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Some Different Results on $\mathcal{MS}$-Groups and $\mathcal{MSN}$-Groups

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Abstract

Let P and Q be different normal Sylow subgroups of the finite group G. If G/P and G/Q are soluble PST-groups (respectively BT-groups), then G is also a soluble PST-group (respectively BT-group). These results have been known for several years. In this paper we establish similar results for MS-groups and MSN-groups.

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1 Introduction

In the following G always denotes a finite group. Recall that a subgroup H of a group G is said to *permute* with a subgroup K of G if HK is a subgroup of G. The subgroup H is said to be *permutable* in G if H permutes with all subgroups of G.

There are many articles in the literature (for instance [1],[3],[4],[9], [10],[15] to name six) where global information about a group G is obtained by assuming that all p-subgroups H, p a prime, of a given

* Jim Beidleman passed away on August 29, 2017. AGTA grieves for a great friend.

order satisfy a sufficiently strong embedding property extending permutability. In many cases, the subgroups H are the maximal subgroups of the Sylow p-subgroups of G, and the embedding assumption is that they are S-semipermutable in G.

Following [11], we say that a subgroup X of a group G is said to be S-semipermutable in G provided that it permutes with every q-Sylow subgroup of G for all primes q not dividing $|H|$. We define the class of *MS-groups* to be the class of groups G in which the maximal subgroups of all the Sylow subgroups of G are S-semipermutable in G. This class was studied in [5], [6] and [10].

Suppose that X is a subnormal S-semipermutable subgroup of a group G. If P is a subgroup (respectively, Sylow subgroup) of G with $\gcd(|X|, |P|) = 1$, then X is a subnormal Hall subgroup of XP, and so X is normalized by P. This observation motivates the following definition (see [9]).

A subgroup X of a group G is said to be *seminormal* (respectively, *S-seminormal*) in G if it is normalized by every subgroup (respectively, Sylow subgroup) K of G such that $\gcd(|X|, |K|) = 1$.

Note that the term *seminormal* has different meanings in the literature. By [9], Theorem 1.2, a subgroup of a group is seminormal if and only if it is S-seminormal. Furthermore, a Sylow 2-subgroup of the symmetric group of degree 3 is an example of an S-semipermutable subgroup which is not seminormal.

We say that a group G is an *MSN-group* if the maximal subgroups of all the Sylow subgroups of G are seminormal in G. It is clear that the class of all MSN-groups is a subclass of the class of all MS-groups. To show that this inclusion is proper is the aim of the following example.

Example 1.1 (see [6]) Let $A = \langle y \rangle \times \langle z \rangle$ be a cyclic group of order 18 with y an element of order 9 and z an element of order 2. Let V be an irreducible A-module over the field of 19 elements such that $C_A(V) = \langle z \rangle$. Then V is a cyclic group of order 19. Let $G = V \rtimes A$ be the semidirect product of V by A. The maximal subgroups of the Sylow subgroups are either trivial or cyclic of order 3. Since V and $\langle z \rangle$ are normal Sylow subgroups of G, it follows that the maximal subgroups of the Sylow 3-subgroups are S-permutable. Hence G is an MS-group. However, the cyclic subgroups of order 3 are not normalized by V and so G is not an MSN-group.

The main purpose of this paper is to provide several new properties of MS-groups and MSN-groups. We now collect the definitions and results which are used to prove our theorems.

The books [4] and [16] will be the main reference for terminology and results on permutability. S-semipermutability and seminormality are closely related to the following subgroup embedding property introduced by Kegel in [12].

A subgroup H of G is said to be *S-permutable* in G if H permutes with every Sylow p-subgroup of G for every prime p.

The following classes of groups have been extensively studied in recent years. They play an important role in the structural study of groups.

1. A group G is a T-*group* if normality is a transitive relation in G. That is, if every subnormal subgroup of G is normal in G.

2. A group G is a PT-*group* if permutability is a transitive relation in G. That is, if H is permutable in K and K is permutable in G, then H is permutable in G.

3. A group G is a PST-*group* if S-permutability is a transitive relation in G. That is, if H is S-permutable in K and K is S-permutable in G, then H is S-permutable in G.

