits, say* O_1, O_2, *and* O_3. *Set* $e^2 = q^{2n} = |V|$. *Then* $v \in O_i$ *implies that* $-v \in O_i$ *for* $1 \leqslant i \leqslant 3$.

Proof — If $q = 2$, then our conclusion is immediate since V is an extra-special 2-group and $v = -v$ in this case. Thus, we will assume that p is an odd prime. Our hypotheses imply those of Theorem 3.4, so we will examine the three situations there. Notice that v and $-v$ must be in orbits of equal size. So, if all three nontrivial orbits have distinct sizes, the result is trivial. Hence, we may assume that two orbits have the same size, which implies that the orbit sizes are $a = |O_2| = |O_3|$ and $e^2 - 1 - 2a = |O_1|$, both of which must divide $|G|$.

Assume Theorem 3.4(i). Then $G \leqslant \Gamma(V)$ and $N = G \cap \Gamma_0(V)$ is a cyclic normal subgroup of G and G/N acts faithfully on N. This means that $|\mathrm{Aut}(N)| = \varphi(|N|)$, where φ is the Euler φ-function, and $|G| \leqslant |N|\varphi(|N|)$. First assume that $|N|$ is not prime and is not equal to 4. Then $\varphi(|N|) \leqslant |N| - 3$ and $|N| \leqslant e + 1$ by Lemma 2.2.3 of [2]. Thus, $|G| < e^2 + 1$.

Claim $|G| \geqslant \frac{1}{3}(e^2 - 1)$.
Since $\max(a, e^2 - 1 - 2a) \leqslant \operatorname{lcm}(a, e^2 - 1 - 2a) \leqslant |G|$, it suffices to show that

$$\max(a, e^2 - 1 - 2a) \geqslant \frac{1}{3}(e^2 - 1).$$

If $a \geqslant \frac{1}{3}(e^2 - 1)$, then we are finished. So assume $a < \frac{1}{3}(e^2 - 1)$. Then $-2a > -\frac{2}{3}(e^2 - 1)$ and $e^2 - 1 - 2a > e^2 - 1 - \frac{2}{3}(e^2 - 1) = \frac{1}{3}(e^2 - 1)$. This completes the claim.

By the claim, $\frac{1}{3}(e^2 - 1) \leqslant |G| < e^2 - 1$. Then

$$2n \geqslant \frac{|G|}{|N|} \geqslant \frac{\frac{1}{3}(e^2 - 1)}{e + 1} = \frac{e - 1}{3} = \frac{q^n - 1}{3}.$$

Thus, we need prime powers q^n such that this inequality holds. This implies that $q^{2n} = e^2 \in \{3^2, 3^4, 5^2\}$. Now, $|N|$ divides $|\Gamma_0(V)| = e^2 - 1$ and $|G : N|$ divide the size of the Galois group. Here, we may assume that $|N|$ is odd as well since otherwise, it contains the central involution of the field and we are finished.

If $q^{2n} = 3^4$, then $|N|$ divides 80, and in particular, $|N| = 5$. This implies that $|G : N|$ divides 4 and $|G| \leqslant 20$. But we know that

$$20 < \frac{80}{3} = \frac{1}{3}(e^2 - 1) \leqslant |G|.$$

This is a contradiction. If $q^{2n} = 3^2$, then $|N|$ divides 8. Since N cannot be the trivial group, it has even order in this case and we are finished. If $q^{2n} = 5^2$, then $|N| = 3$ and $|G| \leqslant 6$. However, our claim implies that $|G| \geqslant 8$, another contradiction. Lastly, if $q^{2n} = 7^2$, then $|N| = 3$ and $|G| \leqslant 6$. But since our claim implies that $|G| \geqslant 16$, we have a contradiction here as well.

Next assume that $|N| = p$ for a prime p. Then $|N| = p \leqslant e + 1$ and $\varphi(|N|) = p - 1$. Assume first that $p < e + 1$, i.e., $p \leqslant e$. Then we get that $|G| < e^2 - 1$ and the claim yields that

$$\frac{1}{3}(e^2 - 1) \leqslant |G| < e^2 - 1.$$

This yields the same results as the last paragraph. So, assume that $p = e + 1$. Then p is a Fermat prime and $e = q^n$ is a power of 2, which is finished by our earlier remark.

Now assume that $|N| = 4$. Then since $|N|$ is even, it contains the central involution of $\Gamma_0(V)$ and this element sends elements to their inverses. This completes this case.

Assume Theorem 3.4 (ii). Then G is an imprimitive linear group and

$$V = \bigoplus_{j=1}^{r} V_j$$

where the V_j are imprimitivity spaces and $r = 2$ or 3. We know that $H = N_G(V_1)$ has index r in G, and that $\overline{H} = H/C_H(V_1)$ acts transitively on $V_1^\#$. Thus, Lemma 2.8 of [10] implies that $G \leqslant \overline{H} \wr \mathbb{Z}_2$, and we may apply Theorem 6.8 of [10] to $\overline{H}$.

