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On the ESQ Property of Certain Representations of Metacyclic Groups *

János Wolosz

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Abstract

A group representation is said to have the ESQ property if it is isomorphic to a quotient of its own exterior square. Let us denote the semidirect product of cyclic groups $Z_p \rtimes Z_q$ by $F_{p,q}$, where p is a prime and $q \mid p-1$. We investigate whether $F_{p,q}$ has an irreducible representation with the ESQ property. Fixing one of the parameters q or $\frac{p-1}{q}$, we will be able to give an asymptotic answer to this question.

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1 Introduction

Let G be a group which acts linearly on the vector space V defined over the field $\mathbb{F}$. Considering V as an $\mathbb{F}G$-module, we say that this module has the ESQ property if it is isomorphic to a quotient of its exterior square (ESQ stands for Exterior Self-Quotient).

This concept in representation theory first appeared in an article by Glasby, Pálfy and Schneider [2]. The authors of this paper have investigated the problem of determining those finite groups

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which have low dimensional irreducible faithful representations with the ESQ property. They have considered representations over arbitrary fields but in the present paper our base field will be the complex numbers. Let us suppose that V is an irreducible $\mathbb{C}G$ module with character χ. In this classical case the ESQ property of V is equivalent to the relation $V \leqslant \Lambda^2 V$, which can be expressed through the inner product of characters as $\langle \chi, \widehat{\chi} \rangle \neq 0$, where

$$\widehat{\chi}(g) = \frac{\chi^2(g) - \chi(g^2)}{2}.$$

Over the complex field the results of [2] on low dimensional irreducible representations with the ESQ property can be summarized as follows:

– the only finite group which has a four dimensional irreducible faithful representation with the ESQ property is $AGL_1(5)$.

– the unique minimal finite group which has a five dimensional irreducible faithful representation with the ESQ property is the nonabelian group of order 55.

All finite subgroups of $SO(3)$ give rise to a three dimensional faithful representation with the ESQ property since the natural representation of $SO(3)$ is isomorphic to its own exterior square. This observation implies that there is no such a fine classifying result in dimension three as in dimensions four and five. The groups which have appeared in the low dimensional results are semidirect products of two cyclic groups $F_{p,q} = Z_p \rtimes_\varphi Z_q$ where p is a prime, $q \mid p-1$ and φ is an injective $Z_q \longrightarrow Z_p^\times$ homomorphism. We denote the fraction $\frac{p-1}{q}$ by r. This notation remains fixed throughout this article. The groups $F_{p,q}$ are unique up to isomorphism for any given values of p and q. The present paper will investigate the problem whether $F_{p,q}$ has an irreducible representation with the ESQ property. This investigation will produce an infinite number of different irreducible ESQ representations even in dimension six. Our main results are the following.

Theorem 1.1 *Let q be fixed. If $6 \nmid q$ then for any sufficiently large p with $q \mid p-1$ the metacyclic group $F_{p,q}$ does not have any irreducible representation with the ESQ property. If $6 \mid q$ then every $F_{p,q}$ has an irreducible representation with the ESQ property.*

Theorem 1.2 *If* r *is fixed then for any sufficiently large* p *the metacyclic group* $F_{p,q}$ *has an irreducible representation with the ESQ property.*

The character table of an arbitrary $F_{p,q}$ is well known. All nonlinear irreducible characters of $F_{p,q}$ are associated with q dimensional faithful representations. As it will turn out, either all of these representations will have the ESQ property or none of them. The proof of Theorem 1.1 and 1.2 will be based on the decomposition of the exterior square of a nonlinear character of $F_{p,q}$. Considering this decomposition we will see that the ESQ property of a q dimensional irreducible representation of $F_{p,q}$ will be equivalent to the solvability of a Fermat type equation in $\mathbb{F}_p^\times$.

2 The proof of Theorem 1.1 and 1.2

To prove the Theorem 1.1 and 1.2 we will make use of the irreducible character values of $F_{p,q}$. Let us fix an element u of order q in $\mathbb{F}_p^\times$, some coset representatives $v_1, v_2, \ldots, v_r$ of $\langle u \rangle$ in $\mathbb{F}_p^\times$ and $\varepsilon = e^{2\pi i/p}$. Finally, we fix the generators a and b in the cyclic groups Z_p and Z_q respectively. A full description of the irreducible characters of $F_{p,q}$ can be found in [3], Theorem 25.10, which we cite here.

Theorem 2.1 *The group* $F_{p,q} = Z_p \rtimes_\varphi Z_q$ *has* $q + r$ *irreducible characters. There are* q *linear characters and* r *characters of degree* q *given by*

$$\chi_t((a^x, b^y)) = 0 \text{ if } b^y \neq 1,$$

$$\chi_t((a^x, 1)) = \sum_{s \in v_t \langle u \rangle} \varepsilon^{sx}$$

for $t = 1, 2, \ldots, r$.