A classical result of Kegel shows that every S-permutable subgroup must be subnormal [12], Theorem 1.2.14 (3). Therefore, a group G is a PST-group (respectively a PT-group) if and only if every subnormal subgroup is S-permutable (respectively permutable) in G.

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular p-groups) and PST does not imply PT (non-modular p-groups).

Another interesting class of groups in this context is the class of T_0-groups studied in [2],[7] and [17]: a group G is called a T_0-*group* if the Frattini factor group $G/\Phi(G)$ is a T-group.

The following example shows that the class of all T_0-groups properly contains the class of all T-groups.

Example 1.2 Let $E = \langle x, y \rangle$ be an extraspecial group of order 27 and exponent 3. Let a be an automorphism of order 2 of G given by $x^a = x^{-1}$, $y^a = y^{-1}$. Let $G = E \rtimes \langle a \rangle$ be the corresponding semidirect product. Clearly G is a T_0-group. The subgroup $H = \langle x \rangle$ is a subnormal subgroup of G which does not permute with the Sylow 2-subgroup $\langle ay \rangle$. Therefore H is not S-permutable. Hence G is not a PST-group and so is not a T-group either.

The following theorem shows that soluble T_0-groups are closely related to PST-groups.

Theorem 1.3 (see [13], Theorems 5, 7 and Corollary 3) *Let* G *be a soluble* T_0*-group with nilpotent residual* $L = \gamma_\infty(G)$. *Then:*

1. G *is supersoluble.*
2. L *is nilpotent Hall subgroup of* G.
3. *If* L *is abelian, then* G *is a* PST*-group.*

Here the *nilpotent residual* $\gamma_\infty(G)$ of a group G is the smallest normal subgroup N of G such that G/N is nilpotent, that is, the limit of the lower central series of G defined by

$$\gamma_1(G) = G \quad \text{and} \quad \gamma_{i+1}(G) = [\gamma_i(G), G]$$

for $i \geqslant 1$.

Let G be a group whose nilpotent residual $L = \gamma_\infty(G)$ is a Hall subgroup of G. Let $\pi = \pi(L)$ and let $\theta = \pi'$, the complement of π in the set of all prime numbers. Let θ_N denote the set of all primes p in θ such that if P is a Sylow p-subgroup of G, then P has at least two maximal subgroups. Further, let θ_C denote the set of all primes q in θ such that if Q is a Sylow q-subgroup of G, then Q has only one maximal subgroup, or equivalently, Q is cyclic.

Throughout this paper we will use the notation presented above concerning π, $\theta = \pi'$, θ_N, and θ_C. Note that $\theta = \theta_N \cup \theta_C$ and $\theta_N \cap \theta_C = \emptyset$.

We now present a characterization of MS-groups established in [5].

Theorem 1.4 *Let* G *be a group with nilpotent residual* $L = \gamma_\infty(G)$. *Then* G *is an MS-group if and only if* G *satisfies the following:*

1. G *is a* T_0*-group.*
2. L *is a nilpotent Hall subgroup of* G.
3. *If* $p \in \pi$ *and* $P \in Syl_p(G)$, *then a maximal subgroup of* P *is normal in* G.
4. *Let* p *and* q *be distinct primes with* $p \in \theta_N$ *and* $q \in \theta$. *If* $P \in Syl_p(G)$ *and* $Q \in Syl_q(G)$, *then* $[P, Q] = 1$.

5. *Let* p *and* q *be distinct primes with* $p \in \theta_C$ *and* $q \in \theta$. *If* $P \in Syl_p(G)$ *and* $Q \in Syl_q(G)$ *and* M *is the maximal subgroup of* P, *then* $QM = MQ$ *is a nilpotent subgroup of* G.

Our next result (see [5]) gives precise conditions for an MS-group to be a MSN-group. It is, therefore, a characterization theorem.

Theorem 1.5 *A group* G *is an* MSN*-group if and only if* G *satisfies the following conditions:*

1. G *is a* MS*-group.*

2. *Let* p *and* q *be distinct primes with* $p \in \pi$ *and* $q \in \theta_N$. *If* $P \in Syl_p(G)$ *and* $Q \in Syl_q(G)$ *then* $[P, Q] = 1$.