First, suppose that $r = 2$ and that Theorem 6.8 (i) of [10] holds, i.e., $\overline{H} \leqslant \Gamma(V_1)$. Since q is an odd prime, we know that $|V_1|$ is odd, and $|V_1^\#|$ is even. Hence, $\overline{H}$ is also even and contains some involution which inverts some element of V_1. This handles the orbit which has form $V_1^\# \cup V_2^\#$. Now, let $v = (a, 0) \in V_1 \cup V_2$ with $a \neq 0$. Then, for an involution $g \in G$, we know that $v \cdot g = (0, b) = w$. Now, consider $v - w = (a, -b)$. Then

$$(v - w) \cdot g = v \cdot g - w \cdot g = w - v = -(v - w)\,.$$

Thus, the orbit containing $v - w \in V - (V_1 \cup V_2)$ contains inverses. Since $v - w$ is in one of the orbits that make up $V - (V_1 \cup V_2)$, it follows that the second orbit contains inverses as well.

Next, assume Theorem 6.8 (ii), (iii) of [10]. Then, $\overline{H} \leqslant GL(V_1)$ where $|V_1| = 3^2, 5^2, 7^2, 11^2, 23^2, 3^4$, and Lemma 2.1 yields that G contains the central involution. Therefore, the orbits contain inverses, as desired.

So assume that there are three imprimitivity spaces, i.e.,

$$V = V_1 \oplus V_2 \oplus V_3.$$

Then, elements of V are 3-tuples, meaning they have form (v_1, v_2, v_3), where $v_i \in V_i$ for $1 \leqslant i \leqslant 3$. Since $0 \in V_i$ is centralized by all $g \in G$, if $v \cdot g = w$, it is impossible to v to have a different number of nonzero coordinates than w. Hence, we obtain the following orbits:

$$\{(0,0,0)\}$$
$$\{(a,b,c) | \text{exactly one of } a,b,c \text{ is nonzero}\}$$
$$\{(a,b,c) | \text{exactly two of } a,b,c \text{ are nonzero}\}$$
$$\{(a,b,c) | a,b,c \neq 0\}\,.$$

By their definition, v and $-v$ must be in the same orbit, since they will have the same number of nonzero coordinates. Thus, we are finished in this case.

Finally, assume Theorem 3.4 (iii). Then by Lemma 2.1, we have that G contains the central involution, and we are finished. □

We will use the last lemma to prove the following proposition.

Proposition 3.6 *Assume Hypothesis 3.1. Then* $\ker(\alpha_1)$ *and* $\ker(\alpha_2)$ *are distinct.*

Proof — Suppose that $Z = \ker(\alpha_1) = \ker(\alpha_2)$ and let E/Z be a chief factor of G. Let $\lambda \in \mathrm{Irr}(Z)$ such that $[\chi_Z, \lambda] \neq 0$. By Lemma 3.2, $\chi_E \in \mathrm{Irr}(E)$ since $E \not\leqslant \ker(\alpha_i)$ for $i = 1, 2$. Then Theorem 6.18 of [7] implies that E/Z is a fully ramified section with respect to χ_E and λ. By appealing to Lemma 3.2.1 of [2], we see that $C_G(E/Z) = EC_G(E)$ and E/Z is a symplectic vector space. Notice that since $\chi \in \mathrm{Irr}(G)$ is faithful, $Z(\chi) = Z$. Also, $\chi_E \in \mathrm{Irr}(E)$ implies that $C_G(E) \leqslant Z(\chi) = Z$ by Lemma 3.2.2 of [2]. Hence, $C_G(E/Z) = E$ and G/E acts faithfully on E/Z. Furthermore,

$$(\chi\overline{\chi})_E = 1_Z^E,$$

by Lemma 3.1.1 of [2].

Now, we have that

$$1_Z^E = (\chi\overline{\chi})_E = 1_E + m_1\,(\alpha_1)_E + m_2\,(\alpha_2)_E + m_2\,(\overline{\alpha_2})_E\,.$$

By Clifford Theory, the irreducible constituents of $(\alpha_1)_E$ form an orbit. If the irreducible constituents of $(\alpha_1)_E$, $(\alpha_2)_E$, and $(\overline{\alpha_2})_E$ are all the same, then there is a single orbit under the action of G/E on $\mathrm{Irr}(E/Z)^\#$ and under the action of G/E on $(E/Z)^\#$ by Brauer's Theorem (Theorem 6.32 of [7]). However, by Theorem A of [2], this means that $\chi\overline{\chi} = 1_G + \alpha$ for some $\alpha \in \mathrm{Irr}(G)^\#$, which is a contradiction. So, assume that two of the orbits given by $(\alpha_1)_E$, $(\alpha_2)_E$, and $(\overline{\alpha_2})_E$ are the same. Then by Proposition 4.3.3 of [2], this implies that $\chi\overline{\chi} = 1_G + \alpha + \beta$ for some $\alpha, \beta \in \mathrm{Irr}(G)^\#$, which is also a contradiction. So, we suppose that the orbits given by $(\alpha_1)_E$, $(\alpha_2)_E$, and $(\overline{\alpha_2})_E$ are all distinct, namely the orbits of α_2 and $\overline{\alpha_2}$ are not the same. Then, their orbits must be inverses of each other. However, by Lemma 3.5, we know that since we have three orbits, if $x \in \mathcal{O}$ for some orbit, then $x^{-1} \in \mathcal{O}$ also. This is a contradiction. Thus, $\ker(\alpha_1) \neq \ker(\alpha_2)$, and the proof is complete. □

Next, we will discuss the option that one of the kernels is an abelian group strictly containing the center of G.