The linear representations of $F_{p,q}$ obviously cannot be ESQ. For the nonlinear irreducible representations of $F_{p,q}$ the following lemma will provide a reformulation of the ESQ property. We will call a group character ESQ if and only if it corresponds to a representation with the ESQ property.

Lemma 2.2 *Let χ_t be a nonlinear character of $F_{p,q}$. Then χ_t is ESQ if and only if there exist natural numbers k, l, and an element $z \in \mathbb{F}_p^\times$ satisfying*

$$z^k = z^l + 1,\ z^l \neq 1,\ z^q = 1.$$

Proof — We denote the subgroup $\langle (a, 1) \rangle$ in $F_{p,q}$ by A and the exterior square of the character χ_t by $\widehat{\chi_t}$. Using Theorem 2.1, we can determine the character values of $\widehat{\chi_t}$ on A as follows:

$$\widehat{\chi_t}((a^x, 1)) = \frac{\chi_t^2((a^x, 1)) - \chi_t((a^{2x}, 1))}{2}$$

$$= \frac{1}{2}\Big(\sum_{s,s' \in \nu_t \langle u \rangle} \varepsilon^{(s+s')x} - \sum_{s \in \nu_t \langle u \rangle} \varepsilon^{2sx} \Big) = \sum_{0 \leqslant i < j < q} \varepsilon^{\nu_t (u^i + u^j) x} \tag{2.1}$$

Let us take any term $\varepsilon^{\nu_t(u^i+u^j)x}$ of (2.1). If $u^i + u^j = 0$ in $\mathbb{F}_p$ then this term will be 1, which can be interpreted as $\mathbb{1}((a^x, 1))$, where $\mathbb{1}$ denotes the trivial character of $F_{p,q}$. If $u^i + u^j \neq 0$ then we isolate the following partial sum of (2.1):

$$\sum_{k=0}^{q-1} \varepsilon^{\nu_t (u^{i+k} + u^{j+k}) x}.$$

In this case the exponents

$$\nu_t(u^{i+k} + u^{j+k}) \text{ for } k = 0, 1, \ldots, q-1$$

form a coset of $\langle u \rangle$ in $\mathbb{F}_p^\times$, so we may assume that this coset corresponds to some coset representative ν_l. This observation implies that the isolated partial sum equals $\chi_l((a^x, 1))$. By repeating this procedure we will get a decomposition of the restricted character $\widehat{\chi_t}|_A$ to some nonlinear irreducible characters of $F_{p,q}$ restricted to A and $\mathbb{1}|_A$. We denote the set of constituents in this decomposition by D.

The restrictions of nonlinear characters of $F_{p,q}$ to A and $\mathbb{1}|_A$ are pairwise orthogonal characters of A, since they are orthogonal in $F_{p,q}$, and with the exception of $\mathbb{1}$ all of them vanish on $F_{p,q} \setminus A$. This observation implies the following equivalence

$$\chi_t|_A \in D \iff \langle \chi_t|_A, \widehat{\chi_t}|_A \rangle_A \neq 0 \iff \chi_t \text{ is ESQ}.$$

We have seen from the decomposition procedure that $\chi_t|_A \in D$ if and only if $u^i + u^j \in \langle u \rangle$ for some $0 \leqslant i < j < q$. Choosing k appropriately, we can write this relation as $u^k = u^l + 1$ where $l = j - i$. The lemma follows. □

An immediate consequence of 2.2 is that either all of the nonlinear representations of $F_{p,q}$ have the ESQ property or none of them has it. Since the elements which satisfy the equation $z^q = 1$ are the r^{th} powers in $\mathbb{F}_p^\times$, we can characterize the ESQ property by the solvability of a Fermat type equation as follows.

Lemma 2.3 *Let χ_t be a nonlinear character of $F_{p,q}$. Then χ_t is ESQ if and only if the equation*

$$x^r + y^r = z^r$$

has a solution in $\mathbb{F}_p^\times$, such that $x^r \neq y^r$.

Now we prove Theorem 1.1 with the help of Lemma 2.2.

Proof — We start our proof with the assumption that $6 \nmid q$. We will show that the equation system $z^k = z^l + 1$, $z^q = 1$ has no solution in $\mathbb{F}_p$, where p is a sufficiently large prime, $q \mid p - 1$, and k, l are arbitrary nonnegative integers. Let us fix some values for k and l. As $z^q = 1$ we may assume that $0 \leqslant k, l < q$.