3. *Let* p *and* q *be distinct primes with* $p \in \pi$ *and* $q \in \theta_C$. *If* $P \in Syl_p(G)$ *and* $Q \in Syl_q(G)$ *and* T *is a maximal subgroup of* Q *then* $[P, T] = 1$.

A group G is called a BT-*group* if semipermutability is a transitive relation in G. In a very interesting paper, Yangming Li, Lifang Wang, and Yanming Wang [18] prove the following theorem about solvable BT-groups.

Theorem 1.6 *Let* G *be a group with nilpotent residual* L. *The following statements are equivalent:*

1. G *is a solvable* BT*-group.*

2. *Every subgroup of* G *of prime power order is semipermutable in* G.

3. *Every subgroup of* G *is semipermutable in* G.

4. G *is a solvable* PST*-group and if* p *and* q *are distinct primes not dividing the order of* L *with* $G_p \in Syl_p(G)$ *and* $G_q \in Syl_q(G)$ *then* $[G_p, G_q] = 1$.

Note that the class of solvable BT-groups is subgroup and quotient closed.

The next theorem shows that under certain assumptions homomorphic images can be used to find conditions on such images that yield information for soluble PST-groups, soluble BT-groups, soluble PT-groups, and soluble T-groups.

Theorem 1.7 (see [2]) *Let* G *be a group with normal Sylow subgroups* $P \in Syl_p(G)$ *and* $Q \in Syl_q(G)$ *with* p *and* q *distinct primes. Then*

1. *If* G/P *and* G/Q *are soluble* PST*-groups, then also* G *is a soluble* PST*-group.*

2. *If* G/P *and* G/Q *are soluble* BT*-groups, then* G *is a soluble* BT*-group.*

3. *If* G/P *and* G/Q *are soluble* T*-groups* (PT*-groups*)*, then* G *is a soluble* T*-group* (PT*-group*)*.*

One of our purposes in this paper is to determine if theorems like Theorem 1.7 can be established for MS-groups and MSN-groups. These questions are answered in our next two theorems, namely Theorems A and B.

Theorem A *Let* G *be a group with distinct normal Sylow subgroups* P *and* Q*. If* G/P *and* G/Q *are* MS*-groups, then* G *is an* MS*-group.*

Theorem B *Let* G *be a group with distinct normal Sylow subgroups* P *and* Q*. If* G/P *and* G/Q *are* MSN*-groups, then* G *is an* MSN*-group.*

We next provide three results which give some information about MS-groups, MSN-groups and BT-groups. Theorems A and B are used in the proofs of these results.

Theorem C *Let* L *be the nilpotent residual of a group* G *and let* N *be a nilpotent normal Hall subgroup of* G *such that* G/N'' *is an* MS*-group. If* L *is abelian, then* G *is an* MS*-group.*

Theorem D *Let* L *be the nilpotent residual of a group* G *and let* N *be a nilpotent normal Hall subgroup of* G *such that* G/N'' *is an* MSN*-group. If* L *is abelian, then* G *is an* MSN*-group.*

Theorem E *Let* L *be the nilpotent residual of a group* G *and let* N *be a nilpotent normal Hall subgroup of* G *such that* G/N'' *is a soluble* BT*-group. If* L *is abelian, then* G *is a soluble* BT*-group.*

We remark that theorems like Theorem E can be established for soluble PST-groups and soluble PT-groups using Theorem 1.7.

2 Preliminary lemmas and some examples

In this section we list several lemmas which will be useful in the proofs of our main theorems.

Lemma 2.1 (see [10]) *Let* N *be a normal subgroup of a group* G*. Then:*

1. *If* G *is an* MS*-group, then* G/N *is also an* MS*-group.*
2. *If* G *is an* MSN*-group, then* G/N *is also an* MSN*-group.*

Example 2.2 Let

$$G = \langle a, x, y \mid a^2 = x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1,\\ x^a = x^{-1}, y^a = y^{-1}\rangle.$$

Then $H = \langle x, y\rangle$ is an extraspecial group of order 27 and exponent 3. Let $z = [x, y]$, so $z^a = z$. Then

$$\Phi(G) = \Phi(H) = \langle z\rangle = Z(G) = Z(H).$$

Note that $G/\Phi(G)$ is a T-group so that G is a T_0-group. The maximal subgroups of H are normal in G and it follows that G is an MS-group. Let $K = \langle x, z, a\rangle$. Then $\langle xz\rangle$ is a maximal subgroup of $\langle x, z\rangle$, the Sylow 3-subgroup of K. However, $\langle xz\rangle$ does not permute with $\langle a\rangle$ and hence $\langle xz\rangle$ is not an S-semipermutable subgroup of K. Therefore, K is not an MS-subgroup of G. Also note that $\Phi(K) = 1$ and so K is not a T-subgroup and K is not a T_0-subgroup of G. Hence the class of soluble T_0-groups is not closed under taking subgroups. Note that G is a soluble group which is not a PST-group.