Lemma 3.7 *Let* G *be a finite solvable group with* $\chi \in \mathrm{Irr}(G)$ *faithful and*

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} m_i \alpha_i,$$

where the $\alpha_i \in \mathrm{Irr}(G)^{\#}$ *are distinct,* $m_i \in \mathbb{N}$*, and* $n \geqslant 2$*. Fix* j *such that* $K = \ker(\alpha_j)$ *is maximal among the* $\ker(\alpha_i)$*. Suppose that* K *abelian, with* $K > Z(G) = Z$*. Let* $\theta \in \mathrm{Irr}(K)$ *such that* $[\chi_K, \theta] \neq 0$*. Denote the inertia group of* θ *by* G_θ*. Let* L/K *be a chief factor of* G*. Then*

(i) $G = G_\theta L$ *and* $K = G_\theta \cap L$*.*

(ii) G/L *acts faithfully and irreducibly on* L/K*.*

(iii) $\chi(1) = |L : K|$*, and thus is a prime power.*

Proof — Since $L > \ker(\alpha_i)$ for all i, $\chi_L \in \mathrm{Irr}(L)$ by Lemma 3.2. Since χ_K is reducible, K is abelian, and $Z(\chi) = Z \neq \ker(\alpha_1)$, Theorem 6.18 of [7] implies that $\chi_L = \theta^L$ and G_θ is proper in G. By Exercise 5.7 of [7], we have that $G_\theta L = G$. Also, we know that $\theta^L = \chi_L$ belongs to $\mathrm{Irr}(L)$ and so by Exercise 6.1 of [7], we know that $L_\theta = K$. But $L_\theta = G_\theta \cap L$, and thus $K = G_\theta \cap L$.

Now, observe that $L \leqslant C_G(L/K) \lhd G$. If we let $C = C_G(L/K)$, then $G = G_\theta L$ implies that $C = (C \cap G_\theta)L$. Also notice that $C \cap G_\theta$ is normal in G_θ and that $[C \cap G_\theta, L] \leqslant [C, L] \leqslant K$. Thus, $C \cap G_\theta$ is normal in G. If $L < C$ then K is proper in the normal subgroup $C \cap G_\theta$ and $\chi_{C \cap C_\theta}$ is an irreducible character. However, since χ is induced from some character of G_θ and $C \cap G_\theta \leqslant G_\theta < G$, it is impossible for $\chi_{C \cap G_\theta}$ to be irreducible. So it must be the case that $C = L$. This proves that the action is faithful.

Since $\chi_L = \theta^L$ and $\theta(1) = 1$,

$$\chi(1) = \theta^L(1) = |L : K|\theta(1) = |L : K|,$$

which is a prime power since L/K is a chief factor of a solvable group. Thus, $\chi(1)$ is also a prime power. Lastly, since K is maximal among the kernels, Lemma 3.1.3 of [2] yields that $m_j = 1$. □

Proposition 3.8 *Assume Hypothesis 3.1. Assume that* $K = \ker(\alpha_1)$ *is an abelian group properly containing* $Z(G) = \ker(\alpha_2)$*. Then* $\mathrm{dl}(G) \leqslant 6$*.*

Proof — Let $\theta \in \mathrm{Irr}(K)$ such that $[\chi_K, \theta] \neq 0$ and let L/K be a chief factor of G. Then by Lemma 3.7, we know that $G = G_\theta L$, where G_θ is the inertia group of θ in G, and G/L acts faithfully and irreducibly on L/K. Also by that lemma, $m_1 = 1$.

By Lemma 3.1.1 of [2], $(\chi\overline{\chi})_L = 1_K^L + \Phi$ where Φ is a character of L and $[\Phi_K, 1_K] = 0$. Also, since

$$(\chi\overline{\chi})_L = 1_L + (\alpha_1)_L + m_2(\alpha_2)_L + m_2(\overline{\alpha_2})_L,$$

we have that $1_K^L = 1_L + (\alpha_1)_L$ and $\Phi = m_2(\alpha_2)_L + m_2(\overline{\alpha_2})_L$. This implies that G/L acts transitively on $\mathrm{Irr}(L/K)^\#$ and hence on $(L/K)^\#$ by Brauer's Theorem (Theorem 6.32 of [7]). Therefore, G/L is one of the groups in Theorem 6.8 of [10]. Hence, $\mathrm{dl}(G/L) \leqslant 4$. Since L/K is a chief factor of G and K is abelian, we have that $\mathrm{dl}(L) = 2$. Therefore, $\mathrm{dl}(G) \leqslant \mathrm{dl}(G/L) + \mathrm{dl}(L) \leqslant 4 + 2 = 6$. □

Before we begin the next proposition, we need the following lemma.

Lemma 3.9 *Let G be a finite solvable group. Let V be a vector space of dimension n over $GF(q)$, where q is a prime power. Assume that V is a faithful G-module and that G acts on $V^\# = V - \{0\}$ with two orbits of equal size. Also assume one of the following situations:*

(i) *G is an imprimitive linear group with imprimitivity spaces V_1, V_2, where $V = V_1 \oplus V_2$. Here $H = N_G(V_1)$ has index 2 and $H/C_H(V_1)$ is a linear group that acts transitively on $V_1 - \{0\}$.*

(ii) *G acts as a primitive linear group on V and $\overline{G}$ has one of the degrees 7^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 47^2, 3^4, 7^4, 2^6, or 3^6.*

Then v and $-v$ belong to the same orbit.