Our aim is to show that the polynomials

$$f(z) = z^k - z^l - 1 \text{ and } g(z) = z^q - 1$$

have no common root in $\mathbb{F}_p$. First we will show that f and g have no common root over $\mathbb{C}$. For a proof of contradiction , assume that there is a $t \in \mathbb{C}$ which satisfies $f(t) = g(t) = 0$. As $t^q - 1 = 0$, it is clear that t and t^k are on the complex unit circle centered in 0. On the other hand, $t^k = t^l + 1$ implies that t^k is also on the unit circle centered in 1. These circles have two common points, which are the primitive sixth roots of unity so the order of t^k is 6. The order of t^k divides the order of t. Using the equation $t^q = 1$, we get $6 \mid q$, which contradicts our starting assumption.

The resultant of two polynomials can be computed by a determinant whose entries are either zeroes or coefficients of the polynomials. This value will be zero if and only if the polynomials have a common root. We conclude that the resultant of f and g over $\mathbb{C}$ is a nonzero integer R_{kl}. The resultant of f and g over $\mathbb{F}_p$ is given by the same determinant as in the complex case, so its value will be the residue of R_{kl} modulo p. This will be nonzero if we choose p to be greater than $|R_{kl}|$. Because of the condition $0 \leqslant k, l < q$ there are only finitely many pairs k, l. Using this observation, we may define C to be $\max_{0 \leqslant k,l < q} |R_{kl}|$. This choice ensures that for any prime p greater than C, the equation system $z^k = z^l + 1,\ z^q = 1$ will have no solution in $\mathbb{F}_p$. Hence by Lemma 2.2 the first statement of Theorem 1.1 follows. Now we deal with the case $6 \mid q$. As $q \mid p-1$, there exists an element g of order 6 in $\mathbb{F}_p^\times$. Since the sixth cyclotomic polynomial is $\Phi_6(z) = z^2 - z + 1$, we conclude that $g^2 - g + 1 = 0$. This implies that the choice $z = g, k = 1$ and $l = 2$ satisfies the equation system in Lemma 2.2. The second statement of Theorem 1.1 follows. $\square$

Finally, we prove Theorem 1.2. Without the assumption $x^r \neq y^r$ the solvability of a Fermat type equation in Lemma 2.3 is a well investigated problem. Schur proved in 1916 that a fixed degree Fermat equation has a solution in $\mathbb{F}_p^\times$ for almost every prime p (see [4]). Now we briefly give the proof for this statement.

Lemma 2.4 *For every positive integer* c*, there exists* $s(c) \in \mathbb{N}^+$ *, so that for an arbitrary coloring of the set* $S(c) = \{1, 2, \ldots, s(c)\}$ *by* c *colors, there will be a monochromatic solution for the equation* $x + y = z$ *(here* x *and* y *can be equal).*

Proof — For an arbitrary coloring of the set $T = \{1, 2, \ldots, t\}$ we assign an edge coloring of the complete graph K_t as follows. The edge e_{ij} will get the color of $|i - j| \in T$. As c is fixed, if t is greater then some constant $R(c, 3)$, then the Ramsey's theorem ensures that K_t will have a monochromatic triangle. Let us denote the vertices of such a monochromatic triangle by $p < q < r$. Now the equation

$$(q - p) + (r - q) = r - p$$

is a monochromatic solution for $x + y = z$, hence the lemma is now demonstrated. $\square$

For $s(r) < p$ let us denote a primitive root of $\mathbb{F}_p^\times$ by g. We color the elements in the coset $g^i \langle g^r \rangle \subset \mathbb{F}_p^\times$ with color i where $i = 1, 2, \ldots r$. For this coloring Lemma 2.4 provides a monochromatic triple in $\mathbb{F}_p^\times$, which gives us a solution for $x^r + y^r = z^r$ in $\mathbb{F}_p^\times$. We will strengthen the statement of Lemma 2.4 in such a way that we will prove the existence of a monochromatic solution of $x + y = z$ with $x \neq y$. After this, Theorem 1.2 will follow from the previous argument of Schur and Lemma 2.3. The proof of this can be found in [1], nevertheless, we briefly present this result.

Proof — From an arbitrary coloring of the set $T = \{1, 2, \ldots, t\}$ we construct the edge coloring of K_t just as before. Now we choose t to be greater than $R(c, 4)$ so there will be an edge monochromatic K_4 with vertices $w < x < y < z$. If

$$x - w = y - x = z - y$$

holds, then $(z - y) + (y - w) = z - w$ is a monochromatic solution with $z - y \neq y - w$. If $x - w \neq y - x$, then $(x - w) + (y - x) = y - w$ will be an appropriate monochromatic solution in T. Finally, if

$$y - x \neq z - y,$$

then $(y - x) + (z - y) = z - x$ shows that the strengthened form of Lemma 2.4 is true. □

The proof of Theorem 1.2 is now complete.

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János Wolosz
ELTE Eötvös Loránd University,
Budapest (Hungary)
e-mail: janos.wolosz@gmail.com