Example 2.3 Let

$$G = \langle y, z, x \mid y^9 = z^2 = x^{19^2} = 1, [y, z] = 1, x^y = x^{62}, x^z = x^{-1}\rangle.$$

Then the soluble group G is a PST-group, but G is not an MS-group since $\left[\langle y^2\rangle^x, z\right] \neq 1$.

Example 2.2 shows that the class of MS-groups and the class of T_0-groups are not subgroup closed. Example 2.3 shows that a soluble PST-group need not be an MS-group.

Lemma 2.4 (see [14]) *If* G *is an* MS*-group, then* G *is supersoluble.*

Lemma 2.5 (see [8]) *Let* N *be a normal subgroup of the group* G *such that* N *and* G/N'' *are supersoluble. Then* G *is supersoluble.*

3 Proofs of the main theorems

Proof of Theorem A — Most of the proof given here was in my paper with Ragland (see [10]). However for the sake of completeness we give a somewhat new proof.

Let G be a group with distinct normal Sylow subgroups P and Q such that G/P and G/Q are MS-groups. We are to prove that G is a MS-group. By Lemma 2.4 G/P and G/Q are supersoluble. Since this class of supersoluble groups is a formation and the fact that $P \cap Q = \langle 1 \rangle$, it follows that G is supersoluble.

Let H be a Hall $\{p, q\}'$-subgroup of G then $G = (P \times Q) \rtimes H$, the semidirect product of $(P \times Q)$ by H.

Let M be a maximal subgroup of P. Then $[M, Q] \leqslant P \cap Q = 1$ so that $[M, Q] = 1$. Likewise, if W is a maximal subgroup of Q, then $[W, P] = 1$. Hence M permutes with Q and W permutes with P.

Let r be a prime divisor of $|G|$ with $r \in \{p, q\}'$ and let R be a Sylow r-subgroup of G. We may assume that R is a subgroup of H. Now $G/Q \simeq P \rtimes H$ which is an MS-group so that $MR = RM$. Hence M permutes with R.

We also note W permutes with R. Let Y be a maximal subgroup of R. Then since P and Q are normal subgroups of G, YP and YQ are subgroups of G. Hence Y permutes with P and Q. Let $t \in \{p, q\}'$ and we assume that t divides the order $|G|$ of G. Let $T \in \mathrm{Syl}_t(G)$. Note that YP/P is a maximal subgroup of G/P. Also TP/P is a Sylow t-subgroup of G/P.

Assume $t \neq r$ and note that YP/P and TP/P permute. We deduce that $(YT)P = (TY)P$. Similarly, we can deduce $(YT)Q = (TY)Q$. It now follows that

$$YT \subseteq (TY)P \cap (TY)Q = TY \quad \text{and} \quad TY \subseteq (YT)P \cap (YT)Q = YT.$$

We see that $YT = TY$. Hence, the maximal subgroups of every Sylow subgroup of G is S-semipermutative in G. Therefore, G is an MS-group. □

Proof of Theorem B — Let G be a group with distinct normal Sylow subgroups P and Q such that G/P and G/Q are MSN-groups. We note that G/P and G/Q are MS-groups so by Lemma 2.4 G/P and G/Q are supersoluble. Hence G is supersoluble. Let H be a Hall $\{p, q\}'$-subgroup of G and note $G = (P \times Q) \rtimes H$, the semidirect product of $(P \times Q)$ by H. Let M be a maximal subgroup of P.

Then $[M, Q] \leqslant [P, Q] = 1$ so that Q normalizes M. Likewise P normalizes W.