Proof — Suppose case (i). Then the orbits are $\{0\}$, $V_1^\# \cup V_2^\#$, and $V - \left(V_1^\# \cup V_2^\#\right)$. In all these cases, if v belongs to of them, so does its inverse $-v$. So, suppose case (ii). Then $|V| = q^n$ is one of the prime-powers listed in that case, and by Lemma 2.1, G has the central involution of $GL(V)$. This completes the proof. □

Proposition 3.10 *Assume Hypothesis 3.1. Assume also that $K = \ker(\alpha_2)$ is an abelian group properly containing Z. Then $\mathrm{dl}(G) \leqslant 6$.*

Proof — Let L/K be a chief factor of G. By Lemma 3.7, we know that if $\theta \in \mathrm{Irr}(K)$ such that $[\chi_K, \theta] \neq 0$, and L/K is a chief factor of G, then $G = G_\theta L$, $K = G_\theta \cap L$, and G/L acts faithfully and irreducibly on L/K. It remains to show how many orbits are in the action of G/L on L/K. By Lemma 3.1.1 of [2], we know that

$$(\chi\overline{\chi})_L = 1_K^L + \Phi,$$

where Φ is either the zero function or a character of L with $[\Phi_K, 1_K] = 0$. Also,

$$(\chi\overline{\chi})_L = 1_L + m_1(\alpha_1)_L + m_2(\alpha_2)_L + m_2(\overline{\alpha_2})_L.$$

Since $\ker(\alpha_1) = Z < \ker(\alpha_2)$, we know that $[(\alpha_1)_K, 1_K] = 0$. This means that $\Phi = m_1(\alpha_1)_L$ and $1_K^L = 1_L + m_2(\alpha_2)_L + m_2(\overline{\alpha_2})_L$. So the irreducible constituents of $(\alpha_2)_L + (\overline{\alpha_2})_L$ form $\mathrm{Irr}(L/K)^\#$. If the orbits of $(\alpha_2)_L$ and $(\overline{\alpha_2})_L$ are the same, then G/L acts transitively on $\mathrm{Irr}(L/K)^\#$, and hence on $(L/K)^\#$ by Brauer's Theorem, implying that $\mathrm{dl}(G/L) \leqslant 4$ by Theorem 6.8 of [10]. If they are different then G/L acts on $(L/K)^\#$ with two orbits of equal size that are inverses of each other. By Lemma 3.9 and Theorem 4 of [6], this means that G/L is isomorphic to a subgroup of $\Gamma(L/K)$. However, these subgroups are metacyclic, which implies that $\mathrm{dl}(G/L) \leqslant 2$. Finally, since L/K is a chief factor of G and K is abelian by assumption, we know that $\mathrm{dl}(L) = 2$ and thus $\mathrm{dl}(G) \leqslant 6$. □

4 The nonabelian case

The last situation we must deal with is the possibility that one of the kernels is a nonabelian group. In fact, this cannot occur, and to show this, we need several lemmas. Note that these lemmas are more general than necessary for this paper.

Lemma 4.1 *Let G be a finite solvable group with $\chi \in \mathrm{Irr}(G)$ a faithful character. Suppose also that*

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} m_i\alpha_i,$$

where the $\alpha_i \in \mathrm{Irr}(G)^\#$ are distinct, the $m_i \in \mathbb{N}$, and $n \geqslant 2$. Also, for all i, define $K_i = \ker(\alpha_i)$. Let $\mathscr{I}$ be a set of indices such that $\mathscr{I} < \{1, 2, \ldots, n\}$

and such that $Z(G) < L$, *where* $L = \bigcap_{i \in \mathscr{I}} K_i$. *Let* $\theta \in \mathrm{Irr}(L)$. *Then for* $E \leqslant L$ *such that* $E \nleqslant L \cap K_j$ *for* $j \notin \mathscr{I}$, $\theta_E \in \mathrm{Irr}(E)$.

Proof — Suppose that θ_E is reducible. Then $[\theta_E, \theta_E] > 1$. Let

$$\chi_L = f\theta + \Delta$$

for some character Δ of L. Then since $f = [\chi_L, \theta] \neq 0$, we have

$$\begin{aligned}[\chi_L, \chi_L] &= [f\theta + \Delta, f\theta + \Delta] = f^2[\theta, \theta] + [\Delta, \Delta] = f^2 + [\Delta, \Delta] \\ &< f^2[\theta_E, \theta_E] + 2f[\theta_E, \Delta_E] + [\Delta_E, \Delta_E] = [\chi_E, \chi_E].\end{aligned}$$

Thus,

$$[(\chi\overline{\chi})_E, 1_E] = [\chi_E, \chi_E] > [\chi_L, \chi_L] = [(\chi\overline{\chi})_L, 1_L].$$

But since E and L are contained in precisely the same K_i, it must be the case that $[(\chi\overline{\chi})_E, 1_E] = [(\chi\overline{\chi})_L, 1_L]$. Therefore, $\theta_E \in \mathrm{Irr}(E)$, as desired. □

Lemma 4.2 *Let* G *be a finite solvable group with* $\chi \in \mathrm{Irr}(G)$ *a faithful character. Suppose also that*

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} m_i\alpha_i,$$

where the $\alpha_i \in \mathrm{Irr}(G)^\#$ *are distinct, the* $m_i \in \mathbb{N}$, *and* $n \geqslant 2$. *Also, for all* i, *define* $K_i = \ker(\alpha_i)$. *Let* $K \leqslant G$ *be a nonabelian subgroup of* G. *Suppose that for all* K_i, *either* $K_i = K$ *or* $K_i = Z$, *where* $Z = Z(G)$. *Let* E/Z *be a chief factor of* G *such that* $E \leqslant K$. *Then* $C_G(E/Z) = E$ *and* G/E *acts faithfully on the symplectic vector space* E/Z.

Proof — See Claim 4.2.17 of [2]. □

The next lemma is a generalization of two claims from [2]. In this situation, we can use Lemmas 4.1 and 4.2 to get that G/E acts transitively on $(E/Z)^\#$, as Adan-Bante does in [2]. Then, it is possible to use Theorem 2.2.1 of [2] to restrict the possible values of $e = |E : Z|^{1/2}$.

Lemma 4.3 *Let* G *be a finite solvable group with* $\chi \in \mathrm{Irr}(G)$ *a faithful character. Suppose also that*

$$\chi\overline{\chi} = 1_G + \sum_{i=1}^{n} m_i\alpha_i,$$

where the $\alpha_i \in \mathrm{Irr}(G)^\#$ *are distinct, the* $m_i \in \mathbb{N}$*, and* $n \geqslant 2$*. Also, for all* i*, define* $K_i = \ker(\alpha_i)$*. Let* $K \leqslant G$ *be a nonabelian subgroup of* G*. Suppose that for all* K_i*, either* $K_i = K$ *or* $K_i = Z$*, where* $Z = Z(G)$*. Suppose also that* E/Z *is a chief factor of* G *such that* $E \leqslant \ker(\alpha_1) = K$ *and* G/E *acts transitively on* E/Z*. Let* $\theta \in \mathrm{Irr}(K)$ *be such that* $[\chi_K, \theta] \neq 0$ *and* $\varphi = \theta_E \in \mathrm{Irr}(E)$*. Also let* $e^2 = |E : Z|$*. Then* $e \in \{2, 3, 5, 7, 9\}$ *and the following occur:*

(a) *if* $e \in \{2, 3, 5, 7\}$*, then* φ *extends to* G*, i.e., there exists some* $\delta \in \mathrm{Irr}(G)$ *such that* $\delta_E = \varphi$*.*

(b) *if* $e = 9$*, then* $E < K$ *and* α_1 *is not a faithful character.*

Proof — For part (a), see Claim 4.2.20 of [2]. For part (b), see Claim 4.2.39 of [2]. □

Proposition 4.4 *Assume Hypothesis 3.1. Then* $\ker(\alpha_1)$ *is abelian.*

Proof — Assume that $\ker(\alpha_1)$ is not abelian and let E/Z be a chief factor of G such that $E \leqslant K = \ker(\alpha_1)$. Let $\theta \in \mathrm{Irr}(K)$ be such that $[\chi_K, \theta] \neq 0$. Since K is nonabelian and χ is faithful, we know that χ_K is a sum of G-conjugate nonlinear characters, one of which is θ. Hence, $\theta(1) > 1$. By Proposition 3.3, $Z = \ker(\alpha_2)$ and by Lemma 4.1, $\theta_E = \varphi \in \mathrm{Irr}(E)$. Since $\varphi(1) > 1$, E/Z is a chief factor of G, and $Z = Z(G)$, we have that E/Z is a fully ramified section with respect to φ and $\lambda \in \mathrm{Irr}(Z)$ such that $[\theta_Z, \lambda] \neq 0$. Thus, φ is G-invariant and $\varphi(1) = |E : Z|^{1/2}$. Also, by Lemma 4.2, G/E acts faithfully and symplectically on E/Z.

Claim 1 G/E *acts transitively on* $(E/Z)^\#$*.*
Since E/Z is a fully ramified section of G and $Z = Z(G)$,

$$(\theta\overline{\theta})_E = 1_Z^E,$$

by Lemma 3.1.1 of [2]. Moreover, since $[\theta_E, \chi_E] \neq 0$, we know that

$$(\chi\overline{\chi})_E = 1_Z^E + \Phi,$$

where Φ is a character of E. Since $K = \ker(\alpha_1) \geqslant E$, our assumption implies that

$$\begin{aligned}(\chi\overline{\chi})_E &= (1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha}_2)_E \\ &= (1 + m_1\alpha_1(1))1_E + m_2(\alpha_2)_E + m_2(\overline{\alpha_2})_E = 1_Z^E + \Phi.\end{aligned}$$

By Clifford theory, it must be that the irreducible constituents of $(\alpha_2)_E$ form a G-orbit, as do those of $(\overline{\alpha_2})_E$. If they are distinct orbits, then they must be inverses of each other. However, this contradicts Lemma 2.2. Therefore, they are the same orbit and G/E acts transitively on $\mathrm{Irr}(E/Z)^\#$ and hence on $(E/Z)^\#$ by Brauer's Theorem (see [7], Theorem 6.32). Thus we can apply Theorem 2.2.1 of [2], which means that $e = |E : Z|^{1/2} \in \{2,3,5,7,9\}$.

Claim 2 $e \notin \{2,3,5,7\}$.

Since $[\varphi, \chi_E] \neq 0$ and φ extends to $\delta \in \mathrm{Irr}(G)$ by Lemma 4.3 (a), Gallagher's Theorem (see [7], Corollary 6.17) states that there exists some $\psi \in \mathrm{Irr}(G/E)$ such that $\chi = \delta\psi$.