Let r be a prime divisor of $|G|$ and let $r \in \{p, q\}'$. Let R be a Sylow r-subgroup of G. We may assume that R is contained in H. Note that $G/Q \simeq P \rtimes H$ which is an MSN-group. Hence, R normalizes M.

We also mention that the maximal subgroup W of Q is normalized by R.

Let Y be a maximal subgroup of R. Since $G/Q \simeq P \rtimes H$ is an MSN-group it follows that P normalizes Y. In the same way, $G/P \simeq Q \rtimes H$ is an MSN-group so that Q normalizes Y.

We note that YP/P is a maximal subgroup of RP/P. Let t be a prime divisor of $|G|$ where $t \in \{p, q\}'$ and let $T \in \mathrm{Syl}_t(G)$. Assume $t \neq r$. Since G/P is an MSN-group TP/P normalizes YP/P so that TP normalizes YP. We note that since G/Q is an MSN-group, TQ normalizes YQ.

Hence we have that T normalizes $YQ \cap YP = Y$. Therefore, every maximal subgroup of every Sylow subgroup of G is S-seminormal in G so that G is an MSN-group. □

Proof of Theorem C — Let L be the nilpotent residual of a group G and let N be a nilpotent normal Hall subgroup of G. If L is abelian and G/N'' is an MS-group, then G is an MS-group. By Lemma 2.5 G is supersoluble. Let p be the largest prime divisor of the order of G and let P be a Sylow p-subgroup of G. Then P is normal in G and G/P satisfies the assumptions of the theorem. Hence G/P is an MS-group.

Now let t be a prime divisor of $|N|$ and let T be a Sylow t-subgroup of N. Since N is a nilpotent Hall normal subgroup of G, T is a normal Sylow t-subgroup of G. Also G/T satisfies the assumptions of the theorem and hence G/T is an MS-group. If $t \neq p$ then G is an MS-group by Theorem A. So we may assume that $t = p$ and hence $P = N$. Note that $L \neq 1$ since if G is nilpotent, then G is an MS-group. So let R be a minimal normal subgroup of G contained in L. Then G/R satisfies the assumptions of the theorem and hence, by induction, G/R is an MS-group. Now L/R is the nilpotent residual of G/R and so L/R is a Hall subgroup of G/R by statement (2) of Theorem 1.4. It follows that L is a Hall subgroup and hence $P = L = N$.

Now $L' = 1$ so that $N'' = 1$ and G is an MS-group. This completes the proof of Theorem C. □

Proof of Theorem D — Let L be the nilpotent residual of a group G and let N be a nilpotent normal Hall subgroup of G such that G/N''

is an MSN-group. We are to show the G is an MSN-group. By Lemma 2.5 G is a supersoluble group. By a proof very much like the proof of Theorem C and using Theorem B instead of Theorem A, we obtain that G is an MSN-group. □

Proof of Theorem E — Let L be the nilpotent residual of a group G and let N be a nilpotent normal Hall subgroup of G such that G/N'' is a soluble BT-group. Then we are to show that G is a soluble BT-group. Note that N is nilpotent and G/N'' is a soluble BT-group and hence by part (4) of Theorem 1.6 G/N'' is a soluble PST-group. By a theorem of Agrawal (see [1]) G/N'' is supersoluble. Hence, by Lemma 2.5, G is supersoluble.

Let p be the largest prime dividing the order of G and let P be a Sylow p-subgroup of G. Then P is normal in G. Notice that G/P satisfies the assumptions of the theorem. Let T be a Sylow t-subgroup of N where t is a prime dividing the order of N. Then T is normal in N and is also normal in G. Note that G/T satisfies the assumptions of the theorem. So, by induction, G/P and G/T are soluble BT-groups.

First assume $p \neq t$. Then G is a soluble BT-group by part (2) of Theorem 1.7. Hence assume $p = t$, then $N = P$. Let R be a minimal normal subgroup of G contained in L. Then G/R satisfies the assumptions of the theorem so that G/R is a soluble BT-group which is a soluble PST-group by statement (4) of Theorem 1.6. Now L/R is the nilpotent residual of the soluble PST-group G/R. Then, by a result of Agrawal (see [1]), L/R is a Hall subgroup of G/R and so L is a Hall subgroup of G. This means the $P = N = L$. Since $L' = 1$ it follows that $N'' = 1$ and so G is a soluble BT-group. □

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