Since δ extends φ and G/E acts faithfully on E/Z, we have that $\ker(\delta) \leqslant E$. Also since χ is faithful, $\ker(\delta) \leqslant E$, and $\chi = \delta\psi$, we have that $\ker(\delta) = \ker(\chi) \cap E = 1$, which implies that δ is a faithful character of G, E/Z is a chief factor of G where $Z = Z(G)$, and G/E acts transitively on $(E/Z)^\#$ by assumption. So Lemma 3.3.2 of [2] gives that $\delta\overline{\delta} = 1_G + c_1\gamma$ for some $\gamma \in \mathrm{Irr}(G)$ and an integer c_1. By Theorem A of [2], $c_1 = 1$ and by Lemma 3.1.2 of [2], $Z = \ker(\gamma)$.

Now let

$$\psi\overline{\psi} = 1_G + \sum_{i=1}^{n} a_i\beta_i$$

for some positive integers a_i and distinct characters $\beta_i \in \mathrm{Irr}(G)$ for $i = 1, 2, \ldots, n$. Since $\psi \in \mathrm{Irr}(G/E)$, we have that $\beta_i \in \mathrm{Irr}(G/E)$ also for all i. Then,

$$1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2} = \chi\overline{\chi} = (\delta\psi)\overline{(\delta\psi)} = (\delta\overline{\delta})(\psi\overline{\psi})$$

$$= (1_G + \gamma)(1_G + \sum_{i=1}^{n} a_i\beta_i) = 1_G + \gamma + \sum_{i=1}^{n} a_i\beta_i + \sum_{i=1}^{n} a_i\beta_i\gamma.$$

Now $\ker(\beta_1) \geqslant E$ and $\ker(\gamma) = Z < E$. Also, notice that $Z = \ker(\alpha_2)$. So, γ and α_2 have the same kernel. But since γ is real and α_2 is complex, this is a contradiction. Thus, $e \notin \{2,3,5,7\}$.

Claim 3 $e \neq 9$.

Recall that $E \leqslant K = \ker(\alpha_1)$. Thus, χ_E is reducible. Also, E/Z is a fully ramified section with respect to θ_E, which implies that $\chi(1) = em$

for $m \geqslant 2$. Also, since $\chi(g) = 0$ for all $g \in E - Z$, we have that

$$0 = \chi\overline{\chi}(g) = 1 + \alpha_1(1) + m_2\alpha_2(g) + m_2\overline{\alpha_2(g)}.$$

This means that

$$\alpha_2(g) + \overline{\alpha_2(g)} = -\frac{1 + \alpha_1(g)}{m_2}$$

is both rational and an algebraic integer. Thus, $\alpha_2(g) + \overline{\alpha_2(g)} \in \mathbb{Z}$. So m_2 divides $1 + \alpha_1(1)$ and in particular $2m_2$ divides $1 + \alpha_1(1)$.

Now, our hypothesis implies that

$$\alpha_2(1) = \frac{\chi(1)^2}{2m_2} - \frac{1 + \alpha_1(1)}{2m_2},$$

which yields that

$$(\alpha_2)_E = s1_Z^E - \frac{1 + \alpha_1(1)}{2m_2}1_E$$

for some integer $s \geqslant \frac{1+\alpha_1(1)}{2m_2}$. Thus,

$$se^2 - \frac{1 + \alpha_1(1)}{2m_2} = \alpha_2(1) = \frac{e^2m^2}{2m_2} - \frac{1 + \alpha_1(1)}{2m_2},$$

and $s = \frac{m^2}{2m_2}$. Therefore, $2m_2$ divides m^2. Also, since $\alpha_2 \in \text{Irr}(G)$ and $\ker(\alpha_2) = Z$, we have $\alpha_2(1)^2 < |G : Z|$ and $\alpha_2(1)$ divides $|G : Z|$. So assume that $e = 9$ and recall that $\chi(1) = em$. We will obtain a contradiction. By Theorem 2.2.1(v) of [2], we have that $|G : Z|$ divides

$$25920 = 10 \times 32 \times 81.$$

By Claim 4 above and Theorem 2.2.1 (v) and Lemma 4.2.12 of [2], we know that $\alpha_1(1) \in \{1, 2, 4, 5, 8, 10\}$. By Lemma 4.3 (b) and Lemma 4.2.12 of [2], we can shorten this list to $\alpha_1(1) \in \{1, 2, 4, 5, 10\}$, which puts $m_2 \in \{1, 3\}$ and gives that

$$\frac{1 + \alpha_1(1)}{2m_2} < \frac{11}{2}.$$

Suppose that $m_2 = 1$. Then since $\frac{1+\alpha_1(1)}{2}$ is an integer, we have that $\alpha_1(1) \in \{1,5\}$. Also, $2m_2$ divides m^2, which implies that 2 divides m. If $m \geqslant 4$, then

$$\alpha_2(1) = \frac{\chi(1)^2}{2} - \frac{1+\alpha_1(1)}{2} \geqslant \frac{4^2 9^2}{2} - \frac{1+5}{2} = 81 \times 8 - 3 = 645.$$

Thus

$$\alpha_2(1)^2 \geqslant 645^2 > 25920 \geqslant |G:Z|,$$

which is a contradiction since we know that $\alpha_2(1)^2$ must be smaller than $|G:Z|$. So, assume that $m = 2$. Then either

$$\alpha_2(1) = 2 \times 81 - 1 = 161 \text{ or } \alpha_2(1) = 2 \times 81 - 3 = 159.$$

However, neither of these divide $|G:Z|$. Since $\alpha_2(1)$ must divide $|G:Z|$, this is a contradiction. Therefore, $m_2 \neq 1$.

Now, assume $m_2 = 3$. We know that $2m_2$ divides m^2, so 6 divides m and $m^2 \geqslant 36$. This implies that

$$\frac{\chi(1)^2}{2m_2} = \frac{e^2 m^2}{6} \geqslant \frac{9^2 36}{6} = 486,$$

and thus $\alpha_2(1) \geqslant 486 - \frac{1+5}{6} = 485$. So,

$$\alpha_2(1)^2 \geqslant 485^2 > 25920 \geqslant |G:Z|,$$

which also contradicts the fact that $\alpha_2(1)^2 < |G:Z|$. So $m_2 \neq 3$ and $e \neq 9$.

Notice that we have considered all the cases given in Theorem 2.2.1 of [2], which completes the proof of the proposition. □

It remains to show that $\ker(\alpha_2)$ must also be abelian.

Proposition 4.5 *Assume Hypothesis 3.1. Then* $\ker(\alpha_2)$ *is abelian.*

Proof — Suppose that $\ker(\alpha_2)$ is not abelian and let E/Z be a chief factor of G such that $E \leqslant K = \ker(\alpha_2)$. Let $\theta \in \mathrm{Irr}(K)$ be such that $[\chi_K, \theta] \neq 0$. Since K is nonabelian and χ is faithful, we have that χ_K is a sum of G-conjugate nonlinear characters, one of which is θ. Hence θ is nonlinear. By Lemma 3.3, $Z = \ker(\alpha_1)$. Also by Lemma 4.1, $\theta_E = \varphi \in \mathrm{Irr}(E)$. Since $\varphi(1) > 1$, E/Z is a chief factor of G, and $Z = Z(G)$, we have that E/Z is a fully ramified section of G

with respect to φ and $\lambda \in \mathrm{Irr}(Z)$ such that $[\theta_Z, \lambda] \neq 0$. Thus, φ is G-invariant and

$$\varphi(1) = \theta_E(1) = |E : Z|^{1/2}.$$

Finally, by Lemma 4.2, $C_G(E/Z) = E$ and G/E acts faithfully on the symplectic vector space E/Z.

Claim 1 G/E *acts transitively on* $(E/Z)^{\#}$.
We know that E and Z are fully ramified with respect to φ and λ. Since $Z = Z(G)$, we have that $(\theta\overline{\theta})_E = 1_Z^E$. Also, since $[\chi_E, \varphi] \neq 0$,

$$(\chi\overline{\chi})_E = 1_Z^E + \Phi,$$

where Φ is a character of E. Since $K = \ker(\alpha_2) \geqslant E$, our hypothesis implies that

$$(\chi\overline{\chi})_E = (1_G + m_1 + m_2\alpha_2 + m_2\overline{\alpha_2})_E = 1_E + m_1(\alpha_1)_E + 2m_2\alpha_2(1)1_E$$

$$= (1 + 2m_2\alpha_2(1))1_E + m_1(\alpha_1)_E = 1_Z^E + \Phi.$$

By Clifford Theory, the irreducible constituents of $(\alpha_1)_E$ are G-conjugate. Thus, $\mathrm{Irr}(E/Z)^{\#}$ is a G-orbit and by Brauer, we have that G/E acts transitively on $(E/Z)^{\#}$.

Now by Theorem 2.2.1 of [2], we have that $e \in \{2, 3, 5, 7, 9\}$ and G/E is isomorphic to different groups for each value of e.

Claim 2 $e \notin \{2, 3, 5, 7\}$.
Since $[\chi_E, \varphi] \neq 0$ and φ extends to $\delta \in \mathrm{Irr}(G)$ by Lemma 4.3 (a), there exists some $\psi \in \mathrm{Irr}(G/E)$ such that $\chi = \delta\psi$ by Gallagher's Theorem. Since δ extends φ and G/E acts faithfully on $(E/Z)^{\#}$, we know that $\ker(\delta) \leqslant E$. Also, since $\chi = \delta\psi$ is faithful and $\ker(\delta) \leqslant E$, it must also be the case that $\ker(\delta) = \ker(\chi) \cap E = 1$ and δ is a faithful character. Thus, by Lemma 3.3.2 and Theorem A of [2],

$$\delta\overline{\delta} = 1_G + \gamma,$$

for some $\gamma \in \mathrm{Irr}(G)$. By Lemma 3.1.2 of [2], $Z = Z(G) = \ker(\gamma)$.

Let

$$\psi\overline{\psi} = 1_G + \sum_{i=1}^{n} c_i\beta_i$$

for some positive integers c_i and $\beta_i \in \mathrm{Irr}(G)$. Since $\psi \in \mathrm{Irr}(G/E)$, we know that $\beta_i \in \mathrm{Irr}(G/E)$ for all i. Also,

$$1_G + m_1\alpha_1 + m_2\alpha_2 + m_2\overline{\alpha_2} = \chi\overline{\chi}$$

$$= 1_G + \gamma + \sum_{i=1}^{n} c_i\beta_i + \sum_{i=1}^{n} c_i\gamma\beta_i.$$

Since $\ker(\beta_i) \geqslant E$ and $\ker(\gamma) = Z < E$, we know that γ is distinct from the β_i. Also, $n \leqslant 2$ since we know that we have exactly three nonprincipal irreducible constituents of $\chi\overline{\chi}$. If $n = 1$, then we get that $\psi\overline{\psi} = 1_G + \alpha_2$. But since $\psi\overline{\psi}$ is a real character and α_2 is not, this is impossible. Thus $n = 2$. As $m_2 = 1$, $\ker(\alpha_1) = Z$, and $\ker(\alpha_2) \geqslant E$, we get that

$$\alpha_1 = \gamma,\ \alpha_2 = \beta_1,\ \overline{\alpha_2} = \beta_2\ \text{, and}\ \ \gamma\beta_1 + \gamma\beta_2 = 2\alpha_2(1)\alpha_1.$$

This implies that $\psi\overline{\psi} = 1_G + \alpha_2 + \overline{\alpha_2}$.

Let $H = \ker(\psi)$ and $Z_H/H = Z(G/H)$. We know that

$$\psi(1)^2 = 1 + 2\alpha_2(1) > 1.$$

Hence $\psi(1) > 1$ and since ψ is a faithful irreducible character of G/H, it follows that G/H is nonabelian. Thus, there exists some $L \lhd G$ such that L/Z_H is a chief factor of G. Notice that $\psi\overline{\psi}$ has two nonprincipal irreducible constituents, which are complex conjugates. Thus Theorem 1.1 finishes the claim.

Claim 3 $e \neq 9$.

Recall that $E \leqslant K = \ker(\alpha_2)$. Thus, χ_E is reducible. Also, since E/Z is fully ramified with respect to φ, we have that $\chi(1) = em$ for $m \geqslant 2$. Now, $\chi(g) = 0$ for all $g \in E - Z$. Thus

$$\chi\overline{\chi}(g) = 0 = 1_G + m_1\alpha_1(g) + \alpha_2(1) + \overline{\alpha_2}(1),$$

and

$$\alpha_1(g) = -\frac{1 + 2\alpha_2(1)}{m_1}.$$

Since $\alpha_1 \in \mathrm{Irr}(G)$, $\alpha_1(g)$ is an algebraic integer for all $g \in G$. Since

$$\frac{1 + 2\alpha_2(1)}{m_1}$$

is a rational algebraic integer, it must be in $\mathbb{Z}$. Thus m_1 divides $1+2\alpha_2(1)$. By hypothesis,

$$\alpha_1(1) = \frac{\chi(1)^2}{m_1} - \frac{1+2\alpha_2(1)}{m_1},$$

which implies that

$$(\alpha_1)_E = s1_Z^E - \frac{1+2\alpha_2(1)}{m_1}$$

for some integer $s \geqslant \frac{1+2\alpha_2(1)}{m_1}$. Therefore

$$se^2 - \frac{1+2\alpha_2(1)}{m_1} = \alpha_1(1) = \frac{m^2e^2}{m_1} - \frac{1+2\alpha_2(1)}{m_1}$$

and $s = m^2/m_1$. In particular, m_1 divides m^2. Also, since $\alpha_1 \in \mathrm{Irr}(G)$ and $Z = \ker(\alpha_1)$, we have that $\alpha_1(1)^2 < |G:Z|$ and $\alpha_1(1)$ divides $|G:Z|$. So assume that $e = 9$ and recall that $\chi(1) = em$. Then by Lemma 4.3 (b), we know that $E < K$ and α_2 is not a faithful character of G/E. Also, we have that $|G:Z|$ divides $10 \times 32 \times 81 = 25920$ by Theorem 2.2.1(v) of [2]. By the above statements and Lemma 4.2.12 of [2], we know that $\alpha_1 \in \{1,2,4,5,10\}$ and $m_1 \in \{1,3,5,7,9,11,21\}$. Thus

$$\frac{1+2\alpha_2(1)}{m_1} \leqslant 21.$$

Now, assume $m_1 = 1$. Then

$$\alpha_1(1) = \frac{\chi(1)^2}{m_1} - \frac{1+2\alpha_2(1)}{m_1} = \chi(1)^2 - (1+2\alpha_2(1))$$

$$\geqslant 4 \times 81 - 21 = 303.$$

But then $\alpha_1(1)^2 \geqslant 303^2 > 25920 \geqslant |G:Z|$, and this is a contradiction. So $m_1 \neq 1$, and hence we may assume that $m_1 \in \{3,5,7,9,11,21\}$. Then $m^2 \geqslant 9$ and

$$\alpha_1(1) = \frac{\chi(1)^2}{m_1} - \frac{1+2\alpha_2(1)}{m_1} \geqslant \frac{9 \times 9^2}{3} - 21 = 3 \times 81 - 21 = 222.$$

However, $\alpha_1(1)^2 \geqslant 222^2 > 25920 \geqslant |G:Z|$, and we again have a contradiction. So $e \neq 9$.

As we have examined all possible values of e, the proof is complete. □

We are now ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2 — Suppose our hypotheses. Then, by Propositions 3.6, 3.8, 3.10, 4.4, and 4.5, we know that both $\ker(\alpha_1)$ and $\ker(\alpha_2)$ must be abelian, with one of them properly containing $Z = Z(G)$. Therefore, by Propositions 3.8 and 3.10, we know that $\mathrm{dl}(G) \leqslant 6$. □

At this point, it is unclear whether there is an example satisfying the hypotheses of Proposition 3.8, and it would be interesting to learn if this proposition is impossible. Attempting to find such a group has only led to cases were $\chi\overline{\chi}$ has three real nonprincipal irreducible constituents and $\mathrm{dl}(G) \leqslant 6$. While this is not helpful in our special situation, it does show that if we generalize to the situation with $\chi\overline{\chi}$ having three nonprincipal irreducible constituents, we can do no better than $\mathrm{dl}(G) \leqslant 6$.

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